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AN ASYMPTOTIC BEHAVIOR OF THE DILATATION FOR A FAMILY OF PSEUDO-ANOSOV BRAIDS

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Abstract

The dilatation of a pseudo-Anosov braid is a conjugacy invariant. In this paper, we study the dilatation of a special family of pseudo-Anosov braids. We prove an inductive formula to compute their dilatation, a monotonicity and an asymptotic behavior of the dilatation for this family of braids. We also give an example of a family of pseudo-Anosov braids with arbitrarily small dilatation such that the mapping torus obtained from such braid has 2 cusps and has an arbitrarily large volume.

1. Introduction

Let $\Sigma = \Sigma_{g,p}$ be an orientable surface of genus g with p punctures, and let $\mathcal{M}(\Sigma)$ be the mapping class group of Σ. The elements of $\mathcal{M}(\Sigma)$, called *mapping classes*, are classified into 3 types: periodic, reducible and pseudo-Anosov [10]. For a pseudo-Anosov mapping class ϕ, the dilatation $\lambda(\phi)$ is an algebraic integer strictly greater than 1. The dilatation of a pseudo-Anosov mapping class is a conjugacy invariant.

Let D_n be an n-punctured closed disk. The mapping class group $\mathcal{M}(D_n)$ of D_n is isomorphic to a subgroup of $\mathcal{M}(\Sigma_{0,n+1})$. There is a natural surjective homomorphism

$$\Gamma : B_n \to \mathcal{M}(D_n)$$

from the n-braid group B_n to the mapping class group $\mathcal{M}(D_n)$ [2]. We say that a braid $\beta \in B_n$ is *pseudo-Anosov* if $\Gamma(\beta)$ is pseudo-Anosov, and if this is the case the dilatation $\lambda(\beta)$ of β is defined equal to $\lambda(\Gamma(\beta))$. Henceforth, we shall abbreviate 'pseudo-Anosov' to 'pA'.

We now introduce a family of braids. Let $\beta_{(m_1, m_2, \ldots, m_{k+1})}$ be the braid as depicted in Figure 1, for each integer $k \geq 1$ and each integer $m_i \geq 1$. These are all pA (Proposition 4.1). We will prove a monotonicity, an inductive formula to

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compute their dilatation and an asymptotic behavior of the dilatation for this family of braids.

PROPOSITION 1.1 (Monotonicity). *For each integer i with $1 \le i \le k+1$, we have*

$$\lambda(\beta_{(m_1,\dots,m_i,\dots,m_{k+1})}) > \lambda(\beta_{(m_1,\dots,m_i+1,\dots,m_{k+1})}).$$

Hence if $m_i \le m_i'$ for each i, then $\lambda(\beta_{(m_1,\dots,m_{k+1})}) \ge \lambda(\beta_{(m_1',\dots,m_{k+1}')})$.

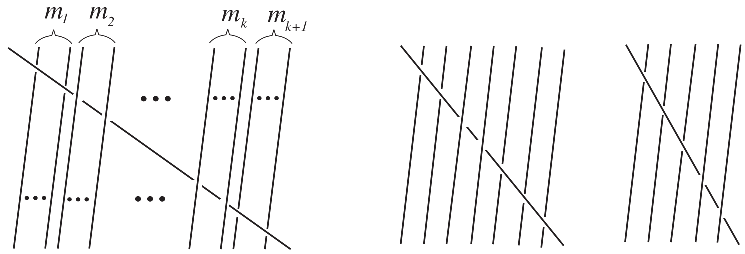

FIGURE 1. (left) $\beta_{(m_1,m_2,\dots,m_{k+1})}$, (center) $\beta_{(2,2,3)}$, (right) $\beta_{(3,2)}$.

For an integral polynomial $f(t)$ of degree d, the *reciprocal* of $f(t)$, denoted by $f_*(t)$, is $t^d f(1/t)$.

