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AN EXPLICIT FORMULA FOR THE ZEROS OF THE RANKIN-SELBERG L-FUNCTION VIA THE PROJECTION OF C^{∞}-MODULAR FORMS

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Abstract

We give an explicit formula for the zeros of the Rankin type zeta-function by using the projection of the C^{∞}-automorphic forms introduced by Sturm (1981). Our theorem gives a correlation of the zeros of the L-functions and the Hecke eigenvalues.

1. Introduction

In this paper, we describe one explicit formula for the zeros of the Rankin-Selberg L-function by using the projection of the C^{∞}-automorphic forms. The projection was introduced by Sturm [12] in the study of the special values of automorphic L-functions. Combining the idea of Zagier [13] (Proposition 3) and the integral transformation of the confluent hypergeometric function, we derive an explicit formula which correlates the zeros of the zeta-function and the Hecke eigenvalues. The main theorem contains the case of the symmetric square L-function, that first appeared in author's previous paper [5].

Let k and l $(k \leq l)$ be positive even integers and S_k (resp. S_l) be the space of cusp forms of weight k (resp. l) on $SL_2(\mathbf{Z})$. Let $f(z) \in S_k$ and $g(z) \in S_l$ be normalized Hecke eigenforms with the Fourier expansions $f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi inz}$ and $g(z) = \sum_{n=1}^{\infty} b(n) e^{2\pi inz}$. For each prime p, we take α_p and β_p such that $\alpha_p + \beta_p = a(p)$ and $\alpha_p \beta_p = p^{k-1}$, and define

$$M_p(f) = \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}.$$

The Rankin-Selberg L-function attached to $f(z)$ and $g(z)$ is defined by

$$L(s, f \otimes g) = \prod_{p:\text{prime}} \det(I_4 - M_p(f) \otimes M_p(g) p^{-s})^{-1}. \tag{1}$$

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Here the product is taken over all rational primes, and I_n is the unit matrix of size n.

In this paper, we prove the following theorem:

THEOREM 1. *Let k and l be positive even integers such that $k, l = 12, 16, 18, 20, 22,$ and 26 respectively. Suppose $k \le l$. Let $\Delta_k(z) = \sum_{n=1}^{\infty} \tau_k(n) e^{2\pi i n z} \in S_k$ be the unique normalized Hecke eigenform. We write ρ as a zero of $L(s - 1 + (k+l)/2, \Delta_k \otimes \Delta_l)$ in the critical strip $0 < \mathrm{Re}(s) < 1$. Assume that $\zeta(2\rho) \neq 0$. Then for each positive integer n,*

$$-\tau_k(n)\left\{\frac{n^{1-2\rho}(-1)^{(l-k)/2}\zeta(2\rho)}{(2\pi)^{2\rho}\Gamma\left(-\rho+\dfrac{k+l}{2}\right)}+\frac{\zeta(2\rho-1)\Gamma(2\rho-1)}{\Gamma\left(\rho-1+\dfrac{k+l}{2}\right)\Gamma\left(\rho+\dfrac{k-l}{2}\right)\Gamma\left(\rho-\dfrac{k-l}{2}\right)}\right\}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(k)\Gamma\left(\rho-\dfrac{k-l}{2}\right)}\sum_{m=1}^{n-1}\tau_k(m)\sigma_{1-2\rho}(n-m) \\
&\quad\times F\left(1-\rho+\frac{k-l}{2}, -\rho+\frac{k+l}{2}; k; \frac{m}{n}\right) \\
&\quad+\frac{1}{\Gamma(l)\Gamma\left(\rho+\dfrac{k-l}{2}\right)}\sum_{m=n+1}^{\infty}\left(\frac{n}{m}\right)^{-\rho+(k+l)/2}\tau_k(m)\sigma_{1-2\rho}(m-n) \\
&\quad\times F\left(1-\rho-\frac{k-l}{2}, -\rho+\frac{k+l}{2}; l; \frac{n}{m}\right),
\end{aligned}$$

where $F(a, b; c; z)$ is the hypergeometric function and $\sigma_s(m)$ is the sum of the s-th powers of positive divisors of m.

COROLLARY 1. *Let $T(n, \rho; k; l)$ be the right-hand side of the equality in Theorem 1. Then, the following equivalence holds:*

$$\mathrm{Re}(\rho) = \frac{1}{2} \iff T(n, \rho; k; l) \asymp \tau_k(n) \quad (\text{as } n \to \infty).$$

Remark 1. By Shimura [10, 11], it is known that the "periods" of the modular form for $L(s, f \otimes g)$ are dominated by the cusp form of large weight, whereas our theorem is expressed by using the Fourier coefficients of the cusp form of small weight.

