K. SEO
KODAI MATH. J.
31 (2008), 113–119

MINIMAL SUBMANIFOLDS WITH SMALL TOTAL SCALAR CURVATURE IN EUCLIDEAN SPACE

Keomkyo Seo

Abstract

Let M be an n-dimensional complete minimal submanifold in $\mathbf{R}^{n+p}$. Lei Ni proved that if M has sufficiently small total scalar curvature, then M has only one end. We improve the upper bound of total scalar curvature. We also prove that if M has the same upper bound of total scalar curvature, there is no nontrivial L^2 harmonic 1-form on M.

1. Introduction and theorems

Let M^n $(n \geq 3)$ be an n-dimensional complete immersed minimal hypersurface in $\mathbf{R}^{n+1}$. Cao, Shen and Zhu [2] proved that if M is stable, then M has only one end. Recall that a minimal submanifold is stable if the second variation of its volume is always nonnegative for any normal variation with compact support. Later Shen and Zhu [8] showed that if M is stable and has finite total scalar curvature, then M is totally geodesic. On the other hand, there are some gap theorems for minimal submanifolds with finite total scalar curvature in $\mathbf{R}^{n+p}$. Recently Lei Ni [6] proved that if M has sufficiently small total scalar curvature then M has only one end. More precisely, he proved the following.

Theorem ([6]). *Let M^n be an n-dimensional complete immersed minimal hypersurface in $\mathbf{R}^{n+p}$, $n \geq 3$. If*

$$\left(\int_M |A|^n \, dv\right)^{1/n} < C_1 = \sqrt{\frac{n}{n-1} C_s^{-1}},$$

then M has only one end. (*Here C_s is a Sobolev constant in* [4].)

In Section 2 we improve the upper bound C_1 of the total scalar curvature as follows.

Mathematics Subject Classification (2000): 53C42, 58C40.
Key Words: minimal submanifolds, total scalar curvature, gap theorem, L^2 harmonic forms.
Received June 25, 2007; revised October 26, 2007.

THEOREM 1.1. *Let M^n be a complete immersed minimal submanifold in $\mathbf{R}^{n+p}$, $n \geq 3$. If*

$$\left(\int_M |A|^n \, dv\right)^{1/n} < \frac{n}{n-1}\sqrt{C_s^{-1}},$$

then M has only one end.

It is well-known that a minimal submanifold with finite total scalar curvature and one end must be an affine n-plane ([1]). Combining this fact, we have

COROLLARY 1.2. *Let M^n be a complete immersed minimal submanifold in $\mathbf{R}^{n+p}$, $n \geq 3$. If*

$$\left(\int_M |A|^n \, dv\right)^{1/n} < \frac{n}{n-1}\sqrt{C_s^{-1}},$$

then M is an affine n-plane.

Moreover, we study L^2 harmonic 1-forms on minimal submanifolds in $\mathbf{R}^{n+p}$. In [7], Palmer proved that if there exists a codimension one cycle C in a complete minimal hypersurface in $\mathbf{R}^{n+1}$, then M is unstable, by using the existence of a nontrivial L^2 harmonic 1-form on such M. Miyaoka [5] showed that if M is a complete stable minimal hypersurface in $\mathbf{R}^{n+1}$, then there are no nontrivial L^2 harmonic 1-forms on M. Recently Yun [10] proved that if M is a complete minimal hypersurface with $(\int_M |A|^n \, dv)^{1/n} < C_2 = \sqrt{C_s^{-1}}$, then there are no nontrivial L^2 harmonic 1-forms on M. We extend Yun's theorem to higher codimensional cases as follows.

THEOREM 1.3. *Let M^n be a complete immersed minimal submanifold in $\mathbf{R}^{n+p}$, $n \geq 3$. If*

$$\left(\int_M |A|^n \, dv\right)^{1/n} < \frac{n}{n-1}\sqrt{C_s^{-1}},$$

then there are no nontrivial L^2 harmonic 1-forms on M.

2. Proofs of the theorems

Before proving Theorem 1.1, we need some useful facts.

LEMMA 2.1 ([4]). *Let M^n be a complete immersed minimal submanifold in $\mathbf{R}^{n+p}$, $n \geq 3$. Then for any $\phi \in W_0^{1,2}(M)$ we have*

$$\left(\int_M |\phi|^{2n/(n-2)} \, dv\right)^{(n-2)/n} \leq C_s \int_M |\nabla\phi|^2 \, dv,$$

where C_s depends only on n.