THEOREM 1.2 (Inductive formula). *The dilatation of the pA braid $\beta_{(m_1,\dots,m_{k+1})}$ is the largest root of the polynomial*

$$t^{m_{k+1}} R_{(m_1,\dots,m_k)}(t) + (-1)^{k+1} R_{(m_1,\dots,m_k)_*}(t),$$

where $R_{(m_1,\dots,m_i)}(t)$ is given inductively as follows:

$$R_{(m_1)}(t) = t^{m_1+1}(t-1) - 2t, \quad \textit{and}$$

$$R_{(m_1,\dots,m_i)}(t) = t^{m_i}(t-1)R_{(m_1,\dots,m_{i-1})}(t) + (-1)^i 2t R_{(m_1,\dots,m_{i-1})_*}(t) \quad \textit{for } 2 \le i \le k.$$

THEOREM 1.3 (Asymptotic behavior). *We have*

(1) $\lim_{m_1,\dots,m_{k+1}\to\infty} \lambda(\beta_{(m_1,\dots,m_{k+1})}) = 1$ *and*

(2) $\lim_{m_i,m_{i+1},\dots,m_{k+1}\to\infty} \lambda(\beta_{(m_1,\dots,m_{k+1})}) = \lambda(R_{(m_1,\dots,m_{i-1})}(t)) > 1$ *for $i \ge 2$, where $\lambda(f(t))$ denotes the maximal absolute value of the roots of $f(t)$.*

For a pA braid β, let ϕ be the pA mapping class $\Gamma(\beta)$. The dilatation $\lambda(\phi)$ can be computed as follows. A smooth graph τ, called a *train track* and a smooth graph map $\hat{\phi} : \tau \to \tau$ are associated with ϕ. The edges of τ are classified into *real* edges and *infinitesimal* edges, and the *transition matrix* $M_{\mathrm{real}}(\hat{\phi})$ with respect to real edges can be defined. Then the dilatation $\lambda(\phi)$ equals the spectral radius of $M_{\mathrm{real}}(\hat{\phi})$. For more details, see Section 2.2.

For the computation of the dilatation of the braid $\beta_{(m_1,\dots,m_{k+1})}$, we introduce *combined trees* and *combined tree maps* in Section 3. For a given $(m_1,\dots,m_{k+1})$, one can obtain the combined tree $\mathcal{Q}_{(m_1,\dots,m_{k+1})}$ and the combined tree map

$q_{(m_1,\dots,m_{k+1})}$ inductively. For example, for $(m_1, m_2, m_3) = (4, 2, 1)$, the combined tree $\mathcal{Q}_{(m_1,m_2,m_3)}$, depicted in Figure 2, is obtained by gluing the combined tree $\mathcal{Q}_{(m_1,m_2)}$ and another tree which depends m_3. The combined tree map $q_{(m_1,m_2,m_3)}$, as shown in Figure 3, is defined by the composition of an extension of the combined tree map $q_{(m_1,m_2)}$ and another tree map which depends on m_3.

By the proof of Proposition 4.1, it turns out that the spectral radius of the transition matrix $M(q_{(m_1,\dots,m_{k+1})})$ obtained from $q_{(m_1,\dots,m_{k+1})}$ equals that of $M_{\mathrm{real}}(\hat{\phi})$, where $\phi = \Gamma(\beta_{(m_1,\dots,m_{k+1})})$, that is the spectral radius of $M(q_{(m_1,\dots,m_{k+1})})$ equals the dilatation $\lambda(\beta_{(m_1,\dots,m_{k+1})})$. Proposition 1.1 and Theorems 1.2, 1.3 will be shown by using the properties of combined tree maps.

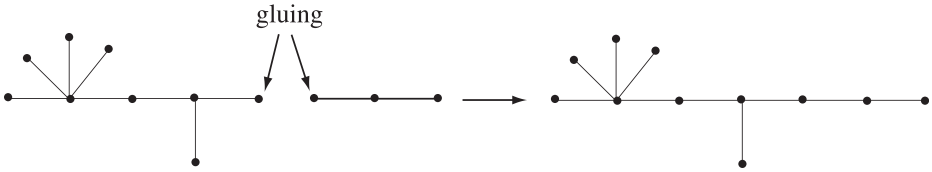

FIGURE 2. $\mathcal{Q}_{(4,2,1)}$ (right) is obtained by gluing $\mathcal{Q}_{(4,2)}$ (left) and another tree (center).