Remark 2. The formula for the symmetric square L-function is given in [5], where the factor $(-n/m)^{k-\rho}$ is not correct. It should be replaced by $(n/m)^{k-\rho}$.

2. Eisenstein series

In this section we recall the fundamental properties of the Eisenstein series, and introduce the C^∞-modular forms.

Let $k \geq 0$ be an even integer, $i = \sqrt{-1}$ and H be the upper half plane. The non-holomorphic Eisenstein series for $SL_2(\mathbf{Z})$ is defined by

$$E_k(z,s) = y^s \sum_{\{c,d\}} (cz+d)^{-k}|cz+d|^{-2s}, \tag{2}$$

where $z = x + iy \in H$, $s \in \mathbf{C}$ and the summation is taken over $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$, a complete system of representation of $\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in SL_2(\mathbf{Z})\right\}\Big\backslash SL_2(\mathbf{Z})$. The right-hand side of (2) converges absolutely and locally uniformly on $\{(z,s) \mid z \in H, \mathrm{Re}(s) > -k/2+1\}$. For $z \in H$ and $\mathrm{Re}(s) > -k/2+1$, $E_k(z,s)$ has an expansion:

$$E_k(z,s) = y^s + a_0(s)y^{1-k-s} + \frac{y^s}{\zeta(k+2s)} \sum_{m \neq 0} \sigma_{1-k-2s}(m) a_m(y,s) \exp(2\pi i m x), \tag{3}$$

where $\sigma_s(m)$ is the s-th powers of positive divisors of m,

$$a_0(s) = (-1)^{k/2} 2\pi \cdot 2^{1-k-2s} \frac{\zeta(k+2s-1)}{\zeta(k+2s)} \frac{\Gamma(k+2s-1)}{\Gamma(s)\Gamma(k+s)}, \tag{4}$$

and

$$a_m(y,s) = \int_{-\infty}^{\infty} \exp(-2\pi i m u)(u+iy)^{-k}|u+iy|^{-2s}\, du. \tag{5}$$

The integral in (5) is the key ingredient in the study of the Eisenstein series, which is known as an entire function in s and of exponential decay in $y|m|$. Therefore, $E_k(z,s)$ is meromorphically continued to the whole s-plane. And there exist positive constants A_1 and A_2 depending only on k and s such that

$$|E_k(z,s)| \leq A_1 y^{\mathrm{Re}(s)} + A_2 y^{1-\mathrm{Re}(s)-k} \quad (y \to \infty), \tag{6}$$

except on the poles.

Let $\Psi(\alpha,\beta;w)$ be the confluent hypergeometric function defined by

$$\Psi(\alpha,\beta;w) := \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-wt} t^{\alpha-1}(1+t)^{\beta-\alpha-1}\, dt \tag{7}$$

for $\mathrm{Re}(w) > 0$ and $\mathrm{Re}(\alpha) > 0$, which is continued holomorphically on $(\alpha,\beta,w) \in \mathbf{C} \times \mathbf{C} \times \{w \mid \mathrm{Re}(w) > 0\}$. Then, the integral in (5) is expressed by

(8)

$$a_m(y,s) = \begin{cases} \dfrac{(-1)^{k/2}(2\pi)^{k+2s} m^{k+2s-1}}{\Gamma(k+s)} e^{-2\pi y m} \Psi(s, k+2s; 4\pi y m) & (m > 0), \\ \dfrac{(-1)^{k/2}(2\pi)^{k+2s} |m|^{k+2s-1}}{\Gamma(s)} e^{-2\pi y |m|} \Psi(k+s, k+2s; 4\pi y |m|) & (m < 0). \end{cases}$$

(See for example [4] §7.2.)

It is also well-known the functional equation:

$$\pi^{-s}\Gamma(s)\zeta(2s)E_k(z,s) = \pi^{-1+s+k}\Gamma(1-s-k)\zeta(2-2s-2k)E_k(z,1-s-k). \tag{9}$$

PROPOSITION 1. *Assume $E_k(z,s)$ is holomorphic at $s \in \mathbf{C}$. Then, there exist positive constants A_1 and A_2 depending only on k and s such that*

$$|E_k(x+iy,s)| \le \begin{cases} A_1(y^{-\mathrm{Re}(s)-k} + y^{\mathrm{Re}(s)}) & \left(\mathrm{Re}(s) > \dfrac{1-k}{2}\right), \\ A_2(y^{-1+\mathrm{Re}(s)} + y^{1-\mathrm{Re}(s)-k}) & \left(\mathrm{Re}(s) \le \dfrac{1-k}{2}\right), \end{cases} \tag{10}$$

for every $y > 0$.

