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KODAI MATH. J.
**31** (2008), 133–149

# ON THE EXISTENCE OF T DIRECTION OF MEROMORPHIC FUNCTION CONCERNING MULTIPLE VALUES*

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**Abstract**

In this paper, by using Ahlfors' theory of covering surface, the existence of T direction of meromorphic function concerning multiple values is obtained. Results are obtained extending the previous results due to Guo, Zheng, Ng in Bull. Austral. Math. Soc., 69 (2004), 277–287. Moreover, we give an affirmative answer to the question by Wang and Gao in Bull. Austral. Math. Soc., 75 (2007), 459–468.

## 1. Introduction and main results

In this paper, *meromorphic function* always means a function meromorphic in the whole complex plane. Assume that basic definitions, theorems and standard notations of the Nevanlinna theory for meromorphic function (see [3] or [15]) are known. The singular direction of meromorphic function $f(z)$ is one of main objects of value distribution theory. Since Julia introduced the concept of Julia direction and showed its existence for a meromorphic function in 1919, several types of singular directions have been introduced and studied. In 1928, Valiron [9] introduced the concept of Borel direction and established its existence for meromorphic function. After that, Q. L. Hiong, A. Rauch, M. Tsuji, C. T. Chuang, L. Yang and G. H. Zhang etc investigated the properties of Borel direction, the details can be found in Chuang [1] and Yang [15]. Recently, J. H. Zheng [18] introduced a new singular direction, namely the T direction and its existence has been established by H. Guo, J. H. Zheng and T. W. Ng in [2]. However, it was not discussed whether there exists a T direction concerning multiple values. In this paper, we investigate this problem.

Suppose that $E$ is a measurable subset of $\mathbf{C}$, let

$$S(E,f) = \frac{1}{\pi}\int_E \left(\frac{|f'(z)|}{(1+|f(z)|^2)}\right)^2 r\, d\theta dr, \quad z = re^{i\theta}.$$

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2000 *Mathematics Subject Classification*. Primary 30D30; Secondary 30D35.

*Key words and phrases*. T direction, Nevanlinna direction, multiple values.

* The work was supported by NSF of China (10471048) and the Research Fund for the Doctoral Program of Higher Education (20050574002) and NSF of Xianning University (KT0623, KZ0629, KY0718).

Received May 1, 2007; revised October 24, 2007.

When $E = \{z \in \mathbf{C}, |z| < r\}$, we denote $S(E, f) = S(r, f)$ and

$$T(r, f) = \int_0^r \frac{S(t, f)}{t} \, dt,$$

where $T(r, f)$ is the *Ahlfors-Shimizu characteristic function.* Deonte the following angular domain by

$$\Omega(\theta, \varepsilon) = \{z \in \mathbf{C}, |\arg z - \theta| < \varepsilon\}.$$

When $E$ is a sector $\{z \in \mathbf{C}, |z| < r\} \cap \Omega(\theta, \varepsilon)$, we denote $S(E, f) = S(r, \Omega(\theta, \varepsilon), f)$ and

$$T(r, \Omega(\theta, \varepsilon), f) = \int_0^r \frac{S(t, \Omega(\theta, \varepsilon), f)}{t} \, dt.$$

For any $a \in \mathbf{C}_\infty$ and $a \neq \infty$, let $n(r, \theta, \varepsilon, a)$ be the number of zeros, counted according to their multiplicities, of $f(z) - a$ in the sector $\{z \in \mathbf{C}, |z| < r\} \cap \Omega(\theta, \varepsilon)$, and $n^{l)}(r, \theta, \varepsilon, a)$ be the number of zeros with multiplicities $\leq l$, of $f(z) - a$ in the sector $\{z \in \mathbf{C}, |z| < r\} \cap \Omega(\theta, \varepsilon)$, where $l$ is any positive integer. Similarly, note the number of poles of $f$ by $n(r, \theta, \varepsilon, \infty)$ and $n^{l)}(r, \theta, \varepsilon, \infty)$. Deonte

$$N(r, \theta, \varepsilon, a) = \int_0^r \frac{n(t, \theta, \varepsilon, a) - n(0, \theta, \varepsilon, a)}{t} \, dt + n(0, \theta, \varepsilon, a) \log r;$$

$$N^{l)}(r, \theta, \varepsilon, a) = \int_0^r \frac{n^{l)}(t, \theta, \varepsilon, a) - n^{l)}(0, \theta, \varepsilon, a)}{t} \, dt + n^{l)}(0, \theta, \varepsilon, a) \log r.$$

In addition, we also need the notations (see [17])

$$L(r, \psi_1, \psi_2) = \int_{\psi_1}^{\psi_2} \frac{|f'(re^{i\psi})|}{(1 + |f(re^{i\psi})|^2)} r \, d\psi, \quad L(r, \psi) = \int_1^r \frac{|f'(te^{i\psi})|}{(1 + |f(te^{i\psi})|^2)} \, dt.$$

Following J. H. Zheng's definitions of T direction of $f(z)$ (see [18]), we give the following definition.

Definition 1. We call $J : \arg z = \theta$ the T direction of $f(z)$, provided that given any $a \in \mathbf{C}_\infty$, possibly with exception of at most two values of $a$, for any positive number $\varepsilon < \pi$, we have

$$\limsup_{r \to \infty} \frac{N(r, \theta, \varepsilon, a)}{T(r, f)} > 0.$$

We call $J : \arg z = \theta$ the T direction of $f(z)$ concerning multiple values, provided that given any $a \in \mathbf{C}_\infty$, possibly with exception of at most $\left[\dfrac{2l + 2}{l}\right]$ values of $a$, for any positive number $\varepsilon < \pi$, we have

$$\limsup_{r \to \infty} \frac{N^{l)}(r, \theta, \varepsilon, a)}{T(r, f)} > 0,$$

where $[x]$ implies the maximum integer number which does not exceed $x$.