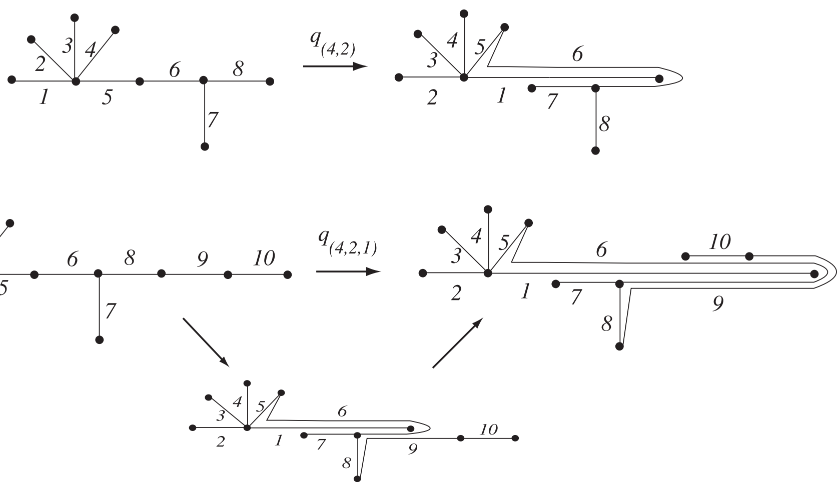

FIGURE 3. (top) $q_{(4,2)}$, (bottom) $q_{(4,2,1)}$.

In the final part, we will consider the two invariants of pA mapping classes, the dilatation and the volume. Choosing any representative $f : \Sigma \to \Sigma$ of a mapping class ϕ, we form the mapping torus

$$\mathbf{T}(\phi) = \Sigma \times [0,1]/\sim,$$

where $\sim$ identifies $(x, 0)$ with $(f(x), 1)$. A mapping class ϕ is pA if and only if $\mathbf{T}(\phi)$ admits a complete hyperbolic structure of finite volume [7]. Since such a structure is unique up to isometry, it makes sense to speak of the *volume* $\mathrm{vol}(\phi)$ of ϕ, the hyperbolic volume of $\mathbf{T}(\phi)$. For a pA braid β, we define the volume $\mathrm{vol}(\beta)$ as equal to $\mathrm{vol}(\Gamma(\beta))$, the volume of the mapping torus $\mathbf{T}(\Gamma(\beta))$.

Theorem 1.3(1) tells us that dilatation of braids can be arbitrarily small. We consider what happen for the volume of a family of pseudo-Anosov mapping classes whose dilatation is arbitrarily small. It is not hard to see the following.

PROPOSITION 1.4. *There exists a family of pA mapping classes ϕ_n of $\mathcal{M}(D_n)$ such that*

$$\lim_{n\to\infty} \lambda(\phi_n) = 1 \quad \textit{and} \quad \lim_{n\to\infty} \mathrm{vol}(\phi_n) = \infty$$

and such that the number of the cusps of the mapping torus $\mathbf{T}(\phi_n)$ goes to ∞ as n goes to ∞.

Proposition 1.4 is not so surprising, because the volume of each cusp is bounded below uniformly. We show the following.

PROPOSITION 1.5. *There exists a family of pA mapping classes ϕ_n of $\mathcal{M}(D_n)$ such that*

$$\lim_{n\to\infty} \lambda(\phi_n) = 1 \quad \textit{and} \quad \lim_{n\to\infty} \mathrm{vol}(\phi_n) = \infty$$

and such that the number of the cusps of the mapping torus $\mathbf{T}(\phi_n)$ is 2 for each n.

Proposition 1.5 is a corollary of the following theorem.

THEOREM 1.6. *For any real number $\lambda > 1$ and any real number $v > 0$, there exist an integer $k \geq 1$ and an integer $m \geq 1$ such that for any integer $m_i \geq m$ with $1 \leq i \leq k+1$, we have*

$$\lambda(\beta_{(m_1,\dots,m_{k+1})}) < \lambda \quad \textit{and} \quad \mathrm{vol}(\beta_{(m_1,\dots,m_{k+1})}) > v.$$

Here we note that for a braid b, the mapping torus $\mathbf{T}(\Gamma(b))$ is homeomorphic to the link complement $S^3\backslash\bar{b}$ in the 3 sphere S^3, where $\bar{b}$ is a union of the closed braid of b and the braid axis (Figure 4). When b is a braid $\beta_{(m_1,\dots,m_{k+1})}$, the link $\bar{b}$ has 2 components, and hence the number of cusps of $\mathbf{T}(\Gamma(b))$ is 2.