Proof. We use (6) and the modularity for $y^{k/2}E_k(z,s)$, and obtain the assertion of Proposition 1. For the detail of the proof, see [7] (Proposition 1). □

Next, we introduce the C^∞-modular form of bounded growth due to Sturm [12]. The function F is called a *C^∞-modular form of weight k*, if F satisfies the following conditions:

(A.1) F is a C^∞-function from H to $\mathbf{C}$,

(A.2) $F((az+b)(cz+d)^{-1}) = (cz+d)^k F(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$.

We denote by $\mathfrak{M}_k$ the set of all C^∞-modular forms of weight k. The function $F \in \mathfrak{M}_k$ is called *of bounded growth* if for every $\varepsilon > 0$

$$\int_0^1\int_0^\infty |F(z)|y^{k-2}e^{-\varepsilon y}\,dydx < \infty.$$

For $F \in \mathfrak{M}_k$ and $f \in S_k$, we define the Petersson inner product

$$\langle f, F\rangle = \int_{SL_2(\mathbf{Z})\backslash H} f(z)\overline{F(z)}y^{k-2}\,dxdy.$$

LEMMA 1. *Assume that $f(z) \in S_k$ and $s \in \mathbf{C}$ in $k/2 - l + 2 < \mathrm{Re}(s) < k/2 - 1$. Then $f(z)E_{l-k}(z,s)$ is a C^∞-modular form of weight l and of bounded growth.*

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, and $\gamma(z) = (az+b)/(cz+d)$. Then, by the definition,

$$f(\gamma(z))E_{l-k}(\gamma(z),s) = (cz+d)^k f(z)\cdot(cz+d)^{l-k}E_{l-k}(z,s).$$

Therefore, $f(z)E_{l-k}(z,s)$ is a C^∞-modular form of weight l in the whole s-plane except on the poles. For $f(z) \in S_k$, it is known that there is a positive constant c_0 such that

$$f(z) \le c_0 y^{-k/2}$$

for every $z = x + iy \in H$. Therefore, by Proposition 1,

$$f(z)E_{l-k}(z,s) \ll \begin{cases} y^{-l+k/2-\mathrm{Re}(s)} & \left(\mathrm{Re}(s) > \dfrac{1-l+k}{2}\right), \\ y^{-k/2-1+\mathrm{Re}(s)} & \left(\mathrm{Re}(s) \le \dfrac{1-l+k}{2}\right), \end{cases}$$

when $y \to 0$. The estimate above implies $f(z)E_{l-k}(z,s)$ is of bounded growth when $k/2 - l + 2 < \mathrm{Re}(s) < k/2 - 1$. $\square$

Remark. Lemma 1 gives immediately that $f(z)E_0(z,s)$ is a C^∞-modular form of weight k, and of bounded growth in the region $-k/2 + 2 < \mathrm{Re}(s) < k/2 - 1$, except on the poles. In [12] (Corollary 2), $1 < \mathrm{Re}(s) < k/2 - 1$ is given as a sufficient condition of bounded growth. Our region $-k/2 + 2 < \mathrm{Re}(s) < k/2 - 1$ includes the critical strip of the symmetric square L-function. This is of advantage because it makes possible to evaluate the L-functions and Riemann's zeta-function at non-trivial zeros (see [5, 6]). The proof of [5], (Lemma 1) in fact is insufficient to imply the assertion, however the insufficiency is corrected in the author's subsequent papers [6] and [7] (Proposition 1).

3. Rankin-Selberg L-function

In this section, we recall the Rankin-Selberg L-function, and quote the projection of the C^∞-modular form due to Sturm [12].