Note that the T direction of meromorphic function concerning multiple values is a refinement of the ordinary T direction since $\left[\frac{2l+2}{l}\right] \to 2$ as $l \to \infty$. For the existence of T direction of meromorphic function $f(z)$, H. Guo, J. H. Zheng and T. W. Ng [2] proved the following Theorem

THEOREM A. *Let $f(z)$ be a meromorphic function and satisfy*

$$\limsup_{r\to\infty} \frac{T(r,f)}{\log^2 r} = \infty. \tag{1}$$

*Then $f(z)$ must have a T direction.*

Theorem A was conjectured by J. H. Zheng [18]. In this paper, we shall study the existence of T direction of $f(z)$ concerning multiple values and prove the following Theorem.

THEOREM 1. *Let $f(z)$ be a meromorphic function and satisfy* (1), *then there at least exists a T direction of $f(z)$ concerning multiple values.*

The following example shows that any smaller number cannot replace $\left[\frac{2l+2}{l}\right]$ $(= 4, 3)$ when $l = 1, 2$ in the definition of T directions concerning multiple values.

*Example.* Consider the Weierstrass $\wp$-function $\wp(z)$ given by the differential equation

$$\{\wp'(z)\}^2 = 4\{\wp(z)^3 - 1\},$$

which has the three cube roots of unity $e^{2k\pi i/3}$ $(k = 0, 1, 2)$ and $\infty$ as completely ramified values, in fact, all these four values are double. Hence this is our desired example for $l = 1$.

A simple calculation gives its derivative $\wp'(z)$ satisfies the equation

$$\{\wp''(z)\}^3 = \frac{27}{2}[\{\wp'(z)^2 + 4\}]^2 = [6\{\wp'(z)^2]^3,$$

so that it assumes three values $\pm 2i$ and $\infty$ with multiplicity three. This is an example for $l = 2$.

## 2. Some lemmas

Our proof requires Ahlfors' theory of covering surface. We firstly introduce the following notations (see Tsuji [8]).

In this paper, the Riemann sphere of diameter 1 is denoted by $K$. Let $F$ be a finite covering surface of $F_0$, and consist of a finite number of sheets and be bounded by a finite number of analytic Jordan curves $\{\Lambda_j\}$ (some of which may reduce to single points), and the spherical distance between any two circular curves $\Lambda_i$ and $\Lambda_j$ is $d(\Lambda_j, \Lambda_j) \geq \delta \in \left(0, \frac{1}{2}\right)$. The part of the boundary of $F$, which does not lie above the boundary of $F_0$, is called the relative boundary of $F$, and denote its spherical length by $L$. Let $D$ be a domain on $F_0$, whose boundary consists a finite number of points or analytic closed Jordan curves, and $F(D)$ be the part of $F$, which lies above $D$. We denote the spherical area of $F$, $F(D)$ and $F_0$ by $|F|$, $|F(D)|$ and $|F_0|$, respectively. We put

$$S = \frac{|F|}{|F_0|}, \quad S(D) = \frac{|F(D)|}{|D|}.$$

Under the above notation, we have the following Ahlfors covering Theorem.

LEMMA 1 (see Tsuji [8]). *For any finite covering surface $F$ of $F_0$, we have*

$$|S - S(D)| < h\frac{L}{|D|},$$

*where $h > 0$ is a constant which depends on $F_0$ only.*

Recently, D. C. Sun [6] has proved a precise version of Lemma 1 and proved that $h = \dfrac{2\pi}{\delta}$, where $0 < \delta < \frac{1}{2}$ is a constant.

LEMMA 2 (see Sun [7]). *Let $F$ be a simply connected finite covering surface of a the unite sphere $K$, and $\{D_v\}$ be $q(>2)$ disjoint spherical disks on $K$, where the spherical distance of any pair of $\{D_v\}$ is at least $\delta$. Let $n_v$ be the number of simply connected islands (see Tsuji [8], P252) in $F(D_v)$, then*

$$\sum_{v=1}^{q} n_v \geq (q-2)S - \frac{C}{\delta^3}L,$$

*where $L$ is the length of the relative boundary of $F$ and $C$ is a constant.*

LEMMA 3. *Let $F$ be a simply connected finite covering surface of a sphere surface $K$, $D_v$ be $q\left(> \left[\dfrac{2l+2}{l}\right]\right)$ disjoint spherical disks with radius $\dfrac{\delta}{3}$ on $K$ and without a pair of $\{D_v\}$ such that their spherical distance is less than $\delta$, $n_v^{l)}$ be the number of simply connected islands in $F(D_v)$, which are consisted of not more than $l$ sheets, then*

$$\sum_{v=1}^{q} n_v^{l)} \geq \left(q - 2 - \frac{2}{l}\right)S - \frac{C + 9qh}{l\delta^3}L,$$