2. Preliminaries

A homeomorphism $\Phi : \Sigma \to \Sigma$ is *pseudo-Anosov* (*pA*) if there exists a constant $\lambda = \lambda(\Phi) > 1$, called the *dilatation of* Φ, and there exists a pair of transverse measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that

$$\Phi(\mathcal{F}^s) = \frac{1}{\lambda}\mathcal{F}^s \quad \text{and} \quad \Phi(\mathcal{F}^u) = \lambda\mathcal{F}^u.$$

A mapping class $\phi \in \mathcal{M}(\Sigma)$ is said to be *pseudo-Anosov* (pA) if ϕ contains a pA homeomorphism. We define the dilatation of a pA mapping class ϕ, denoted by $\lambda(\phi)$, to be the dilatation of a pA homeomorphism of ϕ.

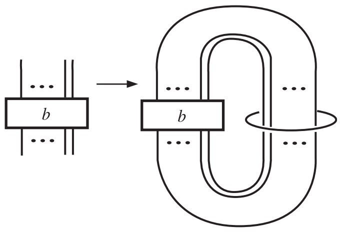

FIGURE 4. link $\bar{b}$.

Let $\mathcal{G}$ be a graph. We denote the set of vertices by $V(\mathcal{G})$ and denote the set of edges by $E(\mathcal{G})$. A continuous map $g : \mathcal{G} \to \mathcal{G}'$ from $\mathcal{G}$ into another graph $\mathcal{G}'$ is said to be a *graph map*. When $\mathcal{G}$ and $\mathcal{G}'$ are trees, a graph map $g : \mathcal{G} \to \mathcal{G}'$ is said to be a *tree map*. A graph map g is called *Markov* if $g(V(\mathcal{G})) \subset V(\mathcal{G}')$ and for each point $x \in \mathcal{G}$ such that $g(x) \notin V(\mathcal{G}')$, g is locally injective at x (that is g has no 'back track' at x). In the rest of the paper we assume that all graph maps are Markov.

For a graph map g, we define the *transition matrix* $M(g) = (m_{i,j})$ such that the i^{th} edge e_i' or the same edge with opposite orientation $(e_i')^{-1}$ of $\mathcal{G}'$ appears $m_{i,j}$-times in the edge path $g(e_j)$ for the j^{th} edge e_j of $\mathcal{G}$. If $\mathcal{G} = \mathcal{G}'$, then $M(g)$ is a square matrix, and it makes sense to consider the spectral radius, $\lambda(g) = \lambda(M(g))$, called the *growth rate* for g. The topological entropy of g is known to be equal to $\log \lambda(g)$.

In Section 2.1 we recall results regarding Perron-Frobenius matrices. In Section 2.2 we quickly review a result from the train track theory which tells us that if a given mapping class ϕ induces a certain graph map, called *train track map*, whose transition matrix is Perron-Frobenius, then ϕ is pA and $\lambda(\phi)$ equals the growth rate of the train track map. In Section 2.3 we consider roots of a family of polynomials to study the dilatation of pA mapping classes and give some results regarding the asymptotic behavior of roots of this family.

2.1. Perron-Frobenius theorem

Let $M = (m_{i,j})$ and $N = (n_{i,j})$ be matrices with the same size. We shall write $M \geq N$ (resp. $M > N$) whenever $m_{i,j} \geq n_{i,j}$ (resp. $m_{i,j} > n_{i,j}$) for each i, j. We say that M is *positive* (resp. *non-negative*) if $M > \mathbf{0}$ (resp. $M \geq \mathbf{0}$), where $\mathbf{0}$ is the zero matrix.

For a square and non-negative matrix T, let $\lambda(T)$ be its spectral radius, that is the maximal absolute value of eigenvalues of T. We say that T is *irreducible* if for every pair of indices i and j, there exists an integer $k = k_{i,j} > 0$ such that the (i, j) entry of T^k is strictly positive. The matrix T is *primitive* if there exists an integer $k > 0$ such that the matrix T^k is positive. By definition, a primitive matrix is irreducible. A primitive matrix T is *Perron-Frobenius*, abbreviated to PF, if T is an integral matrix. For $M \geq T$, if T is irreducible then M is also irreducible. The following theorem is commonly referred to as the Perron-Frobenius theorem.