Let $f(z) \in S_k$ and $g(z) \in S_l$ have the Fourier expansions $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ and $g(z) = \sum_{n=1}^{\infty} b(n)e^{2\pi inz}$. We write

$$\hat{g}(z) = \sum_{n=1}^{\infty} \overline{b(n)} \exp(2\pi inz).$$

Define the Dirichlet series

$$D(s,f,g) = \sum_{n=1}^{\infty} a(n)b(n)n^{-s},$$

for sufficient large $\mathrm{Re}(s)$. Then, by the method of Rankin [8] and Selberg [9],

$$D(s,f,g) = (4\pi)^s \Gamma^{-1}(s) \int_{SL_2(\mathbf{Z})\backslash H} f(z)\overline{\hat{g}(z)} E_{l-k}(z, s-l+1) y^{l-2}\, dxdy. \tag{11}$$

Assume $f(z)$ and $g(z)$ be normalized Hecke eigenforms. By Shimura [10] Lemma 1, we have

$$L(s, f \otimes g) = \zeta(2s + 2 - k - l) D(s,f,g). \tag{12}$$

Put

$$R(s,f,g) = \Gamma(s)\Gamma(s-k+1)L(s, f \otimes g).$$

Then $R(s, f, g)$ is an entire function in s for $l > k$, because $\Gamma(s+k)\zeta(2s+k)E_k(z,s)$ is an entire function in s for $k \geq 2$. For $f = g$, the symmetric square case, it is holomorphic on the whole s-plane except for possible simple poles at $s = k-1$ and $s = k$.

Using (9), we obtain the functional equation,

$$R(k+l-1-s, f, g) = R(s, f, g).$$

The inequalities due to Deligne [1]:

$$|a(p)| \leq 2p^{(k-1)/2} \quad \text{and} \quad |b(p)| \leq 2p^{(l-1)/2},$$

show that the infinite product (1) converges absolutely for $\mathrm{Re}(s) > (k+l)/2$. Hence $L(s, f \otimes g) \neq 0$ for $\mathrm{Re}(s) > (k+l)/2$, and $R(s, f, g) \neq 0$ for $\mathrm{Re}(s) < (k+l-2)/2$. In this paper, we consider the zeros of $L(s-1+(k+l)/2, f \otimes g)$ in the critical strip $0 < \mathrm{Re}(s) < 1$.

Lemma 2. *Let $f(z) \in S_k$ and $g(z) \in S_l$ be normalized Hecke eigenforms. Let ρ be a zero of $L(s-1+(k+l)/2, f \otimes g)$ in the critical strip $0 < \mathrm{Re}(s) < 1$. Assume $\zeta(z\rho) \neq 0$. Then*

$$\left\langle f(z)E_{l-k}\left(z, \rho + \frac{k-l}{2}\right), g(z) \right\rangle = 0. \tag{13}$$

Proof. By the integral representation (11) and (12), we have

$$L(s, f \otimes g) = (4\pi)^s \Gamma^{-1}(s)\zeta(2s-k-l+2)\langle f(z)E_{l-k}(z, s-l+1), \hat{g}(z)\rangle. \tag{14}$$

Here we assumed that $g(z)$ is a normalized Hecke eigenform. Since the Hecke operator is Hermitian, we see all the Fourier coefficients of $g(z)$ are real. This proves Lemma 2. $\square$

Next, we introduce the projection of the C^∞-automorphic forms due to Sturm. In [12], Sturm constructed a certain kernel function by using Poincaré series, and showed the following theorem:

Theorem 2 (Sturm [12]). *Let $F \in \mathfrak{M}_k$ be of bounded growth with the Fourier expansion $F(z) = \sum_{n=-\infty}^{\infty} a(n,y)e^{2\pi inx}$. Assume that $k > 2$. Let*

$$c(n) = (2\pi n)^{k-1}\Gamma(k-1)^{-1}\int_0^\infty a(n,y)e^{-2\pi ny}y^{k-2}\,dy.$$

Then $h(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi inz} \in S_k$ and $\langle g, F\rangle = \langle g, h\rangle$ for all $g \in S_k$.

4. Proof of Theorem 1

Theorem 2 and Lemma 2 in the previous section are the key ingredients in the proof of Theorem 1. In the following, we evaluate the Fourier coefficients of the projection of $f(z)E_{l-k}(z, s-l+1)$ to the space of the holomorphic cusp form.

To begin with, we shall inspect the Laplace-Mellin transform of the confluent hypergeometric function in [2] (6.10.(7)). On this account, we give the analytic continuation and y-estimate of the confluent hypergeometric function (cf. [4] Theorem 7.2.7.):

Lemma 3. *For any compact subset T of $\mathbf{C}\times\mathbf{C}$ and $y>0$, there exist a positive number C_0 and non-negative integers M and N such that*

$$|\Psi(\alpha,\beta;y)| \le C_0\cdot y^{-\mathrm{Re}(\alpha)}(1+y^{-(M+N)}) \quad ((\alpha,\beta)\in T).$$

Here the constants C_0, M and N depend only on T. For every $(\alpha,\beta)\in T$, M is chosen so as $\mathrm{Re}(\alpha+M)>0$, and N is chosen so as $\mathrm{Re}(\beta-\alpha)\le N+1$ respectively.