*$L$ is the length of the relative boundary of $F$.*

*Proof.* It is easy to verify that

$$n_v = n_v^{l)} + n_v^{(l}, \quad S(D_v) \geq n_v^{l)} + (l+1)n_v^{(l},$$

where $n_v^{(l}$ be the number of simply connected islands in $F(D_v)$, which are consisted of not less than $l+1$ sheets. Hence,

$$S(D_v) \geq (l+1)(n_v^{l)} + n_v^{(l)}) - ln_v^{l)} = (l+1)n_v - ln_v^{l)}.$$

Since the spherical area of $D_v$ is $|D_v| \geq \dfrac{\delta^2}{9}$. It follows from Lemma 1 that,

$$S + \frac{9h}{\delta^2}L > S(D_v) \geq (l+1)n_v - ln_v^{l)}.$$

Adding two sides of the above expression from 1 to $q$, we have

$$qS + \frac{9qh}{\delta^2}L + l\sum_{v=1}^{q} n_v^{l)} > (l+1)\sum_{v=1}^{q} n_v.$$

Combining Lemma 2 and the above expression, Lemma 3 follows. □

Lemma 4. *Suppose that $f(z)$ is a meromorphic function and $\{a_v\}$ are $q\left(>\left[\dfrac{2l+2}{l}\right]\right)$ distinct points on $K$ and without a pair of $\{a_v\}$ such that their spherical distance is less than $\delta + \dfrac{2\delta}{3}$. Let $n_v^{l)}$ be the number of zeros of $f(z) - a_v$, which are consisted of not more than $l$ multiplicities, then*

$$\sum_{v=1}^{q} n_v^{l)} \geq \left(q - 2 - \frac{2}{l}\right)S - \frac{C + 9qh}{l\delta^3}L.$$

*Proof.* Let $D_v$ be a spherical disk with the center $a_v$ with radius $\dfrac{\delta}{3}$ on $K$. By Lemma 1, we have

$$\sum_{v=1}^{q} n_v^{l)} \geq \left(q - 2 - \frac{2}{l}\right)S - \frac{C + 9qh}{l\delta^3}L.$$

Note that $n_v^{l)}(D_v) \leq n_v^{l)}(a_v)$, whenever $a_v$ in the island of $D_v$ or in the peninsula of $D_v$. Therefore, Lemma 4 follows. □

We are now in the position to establish our key Lemma by using Lemma 4.

Lemma 5. *Let $f(z)$ be meromorphic in the complex plane. If $\{a_v\}$ are $q\left(>\left[\dfrac{2l+2}{l}\right]\right)$ distinct points on $K$, then we have*

$$(2)\qquad \left(q-2-\frac{2}{l}\right)S(r,\Omega(\theta,\varphi),f) \le \sum_{v=1}^{q} n^{l)}(r,\theta,\delta,a_v) + \frac{2\pi H^2}{\left(q-2-\frac{2}{l}\right)(\delta-\varphi)}\log r$$
$$+\left(q-2-\frac{2}{l}\right)S(1,\Omega(\theta,\varphi),f)$$
$$+HL(1,\theta-\delta,\theta+\delta)+HL(r,\theta-\delta,\theta+\delta)$$

*and*

$$(3)\qquad \left(q-2-\frac{2}{l}\right)T(r,\Omega(\theta,\varphi),f) \le \sum_{v=1}^{q} N^{l)}(r,\theta,\delta,a_v) + \frac{2\pi H^2}{\left(q-2-\frac{2}{l}\right)(\delta-\varphi)}\log^2 r$$
$$+\left(q-2-\frac{2}{l}\right)T(1,\Omega(\theta,\varphi),f)$$
$$+\left(q-2-\frac{2}{l}\right)S(1,\Omega(\theta,\varphi),f)\log r$$
$$+HL(1,\theta-\delta,\theta+\delta)\log r+\chi(r,\theta-\delta,\theta+\delta)$$

*for any $\varphi$, $0<\varphi<\delta$, where $H$ is a constant depending only on $a_v$, $v=1,2,\dots q$ and $\chi(r,\theta-\delta,\theta+\delta)=H\int_1^r \frac{L(t,\theta-\delta,\theta+\delta)}{t}\,dt$.*

*Proof.* Put $D_r=\{z\in\mathbf{C},1<|z|<r\}\cap\Omega(\theta,\varphi)$ and $F_0=K-\{a_v\}$. Using Lemma 4, we have

$$\left(q-2-\frac{2}{l}\right)[S(r,\Omega(\theta,\varphi),f)-S(1,\Omega(\theta,\varphi),f)] \le \sum_{v=1}^{q} n^{l)}(r,\theta,\delta,a_v)+HL(r)$$

where $H=\dfrac{C+9qh}{l\delta^3}$, which depends only on $F_0$, i.e. only on $a_v$, $v=1,2,\dots q$, and

$$L(r)=L(r,\theta-\varphi,\theta+\varphi)+L(1,\theta-\varphi,\theta+\varphi)+L(r,\theta-\varphi)+L(r,\theta+\varphi)$$
$$\le L(r,\theta-\delta,\theta+\delta)+L(1,\theta-\delta,\theta+\delta)+L(r,\theta-\varphi)+L(r,\theta+\varphi)$$

Hence

$$(4)\qquad \left(q-2-\frac{2}{l}\right)[S(r,\Omega(\theta,\varphi),f)-S(1,\Omega(\theta,\varphi),f)]-\sum_{v=1}^{q} n^{l)}(r,\theta,\delta,a_v)$$
$$-HL(r,\theta-\delta,\theta+\delta)-HL(1,\theta-\delta,\theta+\delta)\le H[L(r,\theta-\varphi)+L(r,\theta+\varphi)]$$

Denote the left expression of (4) by $A(r,\varphi)$, thus

$$\frac{d(A(r,\varphi))}{d\varphi} = \left(q-2-\frac{2}{l}\right)\frac{d[S(r,\Omega(\theta,\varphi),f)-S(1,\Omega(\theta,\varphi),f)]}{d\varphi}.$$