THEOREM 2.1 [8]. *Let T be a primitive matrix. Then, there exists an eigenvalue $\lambda > 0$ of T such that*

(1) *λ has strictly positive left and right eigenvectors $\hat{\mathbf{x}}$ and $\mathbf{y}$ respectively, and*

(2) *$\lambda > |\lambda'|$ for any eigenvalue $\lambda' \neq \lambda$ of T.*

If T is a PF matrix, the largest eigenvalue λ in the sense of Theorem 2.1 is strictly greater than 1, and it is called the *PF eigenvalue*. The corresponding positive eigenvector is called the *PF eigenvector*.

The following will be useful.

LEMMA 2.2 [8, Theorem 1.6, Exercise 1.17]. *Let T be a primitive matrix, and let s be a positive number. Suppose that a non-zero vector $\mathbf{y} \geq \mathbf{0}$ satisfies $T\mathbf{y} \geq s\mathbf{y}$. Then,*

(1) *$\lambda \geq s$, where λ is the largest eigenvalue of T in the sense of Theorem 2.1, and*

(2) *$s = \lambda$ if and only if $T\mathbf{y} = s\mathbf{y}$.*

Proof. (1) Let $\hat{\mathbf{x}}$ be a positive left eigenvector of T. Then,

$$\hat{\mathbf{x}}T\mathbf{y} = \lambda\hat{\mathbf{x}}\mathbf{y} \geq s\hat{\mathbf{x}}\mathbf{y}.$$

Hence we have $\lambda \geq s$.

(2) ('Only if' part) Suppose that $s = \lambda$, and suppose that $T\mathbf{y} \geq \lambda\mathbf{y}$ and $T\mathbf{y} \neq \lambda\mathbf{y}$. Premultiplying this inequality by a positive left eigenvector $\hat{\mathbf{x}}$ of T, we have

$$\hat{\mathbf{x}}T\mathbf{y}(= \lambda\hat{\mathbf{x}}\mathbf{y}) > \lambda\hat{\mathbf{x}}\mathbf{y}.$$

Hence $\lambda > \lambda$, which is a contradiction.

('If' part) Suppose that $T\mathbf{y} = s\mathbf{y}$. Premultiplying this equality by a positive left eigenvector $\hat{\mathbf{x}}$ of T, we obtain $\lambda = s$. $\square$

For a non-negative $k \times k$ matrix T, one can associate a *directed graph G_T* as follows. The graph G_T has vertices numbered $1, 2, \ldots, k$ and an edge from the j^{th} vertex to the i^{th} vertex if and only if the (i, j) entry $T_{i,j} \neq 0$. By the definition of G_T, one easily verifies the following.

LEMMA 2.3. *Let T be a non-negative square matrix.*

(1) *T is irreducible if and only if for each i, j, there exists an integer $n_{i,j} > 0$ such that the directed graph G_T has an edge path of length $n_{i,j}$ from the j^{th} vertex to the i^{th} vertex.*

(2) *T is primitive if and only if there exists an integer $n > 0$ such that for each i, j, the directed graph G_T has an edge path of length n from the j^{th} vertex to the i^{th} vertex.*

2.2. Train track maps

A smooth branched 1-manifold τ embedded in D_n is a *train track* if each component of $D_n \backslash \tau$ is either a non-punctured k-gon ($k \geq 3$), a once punctured k-gon ($k \geq 1$) or an annulus such that a boundary component of the annulus coincides with the boundary of D_n and the other component has at least 1 prong. A smooth map from a train track into itself is called a *train track map*.