LEMMA 2.2 ([3]). *Let M^n be a complete immersed minimal submanifold in $\mathbf{R}^{n+p}$. Then the Ricci curvature of M satisfies*

$$\mathrm{Ric}(M) \geq -\frac{n-1}{n}|A|^2.$$

Now let u be a harmonic function on M. Using normal coordinate system $\{x^i\}$ at $p \in M$, we have Bochner formula

$$\frac{1}{2}\Delta(|\nabla u|^2) = \sum u_{ij}^2 + \mathrm{Ric}(\nabla u, \nabla u).$$

Then Lemma 2.2 gives

$$\frac{1}{2}\Delta(|\nabla u|^2) \geq \sum u_{ij}^2 - \frac{n-1}{n}|A|^2|\nabla u|^2.$$

We may choose the normal coordinates at p such that $u_1(p) = |\nabla u|(p)$, $u_i(p) = 0$ for $i \geq 2$. Then we have

$$\nabla_j|\nabla u| = \nabla_j\left(\sqrt{\sum u_i^2}\right) = \frac{\sum u_i u_{ij}}{|\nabla u|} = u_{1j}.$$

Therefore we obtain $|\nabla|\nabla u||^2 = \sum u_{1j}^2$. On the other hand, we know

$$\frac{1}{2}\Delta(|\nabla u|^2) = |\nabla u|\Delta|\nabla u| + |\nabla|\nabla u||^2.$$

Then we have

$$\sum u_{ij}^2 - \frac{n-1}{n}|A|^2|\nabla u|^2 \leq |\nabla u|\Delta|\nabla u| + \sum u_{1j}^2.$$

Hence we get

$$\begin{aligned}
|\nabla u|\Delta|\nabla u| + \frac{n-1}{n}|A|^2|\nabla u|^2 &\geq \sum u_{ij}^2 - \sum u_{1j}^2 \\
&\geq \sum_{i\neq 1} u_{i1}^2 + \sum_{i\neq 1} u_{ii}^2 \\
&\geq \sum_{i\neq 1} u_{i1}^2 + \frac{1}{n-1}\left(\sum_{i\neq 1} u_{ii}\right)^2 \\
&\geq \frac{1}{n-1}\sum_{i\neq 1} u_{i1}^2 = \frac{1}{n-1}|\nabla|\nabla u||^2,
\end{aligned}$$

where we used $\Delta u = \sum u_{ii} = 0$ in the last inequality. Therefore we get

$$|\nabla u|\Delta|\nabla u| + \frac{n-1}{n}|A|^2|\nabla u|^2 - \frac{1}{n-1}|\nabla|\nabla u||^2 \geq 0. \tag{2.1}$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that M has at least two ends. First we note that if M has more than one end then there exists a nontrivial bounded harmonic function $u(x)$ on M which has finite total energy ([2] and [6]). Let $f = |\nabla u|$. From (2.1) we have

$$f\Delta f + \frac{n-1}{n}|A|^2 f^2 \geq \frac{1}{n-1}|\nabla f|^2.$$

Fix a point $p \in M$ and for $R > 0$ choose a cut-off function satisfying $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B_p(R)$, $\varphi = 0$ on $M \backslash B_p(2R)$, and $|\nabla \varphi| \leq \frac{1}{R}$. Multiplying both sides by φ^2 and integrating over M, we have

$$\int_M \varphi^2 f \Delta f \, dv + \frac{n-1}{n}\int_M \varphi^2 |A|^2 f^2 \, dv \geq \frac{1}{n-1}\int_M \varphi^2 |\nabla f|^2 \, dv.$$

Using integration by parts, we get

$$\begin{aligned} -\int_M |\nabla f|^2 \varphi^2 \, dv - 2\int_M f\varphi\langle \nabla f, \nabla \varphi\rangle \, dv + \frac{n-1}{n}\int_M \varphi^2 |A|^2 f^2 \, dv \\ \geq \frac{1}{n-1}\int_M \varphi^2 |\nabla f|^2 \, dv. \end{aligned}$$

Applying Schwarz inequality, for any positive number $a > 0$, we obtain

$$\frac{n-1}{n}\int_M \varphi^2 |A|^2 f^2 \, dv + \frac{1}{a}\int_M f^2 |\nabla \varphi|^2 \, dv \geq \left(\frac{n}{n-1} - a\right)\int_M \varphi^2 |\nabla f|^2 \, dv. \tag{2.2}$$

On the other hand, applying Sobolev inequality (Lemma 2.1), we have

$$\int_M |\nabla (f\varphi)|^2 \, dv \geq C_s^{-1}\left(\int_M (f\varphi)^{2n/(n-2)} \, dv\right)^{(n-2)/n}.$$