We claim the fact

$$[L(r,\theta-\varphi)+L(r,\theta+\varphi)]^2 \le \frac{2\pi}{\left(q-2-\frac{2}{l}\right)}\frac{d(A(r,\varphi))}{d\varphi}\log r. \tag{5}$$

In fact, it follows from the definition of $L(r,\psi)$ and Schwarz's inequality that

$$\begin{aligned}
&[L(r,\theta-\varphi)+L(r,\theta+\varphi)]^2\\
&\le 2\left[\left(\int_1^r \frac{|f'(te^{i(\theta-\varphi)})|}{(1+|f(te^{i(\theta-\varphi)})|^2)}\,dt\right)^2+\left(\int_1^r \frac{|f'(te^{i(\theta+\varphi)})|}{(1+|f(te^{i(\theta+\varphi)})|^2)}\,dt\right)^2\right]\\
&\le 2\pi\frac{d[S(r,\Omega(\theta,\varphi),f)-S(1,\Omega(\theta,\varphi),f)]}{d\varphi}\log r\\
&= \frac{2\pi}{\left(q-2-\frac{2}{l}\right)}\frac{d(A(r,\varphi))}{d\varphi}\log r.
\end{aligned}$$

Noting $A(r,\varphi)$ is an increasing function of $\varphi$, we see that then there exists a $\delta_0>0$, such that $A(r,\varphi)>0$, when $\varphi>\delta_0$; and $A(r,\varphi)\le 0$, when $\varphi\le\delta_0$. For $\varphi>\delta_0$, by (4) and (5),

$$[A(r,\varphi)]^2 \le H^2[L(r,\theta-\varphi)+L(r,\theta+\varphi)]^2 \le \frac{2\pi H^2}{\left(q-2-\frac{2}{l}\right)}\log r\frac{d(A(r,\varphi))}{d\varphi},$$

i.e.

$$d\varphi \le \frac{2\pi H^2}{\left(q-2-\frac{2}{l}\right)}\log r\frac{d(A(r,\varphi))}{[A(r,\varphi)]^2}.$$

Integrating each side of the inequality leads to

$$\delta-\varphi=\int_\varphi^\delta d\varphi \le \frac{2\pi H^2}{\left(q-2-\frac{2}{l}\right)A(r,\varphi)}\log r$$

Thus

$$A(r,\varphi)\le \frac{2\pi H^2}{\left(q-2-\frac{2}{l}\right)(\delta-\tau)}\log r.$$

On the case of $\varphi \le \delta_0$, then the above inequality is obvious valid because of $A(r,\varphi) \le 0$. Replacing $A(r,\varphi)$ in the above inequality by its explicit expression, we see that (2) is established. Therefore

$$\begin{aligned}\left(q-2-\frac{2}{l}\right)T(r,\Omega(\theta,\varphi),f) \le{}& \sum_{v=1}^{q} N^{l)}(r,\theta,\delta,a_v) + \frac{\pi H^2}{\left(q-2-\frac{2}{l}\right)(\delta-\varphi)}\log^2 r \\ &+ \left(q-2-\frac{2}{l}\right)T(1,\Omega(\theta,\varphi),f) \\ &+ \left(q-2-\frac{2}{l}\right)S(1,\Omega(\theta,\varphi),f)\log r \\ &+ HL(1,\theta-\delta,\theta+\delta)\log r + \chi(r,\theta-\delta,\theta+\delta)\end{aligned}$$

where $\chi(r,\theta-\delta,\theta+\delta) = H\int_1^r \frac{L(t,\theta-\delta,\theta+\delta)}{t}\,dt.$ □

Lemma 6 (Zhang [17]). *Under the condition of Lemma* 5, *we have*

$$\begin{aligned}\chi(r,\theta-\delta,\theta+\delta) &= H\int_1^r \frac{L(t,\theta-\delta,\theta+\delta)}{t}\,dt \\ &\le H\sqrt{2\delta\pi S(r,\Omega(\theta,\delta),f)\log r}\end{aligned}$$

*or*

$$\chi(r,\theta-\delta,\theta+\delta) \le H\sqrt{2\delta\pi T(r,\Omega(\theta,\delta),f)}\,\log T(r,\Omega(\theta,\delta),f) \tag{6}$$

*with at most one exceptional set $E_\delta$ of $r$, where $E_\delta$ consists of a series of intervals and satisfies*

$$\int_{E_\delta} \frac{1}{r\log r}\,dr \le \frac{1}{\log T(r,\Omega(\theta,\delta),f)} < \infty.$$

Lemma 7 (Li and Gu [5]). *Suppose that $\Psi(r)$ is a nonnegative increasing function in $(1,\infty)$ and satisfies*

$$\limsup_{r\to\infty} \frac{\Psi(r)}{\log^2 r} = \infty.$$

*Then for any set $E \subset (1,\infty)$ such that $\int_E \frac{1}{r\log r}\,dr < \frac{1}{3}$, we have*