Let $f : D_n \to D_n$ be a homeomorphism. A train track τ is *invariant* under f if $f(\tau)$ can be collapsed smoothly onto τ in D_n. In this case f induces a train track map $\hat{f} : \tau \to \tau$. An edge of τ is called *infinitesimal* if there exists an integer $N > 0$ such that $\hat{f}^N(\tau)$ is a periodic edge under $\hat{f}$. An edge of τ is called *real* if it is not infinitesimal. The transition matrix of $\hat{f}$ is of the form:

$$M(\hat{f}) = \begin{pmatrix} M_{\mathrm{real}}(\hat{f}) & \mathbf{0} \\ A & M_{\mathrm{inf}}(\hat{f}) \end{pmatrix},$$

where $M_{\mathrm{real}}(\hat{f})$ (resp. $M_{\mathrm{inf}}(\hat{f})$) is the transition matrix with respect to real (resp. infinitesimal) edges. The following is a consequence of [1].

PROPOSITION 2.4. *A mapping class $\phi \in \mathcal{M}(D_n)$ is pA if and only if there exists a homeomorphism $f : D_n \to D_n$ of ϕ and there exists a train track τ such that τ is invariant under f, and for the induced train track map $\hat{f} : \tau \to \tau$, the matrix $M_{\mathrm{real}}(\hat{f})$ is PF. When ϕ is a pA mapping class, we have $\lambda(\phi) = \lambda(M_{\mathrm{real}}(\hat{f}))$.*

2.3. Roots of polynomials

For an integral polynomial $S(t)$, let $\lambda(S(t))$ be the maximal absolute value of roots of $S(t)$. For a monic integral polynomial $R(t)$, we set

$$Q_{n,\pm}(t) = t^n R(t) \pm S(t)$$

for each integer $n \geq 1$. The polynomial $R(t)$ (resp. $S(t)$) is called *dominant* (resp. *recessive*) for a family of polynomials $\{Q_{n,\pm}(t)\}_{n \geq 1}$. In case where $S(t) = R_*(t)$, we call $t^n R(t) \pm R_*(t)$ the *Salem-Boyd polynomial associated to* $R(t)$. E. Hironaka shows that such polynomials have several nice properties [3, Section 3]. The following lemma shows that roots of $Q_{n,\pm}(t)$ lying outside the unit circle are determined by those of $R(t)$ asymptotically.

LEMMA 2.5. *Suppose that $R(t)$ has a root outside the unit circle. Then, the roots of $Q_{n,\pm}(t)$ outside the unit circle converge to those of $R(t)$ counting multiplicity as n goes to ∞. In particular, $\lambda(R(t)) = \lim_{n\to\infty} \lambda(Q_{n,\pm}(t))$.*

The proof can be found in [3]. We recall a proof here for completeness.

Proof. Consider the rational function

$$\frac{Q_{n,\pm}(t)}{t^n} = R(t) \pm \frac{S(t)}{t^n}.$$

Let θ be a root of $R(t)$ with multiplicity m outside the unit circle. Let D_θ be any small disk centered at θ that is strictly outside of the unit circle and that contains no roots of $R(t)$ other than θ. Then, $|R(t)|$ has a lower bound on the boundary ∂D_θ by compactness. Hence there exists a number $n_\theta > 0$ depending on θ such that $|R(t)| > \left|\frac{S(t)}{t^n}\right|$ on ∂D_θ for any $n > n_\theta$. By the Rouché's theorem, it follows that $R(t)$ and $R(t) \pm \frac{S(t)}{t^n}$ (hence $R(t)$ and $Q_{n,\pm}(t)$) have the same m roots in D_θ. Since D_θ can be made arbitrarily small and there exist only finitely many roots of $R(t)$, the proof of Lemma 2.5 is complete. □

LEMMA 2.6. *Suppose that $R(t)$ has no roots outside the unit circle, and suppose that $Q_{n,\pm}(t)$ has a real root μ_n greater than* 1 *for sufficiently large n. Then,* $\lim_{n\to\infty} \mu_n = 1$.

Proof. For any $\varepsilon > 0$, let D_ε be the disk of radius $1+\varepsilon$ around the origin in the complex plane. Then, for any sufficiently large n, we have $|R(t)| > \left|\frac{S(t)}{t^n}\right|$ for all t on ∂D_ε. Moreover, $R(t)$ and $\pm\frac{S(t)}{t^n}$ are holomorphic on the complement of D_ε in the Riemann sphere. By Rouché's theorem, $R(t)$ and $R(t) \pm \frac{S(t)}{t^n}$ (hence $R(t)$ and $Q_{n,\pm}(t)$) have no roots outside D_ε. Hence μ_n converges to 1 as n goes to ∞. □

3. Combined tree maps

For an $n \times n$ matrix M, let $M(t)$ be the characteristic polynomial $|tI - M|$ of M, where $I = I_n$ is the $n \times n$ identity matrix. Let $M_*(t)$ be the reciprocal polynomial of $M(t)$. Then,

$$M_*(t) = t^n\left|\frac{1}{t}I - M\right| = |I - tM|,$$

that is $M_*(t)$ equals the determinant of the matrix $I - tM$.