Thus applying Schwarz inequality again, we have for any positive number $b > 0$,

$$\begin{aligned} (1+b)\int_M \varphi^2 |\nabla f|^2 \, dv \geq C_s^{-1}\left(\int_M (f\varphi)^{2n/(n-2)} \, dv\right)^{(n-2)/n} \\ - \left(1 + \frac{1}{b}\right)\int_M f^2 |\nabla \varphi|^2 \, dv. \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3), we get

$$\begin{aligned} \frac{n-1}{n}\int_M \varphi^2 |A|^2 f^2 \, dv \geq \frac{\left(\frac{n}{n-1} - a\right)}{b+1}\left\{C_s^{-1}\left(\int_M (f\varphi)^{2n/(n-2)} \, dv\right)^{(n-2)/n}\right\} \\ - \left(\frac{1}{a} + \frac{\frac{n}{n-1} - a}{b}\right)\int_M f^2 |\nabla \varphi|^2 \, dv. \end{aligned}$$

Using Hölder inequality, we have

$$\int_M \varphi^2 |A|^2 f^2 \, dv \le \left(\int_M |A|^n \right)^{2/n} \left(\int_M (f\varphi)^{2n/(n-2)} \, dv \right)^{(n-2)/n}.$$

Hence we have

$$\begin{aligned} &\left(\frac{1}{a} + \frac{\frac{n}{n-1} - a}{b} \right) \int_M f^2 |\nabla\varphi|^2 \, dv \\ &\quad \ge \left\{ \frac{\left(\frac{n}{n-1} - a \right) C_s^{-1}}{b+1} - \frac{n-1}{n} \left(\int_M |A|^n \, dv \right)^{2/n} \right\} \left(\int_M (f\varphi)^{2n/(n-2)} \, dv \right)^{(n-2)/n}. \end{aligned}$$

By assumption, we choose a and b small enough such that

$$\left\{ \frac{\left(\frac{n}{n-1} - a \right) C_s^{-1}}{b+1} - \frac{n-1}{n} \left(\int_M |A|^n \, dv \right)^{2/n} \right\} \ge \varepsilon > 0.$$

Then letting $R \to \infty$, we have $f \equiv 0$, i.e., $|\nabla u| \equiv 0$. Therefore u is constant. This contradicts the assumption that u is a nontrivial harmonic function. □

Proof of Theorem 1.3. Let ω be an L^2 harmonic 1-form on minimal submanifold M in $\mathbf{R}^{n+p}$. We recall that such ω means

$$\Delta\omega = 0 \quad \text{and} \quad \int_M |\omega|^2 \, dv < \infty.$$

We will use confused notation for a harmonic 1-form ω and its dual harmonic vector field $\omega^{\#}$. From Bochner formula we have

$$\Delta|\omega|^2 = 2(|\nabla\omega|^2 + \mathrm{Ric}(\omega, \omega)).$$

We also have

$$\Delta|\omega|^2 = 2(|\omega|\Delta|\omega| + |\nabla|\omega|\,|^2).$$

Since $|\nabla\omega|^2 \ge \frac{n}{n-1} |\nabla|\omega|\,|^2$ by [9], it follows that

$$|\omega|\Delta|\omega| - \mathrm{Ric}(\omega, \omega) = |\nabla\omega|^2 - |\nabla|\omega|\,|^2 \ge \frac{1}{n-1} |\nabla|\omega|\,|^2.$$

By Lemma 2.2, we have

$$|\omega|\Delta|\omega| - \frac{1}{n-1}|\nabla|\omega||^2 \geq \mathrm{Ric}(\omega,\omega) \geq -\frac{n-1}{n}|A|^2|\omega|^2.$$

Therefore we get

$$|\omega|\Delta|\omega| + \frac{n-1}{n}|A|^2|\omega|^2 - \frac{1}{n-1}|\nabla|\omega||^2 \geq 0.$$

Multiplying both sides by φ^2 as in the proof of Theorem 1.1 and integrating over M, we have from integration by parts that

$$\begin{aligned} (2.4) \qquad 0 &\leq \int_M \varphi^2|\omega|\Delta|\omega| + \frac{n-1}{n}\varphi^2|A|^2|\omega|^2 - \frac{1}{n-1}\varphi^2|\nabla|\omega||^2\,dv \\ &= -2\int_M \varphi|\omega|\langle\nabla\varphi,\nabla|\omega|\rangle\,dv - \frac{n}{n-1}\int_M \varphi^2|\nabla|\omega||^2\,dv \\ &\quad + \frac{n-1}{n}\int_M |A|^2|\omega|^2\varphi^2\,dv. \end{aligned}$$