$$\limsup_{r\to\infty;\, r\in(1,\infty)-E} \frac{\Psi(r)}{\log^2 r} = \infty.$$

## 3. Proof of Theorem 1

*Proof.* Firstly, we prove the following statement. Let $m$ $(m \geq 4)$ be a fixed positive integer, $\theta_0 = 0, \theta_1 = \frac{2\pi}{m}, \dots, \theta_{m-1} = \frac{(m-1)2\pi}{m}, \theta_m = \theta_0$. We put $\triangle(\theta_i) = \left\{ z \mid |\arg z - \theta_i| < \frac{2\pi}{m} \right\}$, $\triangle^o(\theta_i) = \left\{ z \mid |\arg z - \theta_i| < \frac{\pi}{m} \right\}$, $i = 0, 1, \dots, m-1$; $\triangle(\theta_m) = \triangle(\theta_0)$, $\triangle^o(\theta_m) = \triangle^o(\theta_0)$. Then among these $m$ angular domains $\{\triangle(\theta_i)\}$, there is at least an angular domain $\triangle(\theta_i)$ such that the relative expression

$$\limsup_{r \to \infty} \frac{N^{l)}(r, \triangle(\theta_i), a)}{T(r, w)} > 0, \tag{7}$$

holds for all $a \in \mathbf{C}_\infty$ with at most $\left[\frac{2l+2}{l}\right]$ exceptions. Otherwise, for any angular domain $\triangle(\theta_i)$ $(1 \leq i \leq m)$, we have $q = \left[\frac{2l+2}{l}\right] + 1$ distinct points $a_i^j$ $(j = 1, 2, \dots, q)$ in $\mathbf{C}_\infty$ such that

$$\sum_{i=0}^{m-1} \sum_{j=1}^{q} N^{l)}(r, \triangle(\theta_{i+1}), a_{i+1}^j) = o(T(r, w)). \tag{8}$$

Applying Lemma 5 to $\triangle^o(\theta_{i+1})$, $\triangle(\theta_{i+1})$, we have

$$\begin{aligned} &\left(q - 2 - \frac{2}{l}\right) T(r, \triangle^o(\theta_{i+1}), f) \\ &\quad \leq \sum_{j=1}^{q} N^{l)}(r, \triangle(\theta_{i+1}), a_{i+1}^j) + O(\log^2 r) + \chi(r, \triangle(\theta_{i+1})), \end{aligned}$$

Noting $T(r, f) = \sum_{i=0}^{m-1} T(r, \triangle^o(\theta_{i+1}), f)$ and adding two sides of the above expression from $i = 0$ to $m-1$, we can obtain

$$\begin{aligned} &\left(q - 2 - \frac{2}{l}\right) T(r, f) \\ &\quad \leq \sum_{i=0}^{m-1} \sum_{j=1}^{q} N^{l)}(r, \triangle(\theta_{i+1}), a_{i+1}^j) + O(\log^2 r) + \sum_{i=0}^{m-1} \chi(r, \triangle(\theta_{i+1})). \end{aligned} \tag{9}$$

For any $i$, there exists a $r_i$, the inequality $T(r, \triangle^o(\theta_{i+1}), f) > e^{3m}$ would bold for $r > r_i$, while the inequality (3) does not look appropriate here. Put $E_{\triangle^o(\theta_{i+1})}$ is the set of $r$ which consists of a series of intervals and satisfies

$$\int_{E_{\triangle^o(\theta_{i+1})}} \frac{1}{r \log r} \, dr \leq \frac{1}{\log T(r, \triangle^o(\theta_{i+1}), f)} < \frac{1}{3m}.$$

Let $r_0 = \max\{r_i, 1 = 1, 2, \dots, m\}$, we have for any $i$, $T(r_0, \triangle^o(\theta_{i+1}), f) > e^{3m}$, then

$$\int_{\bigcup_{i=0}^{m-1} E_{\triangle^o(\theta_{i+1})}} \frac{1}{r \log r}\, dr \le \sum_{i=0}^{m-1} \frac{1}{\log T(r, \triangle^o(\theta_{i+1}), f)} < \frac{1}{3}.$$

Applying Lemma 7, we have

$$\limsup_{r\to\infty;\, r\in(1,\infty)-E} \frac{T(r,f)}{\log^2 r} = \infty,$$

where $E = \bigcup_{i=0}^{m-1} E_{\triangle^o(\theta_{i+1})}$. Therefore, there exists a sequence $r_n' \in (1,\infty) - E$, such that

$$\lim_{n\to\infty} \frac{T(r_n', f)}{\log^2 r_n'} = \infty. \tag{10}$$

It follows from (6), (9) and (10) that $\left(q - 2 - \frac{2}{l}\right) \le 0$. Hence $\left[\frac{2l+2}{l}\right] + 1 \le \frac{2l+2}{l}$. This is a contradiction. Hence for an arbitrary positive integer $m$, there exists an angular domain $\triangle(\theta_m) = \left\{ z \mid |\arg z - \theta_m| < \frac{2\pi}{m} \right\}$ such that for any $a$, we have

$$\limsup_{r\to\infty} \frac{N^{l)}(r, \triangle(\theta_m), a)}{T(r,f)} > 0, \tag{11}$$

except for $\left[\frac{2l+2}{l}\right]$ exceptions at most. Choosing subsequence of $\{\theta_m\}$, still denote it $\{\theta_m\}$, we assume that $\theta_m \to \theta_0$. Put $L : \arg z = \theta_0$, then $L$ is a T direction that is stated in Theorem 1.