This section introduces *combined tree maps*. Given two trees we combine these trees with another tree of star type having the valence $n+1$ vertex and define a new tree, say $\mathcal{Q}_n$. When two tree maps on $\mathcal{Q}_n$ satisfy certain conditions (**L1**, **L2**, **L3** and **R1**, **R2**, **R3**), we can define the combined tree map q_n on $\mathcal{Q}_n$ and obtain a family of tree maps $\{q_n : \mathcal{Q}_n \to \mathcal{Q}_n\}_{n\geq 1}$. In Section 3.1 we give a sufficient condition that guarantees $M(q_n)$ is PF. In Section 3.2 we consider combined tree maps in a particular setting. Then, we give a formula for $M(q_n)(t)$ and $M(q_n)_*(t)$ and analyze the asymptotic behavior of the growth rate for q_n. This analysis will be applied to train track maps in Section 4.

3.1. Transition matrices and growth rate

We assume that all trees are embedded in the disk D. By the trivial tree $\mathcal{T}_0$, we mean the tree with only one vertex. Let $\mathcal{G}_{n,+}$ and $\mathcal{G}_{n,-}$ be trees of star type as in Figure 5, having one vertex of valence $n+1$.

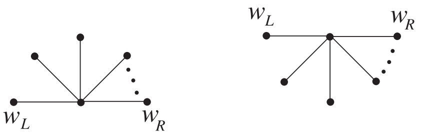

FIGURE 5. trees (left) $\mathcal{G}_{n,+}$ and (right) $\mathcal{G}_{n,-}$ having one vertex of valence $n+1$.

Let $\mathcal{G}_L$ (resp. $\mathcal{G}_R$) be a tree (possibly a trivial tree) with a valence 1 vertex, say v_L (resp. v_R). Let w_L and w_R be vertices of $\mathcal{G}_{n,+}$ as in Figure 5, and glue $\mathcal{G}_L$, $\mathcal{G}_{n,+}$ and $\mathcal{G}_R$ together so that for $S \in \{L, R\}$, v_S and w_S become one vertex (Figure 6). The resulting tree $\mathcal{Q}_{n,+}$ is called the *combined tree*, obtained from the triple $(\mathcal{G}_L, \mathcal{G}_{n,+}, \mathcal{G}_R)$. We define the combined tree $\mathcal{Q}_{n,-}$, obtained from the triple $(\mathcal{G}_L, \mathcal{G}_{n,-}, \mathcal{G}_R)$ in the same manner.

Before we define combined tree maps on $\mathcal{Q}_{n,+/-}$, we label the edges of $\mathcal{Q}_{n,+/-}$. Let ℓ be the number of edges of $\mathcal{G}_L$, and let r be the number of edges of $\mathcal{G}_R$ plus 1. Note that the number of edges of $\mathcal{Q}_{n,+/-}$ is $\ell + n + r$.

- The edges of $\mathcal{G}_{n,+/-}$ are numbered $\ell + 1$ to $\ell + n + 1$ in the clockwise/counterclockwise direction as in Figure 7.
- The edge of $\mathcal{G}_L$ sharing a vertex with the $(\ell + 1)^{\mathrm{st}}$ edge is numbered ℓ and the remaining edges of $\mathcal{G}_L$ are numbered 1 to $\ell - 1$ arbitrarily.
- The edge of $\mathcal{G}_R$ sharing a vertex with the $(\ell + n + 1)^{\mathrm{st}}$ edge is numbered $\ell + n + 2$ and the remaining edges of $\mathcal{G}_R$ are numbered $\ell + n + 3$ to $\ell + n + r$ arbitrarily.

The edge numbered i is denoted by e_i.