Proof. We put $H'=\{z\in\mathbf{C}\,|\,\mathrm{Re}(z)>0\}$. First, we prove the assertion when T is contained in $H'\times\mathbf{C}$. Let $(\alpha,\beta)\in T$ and choose a non-negative integer N such that $\mathrm{Re}(\beta-\alpha)\le N+1$. By (7), we have

$$\begin{aligned}|\Psi(\alpha,\beta;y)| &\le |\Gamma(\alpha)|^{-1}\int_0^\infty e^{-yt}t^{\mathrm{Re}(\alpha)-1}(1+t)^{\mathrm{Re}(\beta-\alpha)-1}\,dt\\ &\le |\Gamma(\alpha)|^{-1}\sum_{k=0}^{N}\binom{n}{k}\Gamma(k+\mathrm{Re}(\alpha))y^{-\mathrm{Re}(\alpha)-k}.\end{aligned}$$

Since $|\Gamma(\alpha)|^{-1}$ and $\Gamma(k+\mathrm{Re}(\alpha))$ are continuous function of α, there exist a positive constant C_1 such that

$$|\Psi(\alpha,\beta;y)| \le C_1\sum_{k=0}^{N} y^{-\mathrm{Re}(\alpha)-k} \le C_1N(y^{-\mathrm{Re}(\alpha)}+y^{-\mathrm{Re}(\alpha)-N}) \quad ((\alpha,\beta)\in T).$$

Thus we obtain the assertion of Lemma 3 when $T\subset H'\times\mathbf{C}$.

In order to remove the assumption on T, we take the integration by parts of the right-hand side in (7). Replacing the variables α and β by $\alpha+1$ and $\beta+1$ respectively, we have

$$\Psi(\alpha+1,\beta+1;y) = y^{-1}\Psi(\alpha,\beta;y)+(\beta-\alpha-1)y^{-1}\Psi(\alpha+1,\beta;y).$$

Namely,

$$\Psi(\alpha,\beta;y) = y\Psi(\alpha+1,\beta+1;y)+(1-\beta+\alpha)\Psi(\alpha+1,\beta;y),$$

that is equivalent to the equation in [4] (7.2.39). The repeated use of the equality above, we have

$$\Psi(\alpha,\beta;y) = \sum_{j=0}^{m}\binom{m}{j}\frac{\Gamma(\alpha-\beta+1+m-j)}{\Gamma(\alpha-\beta+1)}\,y^j\cdot\Psi(\alpha+m,\beta+j;y).$$

For any compact subset T of $\mathbf{C}\times\mathbf{C}$, we take a non-negative integer M such that

$$\{(\alpha+M,\beta)\,|\,(\alpha,\beta)\in T\}\subset H'\times\mathbf{C}.$$

Then, there exist positive constants C_2 and C_3 depending only on T such that

$$\begin{aligned}|\Psi(\alpha,\beta;y)| &\le \sum_{j=0}^{M}\binom{M}{j}\frac{|\Gamma(\alpha-\beta+1+M-j)|}{|\Gamma(\alpha-\beta+1)|}y^j\cdot|\Psi(\alpha+M,\beta+j;y)| \\ &\le C_2\sum_{j=0}^{M}y^j\cdot\int_0^\infty e^{-yt}t^{\mathrm{Re}(\alpha+M)-1}(1+t)^{\mathrm{Re}(\beta+j-\alpha-M)-1}\,dt \\ &\le C_2\int_0^\infty e^{-yt}t^{\mathrm{Re}(\alpha+M)-1}(1+t)^{\mathrm{Re}(\beta-\alpha)-1}\,dt\cdot\sum_{j=0}^{M}y^j \\ &\le C_2\sum_{k=0}^{N}\binom{N}{k}\Gamma(k+\mathrm{Re}(\alpha+M))y^{-\mathrm{Re}(\alpha+M)-k}\cdot\sum_{j=0}^{M}y^j \\ &\le C_3\cdot N\cdot M\cdot(y^{-\mathrm{Re}(\alpha)}+y^{-\mathrm{Re}(\alpha+M)-N}).\end{aligned}$$

This completes the proof of Lemma 3. $\square$

PROPOSITION 2. *Let $F(a,b;c;z)$ be the hypergeometric function. Then*

$$\int_0^\infty \Psi(a,c;y)y^{b-1}e^{-uy}\,dy = \frac{\Gamma(b)\Gamma(b-c+1)}{\Gamma(a+b-c+1)}u^{-b}F\left(a,b;a+b-c+1;1-\frac{1}{u}\right)$$

is valid when $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$, $\mathrm{Re}(b-c+1)>0$, *and* $\mathrm{Re}(u)>0$.