On the other hand, we get the following from Hölder inequality and Sobolev inequality (Lemma 2.1)

$$\begin{aligned} \int_M |A|^2|\omega|^2\varphi^2\,dv &\leq \left(\int_M |A|^n\,dv\right)^{2/n}\left(\int_M (\varphi|\omega|)^{2n/(n-2)}\,dv\right)^{(n-2)/n} \\ &\leq C_s\left(\int_M |A|^n\,dv\right)^{2/n}\int_M |\nabla(\varphi|\omega|)|^2\,dv \\ &= C_s\left(\int_M |A|^n\,dv\right)^{2/n} \\ &\quad \times\left(\int_M |\omega|^2|\nabla\varphi|^2 + |\varphi|^2|\nabla|\omega||^2 + 2\varphi|\omega|\langle\nabla\varphi,\nabla|\omega|\rangle\,dv\right). \end{aligned}$$

Then (2.4) becomes

$$\begin{aligned} (2.5) \qquad 0 &\leq -2\int_M \varphi|\omega|\langle\nabla\varphi,\nabla|\omega|\rangle\,dv - \frac{n}{n-1}\int_M \varphi^2|\nabla|\omega||^2\,dv \\ &\quad + \frac{n-1}{n}C_s\left(\int_M |A|^n\,dv\right)^{2/n} \\ &\quad \times\left(\int_M |\omega|^2|\nabla\varphi|^2 + \varphi^2|\nabla|\omega||^2 + 2\varphi|\omega|\langle\nabla\varphi,\nabla|\omega|\rangle\,dv\right). \end{aligned}$$

Using the following inequality for $\varepsilon > 0$,

$$2\left|\int_M \varphi|\omega|\langle\nabla\varphi,\nabla|\omega|\rangle\,dv\right| \leq \frac{\varepsilon}{2}\int_M \varphi^2|\nabla|\omega||^2\,dv + \frac{2}{\varepsilon}\int_M |\omega|^2|\nabla\varphi|^2\,dv,$$

we have from (2.5)

$$\left\{\frac{n}{n-1}-\frac{n-1}{n}C_s\left(\int_M |A|^n\,dv\right)^{2/n}-\frac{\varepsilon}{2}\left(1+\frac{n-1}{n}C_s\left(\int_M |A|^n\,dv\right)^{2/n}\right)\right\}$$

$$\times\int_M \varphi^2|\nabla|\omega|\,|^2\,dv$$

$$\leq\left\{\frac{2}{\varepsilon}\left(1+\frac{n-1}{n}\left(\int_M |A|^n\,dv\right)^{2/n}\right)+\frac{n-1}{n}C_s\left(\int_M |A|^n\,dv\right)^{2/n}\right\}$$

$$\times\int_M |\omega|^2|\nabla\varphi|^2\,dv.$$

Since $(\int_M |A|^n\,dv)^{1/n}<\frac{n}{n-1}\sqrt{C_s^{-1}}$ by assumption, choosing $\varepsilon>0$ sufficiently small and letting $R\to\infty$, we obtain $\nabla|\omega|\equiv 0$, i.e., $|\omega|$ is constant. However, since $\int_M |\omega|^2\,dv<\infty$ and the volume of M is infinite, we get $\omega\equiv 0$. □

References

[1] M. T. Anderson, The compactification of a minimal submanifold in Euclidean space by the Gauss map, Inst. Hautestudes Sci. Publ. Math., 1984, preprint.

[2] H. Cao, Y. Shen and S. Zhu, The structure of stable minimal hypersurfaces in $\mathbf{R}^{n+1}$, Math. Res. Lett. **4** (1997), 637–644.

[3] P. F. Leung, An estimate on the Ricci curvature of a submanifold and some applications, Proc. Amer. Math. Soc. **114** (1992), 1051–1063.

[4] J. Michael and L. M. Simon, Sobolev and mean-value inequalities on generalized submanifolds of $\mathbf{R}^n$, Comm. Pure. Appl. Math. **26** (1973), 361–379.

[5] R. Miyaoka, L^2 harmonic 1-forms on a complete stable minimal hypersurface, Geometry and global analysis, Tohoku Univ., Sendai, 1993, 289–293.

[6] L. Ni, Gap theorems for minimal submanifolds in $\mathbf{R}^{n+1}$, Comm. Anal. Geom. **9** (2001), 641–656.

[7] B. Palmer, Stability of minimal hypersurfaces, Comment. Math. Helv. **66** (1991), 185–188.

[8] Y. Shen and X. Zhu, On stable complete minimal hypersurfaces in $\mathbf{R}^{n+1}$, Amer. J. Math. **120** (1998), 103–116.

[9] X. Wang, On conformally compact Einstein manifolds, Math. Res. Lett. **8** (2001), 671–685.

[10] G. Yun, Total scalar curvature and L^2 harmonic 1-forms on a minimal hypersurface in Euclidean space, Geom. Dedicata **89** (2002), 135–141.

Keomkyo Seo
School of Mathematics
Korea Institute for Advanced Study, 207-43
Cheongnyangni 2-dong, Dongdaemun-gu
Seoul 130-722
Korea
E-mail: kseo@kias.re.kr