In fact, for any $\varepsilon$ $(0 < \varepsilon < \pi/2)$, when $m$ is sufficiently large, we have $\triangle(\theta_m) \subset \Omega(\theta_0, \varepsilon)$. By (11), we have

$$\limsup_{r\to\infty} \frac{N^{l)}(r, \theta_0, \varepsilon, a)}{T(r,f)} \ge \limsup_{r\to\infty} \frac{N^{l)}(r, \triangle(\theta_m), a)}{T(r,f)} > 0$$

hold for any $a \in \mathbf{C}_\infty$ with at most $\left[\frac{2l+2}{l}\right]$ possible exceptional values of $a$. Hence Theorem 1 holds in this case. □

## 4. Concluding remarks

Let $w = w(z)$ be a $v$-valued algebroid function defined by the irreducible equation

$$A_v(z)w^v + A_{v-1}(z)w^{v-1} + \cdots + A_0(z)w = 0, \tag{12}$$

where $A_j(z)$ $(j = 0, 1, 2, \ldots, v)$ are entire functions without any common zero point. The single valued domain of definition of $w(z)$ is a $v$-sheeted covering of the $z$-plane, a Riemann surface, denoted by $\tilde{R}_z$. It is denoted by $\tilde{z}$ that the point in $\tilde{R}_z$ whose projection in the $z$-plane is $z$. The part of $\tilde{R}_z$, which covers a disk $|z| < r$, is denoted by $|\tilde{z}| < r$. For any $a \in \mathbf{C}_\infty$, put

$$N(r,a) = \frac{1}{v}\int_0^r \frac{n(t,a) - n(0,a)}{t}\,dt + \frac{n(0,a)}{v}\log r,$$

$$m(r,w) = \frac{1}{2\pi v}\int_{|\tilde{z}|=r} \log^+|w(re^{i\theta})|\,d\theta, \quad z = re^{i\theta},$$

where $n(r,a)$ is the number of zeros, counted according to their multiplicities, of $w(z) - a$ in $|\tilde{z}| \le r$. When $a = \infty$, we consider $\frac{1}{w(z)}$ instead of $w(z) - a$. Let

$$T(r,w) = m(r,w) + N(r,w).$$

The Ahlfors-Shimizu characteristic may be written as

$$\frac{1}{v}\int_0^r \frac{S(t,w)}{t}\,dt := T_0(r,w) = T(r,w) + O(1),$$

where

$$S(r,w) = \frac{1}{\pi}\int\int_{|\tilde{z}|\le r}\left(\frac{|w'(z)|}{1+|w(z)|^2}\right)^2 dw.$$

In this paper, the Ahlfors-Shimizu characteristic is the same as the $T(r,w)$ without difference. In general, suppose that $\tilde{E}$ is a subset of $\tilde{R}_z$, we denote

$$S(r,E,w) = \frac{1}{\pi}\int\int_{\tilde{E}}\left(\frac{|w'(z)|}{1+|w(z)|^2}\right)^2 dw,$$

and

$$T(r,E,w) = \frac{1}{v}\int_0^r \frac{S(t,E,w)}{t}\,dt.$$

The order and lower order of the algebroid function $w(z)$ are denoted by

$$\lambda = \limsup_{r\to\infty}\frac{\log T(r,w)}{\log r}, \quad \mu = \liminf_{r\to\infty}\frac{\log T(r,w)}{\log r}.$$

We define an angular domain $\triangle(\theta_0,\delta) = \{z|\arg z - \theta_0| < \delta\}$, $0 \le \theta_0 < 2\pi$, $0 < \delta < \frac{\pi}{2}$. The part of $\tilde{R}_z$ which lies over $\triangle(\theta_0,\delta)$ is denoted by $\tilde{\triangle}(\theta_0,\delta)$. Let $n(r,\theta_0,\delta,a)$ (or $\bar{n}(r,\theta_0,\delta,a)$) be the number of distinct zeros $w(z) - a$ in $\tilde{\triangle}(\theta_0,\delta) \cap \{|\tilde{z}| \le r\}$, counting multiplicities (or ignoring multiplicities). Put

$$N(r,\theta_0,\delta,a) = \frac{1}{v}\int_0^r \frac{n(t,\theta_0,\delta,a) - n(0,\theta_0,\delta,a)}{t}\,dt + n(0,\theta_0,\delta,a)\log r.$$

Let $\bar{n}^{l)}(r,\theta_0,\delta,a)$ be the number distinct zeros with multiplicity $\le l$ of $w(z) - a$ in $\tilde{\triangle}(\theta_0,\delta) \cap \{|\tilde{z}| \le r\}$. Similarly, we can define $\overline{N}(r,\theta_0,\delta,a)$ and $\overline{N}^{l)}(r,\theta_0,\delta,a)$.

Denote $n_\chi(r,w)$, $(n_\chi(r,\triangle(\theta_0,\delta),w))$ by the number of the branch points of $\tilde{R}_z$ on $|z|<r$ (on the region $\triangle(\theta_0,\delta)$), counting the order of branch points. Denote

$$N_\chi(r,w)=\frac{1}{v}\int_0^r\frac{n_\chi(t,w)-n_\chi(0,w)}{t}\,dt+\frac{n_\chi(0,w)}{v}\log r.$$

Similarly, we can define $N_\chi(r,\triangle(\theta_0,\delta),w)$.

For a $v$-valued algebroid function, its ordinary T direction is defined in the same way as above with the corresponding characteristic and counting functions introduced by H. Selberg for algebroid function, provided the maximum number of exceptional values permitted here is $2v$ instead 2. The existence of ordinary T direction for algebroid function has been established by one of the authors in [11] under the condition

$$\limsup_{r\to\infty}\frac{T(r,w)}{\log^2 r}=\infty. \tag{13}$$

Most recently, Wang and Gao [10] confirms the existence of T direction dealing with multiple values for an algebroid function $w(z)$ under the condition (13) and an additional condition of the lower order to be finite. Then they ask whether it is also the case without this additional condition. Here, we shall confirm this problem by proving the following Theorem 2.