Proof. By (7), we have

$$\int_0^\infty \Psi(a,c;y)y^{b-1}e^{-uy}\,dy = \frac{1}{\Gamma(a)}\int_0^\infty\int_0^\infty t^{a-1}(1+t)^{c-a-1}y^{b-1}e^{-(t+u)y}\,dtdy. \tag{15}$$

Here we observe

$$\begin{aligned}&\int_0^\infty\int_0^\infty |t^{a-1}(1+t)^{c-a-1}y^{b-1}e^{-(t+u)y}|\,dydt \\ &\qquad = \Gamma(\mathrm{Re}(b))\int_0^\infty |t^{a-1}(1+t)^{c-a-1}|(\mathrm{Re}(u)+t)^{-\mathrm{Re}(b)}\,dt.\end{aligned} \tag{16}$$

The y-integral in (16) converges absolutely when $\mathrm{Re}(b)>0$ and $\mathrm{Re}(u)+t>0$, and the t-integral converges absolutely when $\mathrm{Re}(a)>0$ and $\mathrm{Re}(b-c+1)>0$. Thus the interchange of the order of integration (15) is justified by Fubini's theorem when $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$, $\mathrm{Re}(b-c+1)>0$, and $\mathrm{Re}(u)>0$. In this region, we have

$$\int_0^\infty \Psi(a,c;y)y^{b-1}e^{-uy}\,dy = \frac{\Gamma(b)}{\Gamma(a)}\cdot u^{-b}\int_0^\infty t^{a-1}(1+t)^{c-a-1}\left(1+\frac{t}{u}\right)^{-b}\,dt. \tag{17}$$

Here we employ the integral representation of the hypergeometric function

$$(18)\qquad F(\alpha,\beta;\gamma;1-z)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^\infty t^{\beta-1}(1+t)^{\alpha-\gamma}(1+zt)^{-\alpha}\,dt$$

valid for $\mathrm{Re}(\gamma)>\mathrm{Re}(\beta)>0$ and $|\arg(z)|<\pi$, (see [2] 2.12.(5)). Combining (17) and (18), we obtain the assertion of Proposition 2. □

We are able to find the Laplace-Mellin transform of the confluent hypergeometric function in [2] (6.10.(7)) and [3] (7.621.6). In [2], it is mentioned that Proposition 2 holds for $\mathrm{Re}(u)>0$ without assumptions for a, b, and c. In this paper, in order to remove the assumptions for a, b, and c in Proposition 2, we apply Lemma 3. As a consequence, we have following:

PROPOSITION 3. *The integral transform*

$$\int_0^\infty \Psi(a,c;y)y^{b-1}e^{-uy}\,dy=\frac{\Gamma(b)\Gamma(b-c+1)}{\Gamma(a+b-c+1)}u^{-b}F\left(a,b;a+b-c+1;1-\frac{1}{u}\right)$$

is valid when $\mathrm{Re}(u)>0$ *and* $\mathrm{Re}(b-a)-M-N>0$. *Here* M *and* N *are non-negative integers so as* $\mathrm{Re}(a+M)>0$ *and* $\mathrm{Re}(c-a)\le N+1$ *respectively.*

Proof. By Lemma 3, the integral on the left-hand side in (15) converges absolutely in the region $\mathrm{Re}(u)>0$ and $\mathrm{Re}(b-a)-M-N>0$. By the identity theorem, the integral has the same expression as in Proposition 2. □

Now, we prove Theorem 1.

Proof of Theorem 1. Let $\Delta_k(z)$ be the unique normalized Hecke eigenform for $k=12,16,18,20,22$, and 26. We write the Fourier expansion as follows:

$$\Delta_k(z)\cdot E_{l-k}(z,s)=\sum_{n=-\infty}^{\infty}b(n,y,s)e^{2\pi inx}.$$

Using the notation $a_0(s)$ and $a_n(y,s)$ defined by (4) and (5), we have

$$(19)\quad b(n,y,s)=\{y^s+a_0(s)y^{1-l+k-s}\}\tau_k(n)e^{-2\pi ny}$$
$$+\frac{y^s}{\zeta(2s+l-k)}\sum_{\substack{m=1,\\ m\neq n}}^{\infty}\tau_k(m)\sigma_{1-l+k-2s}(n-m)a_{n-m}(y,s)e^{-2\pi my}.$$

Here we regard $\tau_k(m)$ as 0 if $m\le 0$.