Theorem 2. *Let $w(z)$ be a $v$-valued algebroid function defined on the whole complex plane and satisfy* (13). *Then there at least exists a ray $L:\arg z=\theta$ such that, for any given $b\in\mathbf{C}_\infty$, possibly with the exception of at most $\left[\frac{2l+2}{l}v\right]$ values of $b$, for an arbitrary small $\varepsilon>0$ we have*

$$\limsup_{r\to\infty}\frac{\overline{N}^{l)}(r,\theta,\varepsilon,b)}{T(r,w)}>0,$$

*for any positive integer $l\geq 3$.*

In order to prove Theorem 2, we also need the following Lemma by Xuan and Gao [13].

Lemma 8 (Xuan and Gao [13]). *Let $w(z)$ be a $v$ valued algebroid function defined by* (12). *If $a_i$, $i=1,2,\dots,q$ $(q\geq 3)$ are distinct complex numbers in $\mathbf{C}_\infty$, then we have*

$$\begin{aligned}&\left(q-2-\frac{2}{l}\right)T(r,\Delta(\varphi_0,\varphi),w)\\&\quad\leq\sum_{i=1}^{q}\overline{N}^{l)}(r,\varphi_0,\varphi,a_i)+\frac{l+1}{l}N_\chi(r,\Delta(\varphi_0,\delta),w)+O(\log^2 r)+X(r,\Delta(\varphi_0,\delta),w),\end{aligned}$$

*for any positive integer $l\geq 3$ and any $\varphi$, $0<\varphi<\delta$, where*

$$X(r, \Delta(\varphi_0, \delta), w) \leq h\sqrt{2\delta\pi T(r, \Delta(\varphi_0, \delta), w) \log r} \log T(r, \Delta(\varphi_0, \delta), w),$$

*outside a set $E_\delta$ of $r$ at most, where $h$ is a constant depending only on the $a_j$ and $E_\delta$ consists of a series of intervals and satisfies*

$$\int_{E_\delta} \frac{1}{r \log r} dr \leq \frac{1}{\log T(r, \Delta(\varphi_0, \delta), w)} < \infty.$$

We are now in the position to prove the Theorem 2.

*Proof.* Firstly, we prove the following statement. Let $w(z)$ be a $v$-valued algebroid function defined by (12) on the whole complex plane and satisfies (13). Let $m$ $(m \geq 4)$ be the positive integer, $\theta_0 = 0, \theta_1 = \dfrac{2\pi}{m}, \dots, \theta_{m-1} = \dfrac{(m-1)2\pi}{m}$, $\theta_m = \theta_0$. and $\triangle(\theta_i) = \left\{z \,|\, |\arg z - \theta_i| < \dfrac{2\pi}{m}\right\}$, $i = 0, 1, \dots, m-1$; $\triangle(\theta_m) = \triangle(\theta_0)$, then among these $m$ angular domains $\{\triangle(\theta_i)\}$, there is at least an angular domain $\triangle(\theta_i)$ such that the relative expression

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, \triangle(\theta_i), a)}{T(r, w)} > 0, \tag{14}$$

holds for all $a \in \mathbf{C}_\infty$ with at most $\left[\dfrac{2l+2}{l} v\right]$ exceptions. If on the contrary, then for any angular domain $\triangle(\theta_i)$ $(1 \leq i \leq m)$, we have $q = \left[\dfrac{2l+2}{l} v\right] + 1$ distinct points $a_i^j$ $(j = 1, 2, \dots, q)$ in $\mathbf{C}_\infty$ such that

$$\sum_{i=0}^{m-1} \sum_{j=1}^{q} \overline{N}^{l)}(r, \triangle(\theta_{i+1}), a_{i+1}^j) = o(T(r, w)). \tag{15}$$

Let $\alpha$ be arbitrary positive integer. Put

$$\theta_{i,k} = \frac{2\pi i}{m} + \frac{2\pi k}{\alpha m}, \quad 0 \leq i \leq m-1,\ 0 \leq k \leq \alpha - 1,\ \theta_{i,0} = \theta_i.$$

For sufficient large $r$, let

$$\triangle_{i,k} = \{z \,|\, |z| < r, \theta_{i,k} \leq \arg z < \theta_{i,k+1}\}.$$

Then

$$\{|z| < r\} = \sum_{k=0}^{\alpha-1} \sum_{i=0}^{m-1} \triangle_{i,k},$$

Hence there must be one $k_0$ $(0 \leq k_0 \leq \alpha - 1)$, such that

$$\sum_{i=0}^{m-1} n(\triangle_{i,k_0}, \tilde{R}_z) \leq \frac{1}{\alpha} n(r, \tilde{R}_z).$$

Define the angular domains

$$\overline{\triangle}_i = \left\{ z \,\middle|\, \frac{\theta_{i,k_0} + \theta_{i,k_0+1}}{2} \le \arg z \le \frac{\theta_{i+1,k_0} + \theta_{i+1,k_0+1}}{2} \right\},$$

$$\triangle_i^0 = \{ z \mid \theta_{i,k_0} < \arg z < \theta_{i+1,k_0+1} \} \subset \triangle(\theta_{i+1}).$$

Since $\triangle_i^0$ only covers $\triangle_{i,k_0}$ twice, we have

$$\sum_{i=0}^{m-1} n_\chi(r, \triangle_i^0, w) \le \left(1 + \frac{1}{\alpha}\right) n_\chi(r, w). \tag{16}$$

Hence,

$$\sum_{i=0}^{m-1} N_\chi(r, \triangle_i^0, w) \le \left(1 + \frac{1}{\alpha}\right) N_\chi(r, w). \tag{17}$$

Applying Lemma 8 to $\triangle_i^0$, $\overline{\triangle}_i$, we have

$$\begin{aligned}&\left(q - 2 - \frac{2}{l}\right) T(r, \overline{\triangle}_i, w)\\ &\quad \le \sum_{j=1}^{q} \overline{N}^{l)}(r, \triangle_i^0, a_{i+1}^j) + \frac{l+1}{l} N_\chi(r, \triangle_i^0, w) + O(\log^2 r) + X(r, \triangle_i^0, w),\end{aligned}$$

By $T(r, w) = \sum_{i=0}^{m-1} T(r, \overline{\triangle}_i, w)$ with the above inequality and (17), we obtain

$$\begin{aligned}&\left(q - 2 - \frac{2}{l}\right) T(r, w)\\ &\quad \le \sum_{i=0}^{m-1} \sum_{j=1}^{q} \overline{N}^{l)}(r, \triangle_i^0, a_{i+1}^j) + \left(1 + \frac{1}{\alpha}\right) \frac{l+1}{l} N_\chi(r, w) + O(\log^2 r)\\ &\qquad + \sum_{i=0}^{m-1} X(r, \triangle_i^0, w).\end{aligned} \tag{18}$$

Applying Lemma 7, as we did in the proof of Theorem 1, we can see that there exists a sequence $r_n \in (1, \infty) - E$, such that

$$\lim_{n \to \infty} \frac{T(r_n, w)}{\log^2 r_n} = \infty, \tag{19}$$

here $E$ is a set of finite measure. Noting that

$$N_\chi(r, w) \le 2(v - 1) T(r, w). \tag{20}$$

From (18)–(20), we have

$$\left(q - 2 - \frac{2}{l}\right) \le 2\left(1 + \frac{1}{\alpha}\right) \frac{l+1}{l} (v - 1)$$

Letting $\alpha \to \infty$, we get $q = \left[\frac{2l+2}{l}v\right] + 1 \leq \frac{2l+2}{l}v$. This contradiction results in that for arbitrary positive integer $m$, there exists an angular domain $\triangle(\theta_m) = \left\{z \mid |\arg z - \theta_m| < \frac{2\pi}{m}\right\}$ such that for any $a$, we have

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, \triangle(\theta_m), a)}{T(r, w)} > 0, \tag{21}$$

except for $\left[\frac{2l+2}{l}v\right]$ exceptions at most. Choosing subsequence of $\{\theta_m\}$, still denote it $\{\theta_m\}$, we assume that $\theta_m \to \theta_0$. Put $L : \arg z = \theta_0$, then $L$ is a T direction as stated in Theorem 2.

In fact, for any $\varepsilon$ $(0 < \varepsilon < \pi/2)$, when $m$ is sufficiently large, we have $\triangle(\theta_m) \subset \triangle(\theta_0, \varepsilon)$. By (21), we have

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, \theta_0, \varepsilon, a)}{T(r, w)} \geq \limsup_{r\to\infty} \frac{N^{l)}(r, \triangle(\theta_m), a)}{T(r, w)} > 0$$

holds for any $a \in \mathbf{C}_\infty$ with at most $\left[\frac{2l+2}{l}v\right]$ possible exceptional values of $a$. Hence Theorem 2 holds in this case. $\square$

When $w(z)$ is an algebroid function of finite and positive order growth, Theorem 2 was obtained by Xuan and Gao [14] for $l \geq 2v+1$. Combining Theorems 1 and 2, we pose the following question.

Question 1. Let $w(z)$ be a $v$-valued algebroid function defined on the whole complex plane and satisfies (13). Does there exist a ray $L : \arg z = \theta$? Such that, for arbitrary small $\varepsilon > 0$ we have

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, \theta, \varepsilon, a)}{T(r, w)} > 0,$$

holds for any given $a \in \mathbf{C}_\infty$, provided the maximum number $q$ of exceptional values satisfying the following relation

$$q = \begin{cases} 4v & \text{if } l = 1, \\ 3v & \text{if } l = 2. \\ 3v-1 & \text{if } l = 3. \\ 3v-2 & \text{if } l = 4. \\ \cdots\cdots & \cdots\cdots \\ 2v+2 & \text{if } l = 2v-1. \\ 2v+1 & \text{if } l = 2v. \\ 2v & \text{if } l \geq 2v+1. \end{cases}$$

Applying the inequality was obtained by Zhang and Sun [16], Wu and Sun [12] confirms that Question 1 is true in the case of $l = 1$. Theorem 2 means that Question 1 is true in the case of $l \geq \max\{3, 2\nu\}$. Theorem 1 means that Question 1 is true for meromorphic function. And above all, with the $\wp(z)$ in example, the irreducible equation $w^{\nu} = 1 + \dfrac{1}{\wp(z)}$ or $w^{\nu} = 1 + \dfrac{1}{\wp'(z)}$ gives us an algebroid function taking such $4\nu$ values as the $\nu$-th roots of $1 + e^{2k\pi i/3}$ $(k = 0, 1, 2)$ and 1 with multiplicity 2, or $3\nu$ values as the $\nu$-th roots of $1 \mp 2i$ and 1 with multiplicity 3, respectively. Hence, any smaller number cannot replace $\left[\dfrac{2l+2}{l}\nu\right]$ $(= 4\nu, 3\nu)$ when $l = 1, 2$ in the Question 1.

*Acknowledgement*. The authors would like to thank referees for providing numerous suggestions and comments.

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