By Lemma 1 and Theorem 2, there exists $h(z,s)=\sum_{n=1}^{\infty}c(n,s)e^{2\pi inz}\in S_l$ such that $\langle f(z)\cdot E_{l-k}(z,s),g(z)\rangle=\langle h(z,s),g(z)\rangle$ for all $g(z)\in S_l$ in the region $k/2-l+2<\mathrm{Re}(s)<k/2-1$. The Fourier coefficients of $h(z,s)$ are given by

$$(20)\qquad c(n,s)=(2\pi n)^{l-1}\Gamma(l-1)^{-1}\int_0^\infty b(n,y,s)e^{-2\pi ny}y^{l-2}\,dy,$$

for $n > 0$. In the following we put

$$\gamma(n,l) = (2\pi n)^{l-1}\Gamma(l-1)^{-1}.$$

Substituting (19) into (20), we have

$$(21)\qquad c(n,s) = \frac{\gamma(n,l)}{\zeta(2s+l-k)}\sum_{\substack{m=1\\ m\neq n}}^{\infty}\tau_k(m)\sigma_{1-l+k-2s}(n-m)$$
$$\times\int_0^\infty a_{n-m}(y,s)y^{s+l-2}e^{-2\pi(m+n)y}\,dy$$
$$+\gamma(n,l)\tau_k(n)\left\{\frac{\Gamma(s+l-1)}{(4\pi n)^{s+l-1}}+(-1)^{(l-k)/2}2^{1-l+k-2s}\cdot 2\pi\right.$$
$$\left.\times\frac{\Gamma(2s+l-k-1)}{\Gamma(s)\Gamma(s+l-k)}\cdot\frac{\zeta(2s+l-k-1)}{\zeta(2s+l-k)}\cdot\frac{\Gamma(k-s)}{(4\pi n)^{k-s}}\right\}.$$

The interchange of summation and integration will be justified later.

We treat the integral in (21), the Laplace-Mellin transform of the Fourier coefficients of the Eisenstein series. By (8), we observe

$$(22)\qquad a_{n-m}(y,s) = \begin{cases}\dfrac{(-1)^{(l-k)/2}(2\pi)^{l-k+2s}(n-m)^{l-k+2s-1}}{\Gamma(l-k+s)}e^{-2\pi y(n-m)} & \\ \qquad\times\Psi(s,l-k+2s;4\pi y(n-m)) & (n>m),\\ \dfrac{(-1)^{(l-k)/2}(2\pi)^{l-k+2s}|n-m|^{l-k+2s-1}}{\Gamma(s)}e^{-2\pi y|n-m|} & \\ \qquad\times\Psi(l-k+s,l-k+2s;4\pi y|n-m|) & (n<m).\end{cases}$$

For $n < m$, by Proposition 2,

$$(23)\qquad \int_0^\infty\Psi(l-k+s,l-k+2s;4\pi(m-n)y)e^{-4\pi my}y^{s+l-2}\,dy$$
$$= (4\pi m)^{1-l-s}\frac{\Gamma(s+l-1)\Gamma(k-s)}{\Gamma(l)}F\left(l-k+s,s+l-1;l;\frac{n}{m}\right)$$
$$= (4\pi m)^{1-l-s}\left(\frac{m-n}{m}\right)^{k-2s-l+1}\frac{\Gamma(s+l-1)\Gamma(k-s)}{\Gamma(l)}F\left(1-s,k-s;l;\frac{n}{m}\right)$$

holds for $\mathrm{Re}\left(\dfrac{m}{m-n}\right) > 0$ and $-l+k < \mathrm{Re}(s) < k$. Here we have used the relations of the hypergeometric function,

$$F(a,b,c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z)$$

and $F(a,b,c;z) = F(b,a,c;z)$, (see for example [2] 2.9.(2) and 2.8.(18)).

For $n > m$, by Proposition 3,

$$\begin{aligned}(24)\qquad &\int_0^\infty \Psi(s, l-k+2s; 4\pi(n-m)y)e^{-4\pi ny}y^{s+l-2}\,dy\\ &= (4\pi n)^{1-l-s}\frac{\Gamma(s+l-1)\Gamma(k-s)}{\Gamma(k)}F\left(s, s+l-1; k; \frac{m}{n}\right)\\ &= (4\pi n)^{1-l-s}\left(\frac{n-m}{n}\right)^{k-2s-l+1}\frac{\Gamma(s+l-1)\Gamma(k-s)}{\Gamma(k)}\\ &\quad\times F\left(1-s+k-l, k-s; k; \frac{m}{n}\right)\end{aligned}$$

valid for $\mathrm{Re}\left(\dfrac{n}{n-m}\right) > 0$ and $l-1-M-N>0$. Here M and N are non-negative integers so as $\mathrm{Re}(s+M)>0$ and $\mathrm{Re}(s+l-k)\le N+1$ respectively. By Lemma 2,

$$(25)\qquad \left\langle h\left(z, \rho+\frac{k-l}{2}\right), g(z)\right\rangle = 0.$$

Because $0<\mathrm{Re}(\rho)<1$ in (13) and (25), we take s in (19)–(24) so as

$$(26)\qquad \frac{k-l}{2} < \mathrm{Re}(s) < \frac{k-l}{2}+1.$$

Then we may choose $M = 1+(l-k)/2$ and $N=(l-k)/2$. Therefore, the condition (26) meets the requirements $-l+k<\mathrm{Re}(s)<k$ in (23) and $l-1-M-N>0$ in (24) respectively. The region (26) is also included in $k/2-l+2<\mathrm{Re}(s)<k/2-1$ that was required in Lemma 1 for the bounded growth condition.

By (22) and (23), the infinite sum in (21) is estimated as

$$\begin{aligned}(27)\qquad &\sum_{m=n+1}^\infty \left|\tau_k(m)\sigma_{1-l+k-2s}(n-m)\int_0^\infty a_{n-m}(y,s)y^{s+l-2}e^{-2\pi(m+n)y}\,dy\right|\\ &\le \left|\frac{\Gamma(s+l-1)\Gamma(k-s)}{2^{s+l-1}(2\pi)^{k-1-s}\Gamma(s)\Gamma(l)}\right|\\ &\quad\times\sum_{m=n+1}^\infty \left|m^{s-k}\tau_k(m)\sigma_{1-l+k-2s}(n-m)F\left(1-s, k-s; l; \frac{n}{m}\right)\right|.\end{aligned}$$

By the result of Deligne [1]:

$$(28)\qquad |\tau_k(m)| \ll m^{(k-1)/2}+\varepsilon$$

for every $\varepsilon>0$. And when $s=\rho+(k-l)/2$ with $0<\mathrm{Re}(\rho)<1$, we have

$$(29)\quad |m^{s-k}|\le m^{\mathrm{Re}(\rho)-(1/2)(k+l)},\quad |\sigma_{1-l+k-2s}(n-m)| = |\sigma_{1-2\rho}(m-n)| \ll (m-n)^{1+\varepsilon}.$$

Using the hypergeometric series

$$F(a,b;c;z) = \sum_{u=0}^{\infty} \frac{(a)_u (b)_u}{(c)_u u!} z^u = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{u=0}^{\infty} u^{a+b-c-1}\{1 + O(u^{-1})\} z^u,$$

(see [2] 2.1.1.(2) and 2.1.1.(5)), we have

$$\begin{aligned}(30)\qquad & \left| F\left(1 - \rho - \frac{k-l}{2}, -\rho + \frac{k+l}{2}; l; \frac{n}{m}\right) \right| \\ & \ll \left| \Gamma(l)\Gamma^{-1}\left(1 - \rho - \frac{k-l}{2}\right)\Gamma^{-1}\left(-\rho + \frac{k+l}{2}\right) \right| \sum_{u=0}^{\infty} |u^{-2\rho}| \left(\frac{n}{m}\right)^u \\ & \ll \frac{m}{m-n},\end{aligned}$$

where the implied constant in the last inequality depends only on k, l and ρ. By using (28), (29) and (30), we observe that the infinite sum (27) converges absolutely in the region (26). Hence the interchange of summation and integration in (21) is justified.

We assumed $\dim S_l = 1$, namely, $g(z)$ in (25) is the unique cusp form of S_l. Therefore, $h\left(z, \rho + \dfrac{k-l}{2}\right)$ is identically zero, namely, $c\left(n, \rho + \dfrac{k-l}{2}\right) = 0$ for every positive integer n. Substituting $\rho + (k-l)/2$ for s in (21), (22), (23) and (24), we complete the proof of Theorem 1. $\square$

Proof of Corollary 1. In the left-hand side of the equation in Theorem 1, we observe

$$n^{2\rho-1} \asymp 1,$$

as $n \to \infty$, if and only if $\mathrm{Re}(\rho) = 1/2$. This proves Corollary 1. $\square$

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