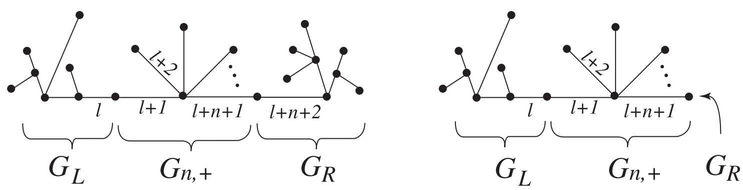

FIGURE 6. combined trees $\mathcal{Q}_{n,+}$: (left) general case, (right) case where $\mathcal{G}_R$ is the trivial tree.

Now we take a tree map $g_L : \mathcal{Q}_{n,+/-} \to \mathcal{Q}_{n,+/-}$ satisfying the following conditions.

L1 The map g_L restricted to the set of vertices of $E(\mathcal{Q}_{n,+/-}) \backslash (E(\mathcal{G}_L) \cup \{e_{\ell+1}\})$ is the identity.

L2 $g_L(\mathcal{G}_L) \subset \mathcal{G}_L$.

L3 The edge path $g_L(e_{\ell+1})$ passes through $e_{\ell+1}$ only once and passes through e_ℓ.

Next, we take a tree map $g_R : \mathcal{Q}_{n,+/-} \to \mathcal{Q}_{n,+/-}$ satisfying the following conditions.

R1 The map g_R restricted to the set of vertices of $E(\mathcal{Q}_{n,+/-}) \backslash (E(\mathcal{G}_R) \cup \{e_{\ell+n+1}\})$ is the identity.

R2 $g_R(\mathscr{G}_R) \subset \mathscr{G}_R$.

R3 The edge path $g_R(e_{\ell+n+1})$ passes through $e_{\ell+n+1}$ only once and passes through $e_{\ell+n+2}$.

Finally, we define the tree map $g_n : \mathscr{Q}_{n,+/-} \to \mathscr{Q}_{n,+/-}$ satisfying the following conditions.

n1 The map g_n restricted to the set of vertices of $E(\mathscr{Q}_{n,+/-}) \setminus (E(\mathscr{G}_{n,+/-}) \cup \{e_\ell, e_{\ell+n+2}\})$ is the identity.

n2 g_n rotates the subtree $\mathscr{G}_{n,+/-}$ as in Figure 7.

n3 The image of each $e \in \{e_\ell, e_{\ell+n+2}\}$ is as in Figure 7. The length of the edge path $g_n(e)$ is 3.

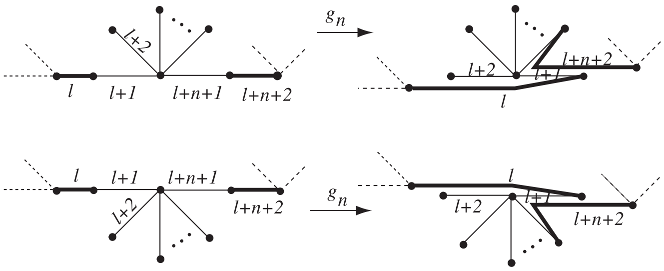

FIGURE 7. (top) g_n rotates $\mathscr{G}_{n,+}$, (bottom) g_n rotates $\mathscr{G}_{n,-}$. The edges e_ℓ and $e_{\ell+n+2}$ and their images are drawn in bold.

The composition

$$q_n = g_R g_n g_L : \mathscr{Q}_{n,+/-} \to \mathscr{Q}_{n,+/-}$$

is called the *combined tree map*, obtained from the triple (g_L, g_n, g_R). It makes sense to consider the transition matrices $M(g_S)$ of $g_S|_{\mathscr{G}_S} : \mathscr{G}_S \to \mathscr{G}_S$, $S \in \{L, R\}$ and $M(g_n)$ of $g_n|_{\mathscr{G}_{n,+/-}} : \mathscr{G}_{n,+/-} \to \mathscr{G}_{n,+/-}$. The transition matrix $M(q_n)$ has the following form:

$$M(q_n) = \begin{pmatrix} M_L & A & \mathbf{0} \\ B & M_n & C \\ D & E & M_R \end{pmatrix}, \quad \text{where } M_n = \begin{pmatrix} * & 1 & & \\ & & \ddots & \\ & & & 1 \\ * & & & \end{pmatrix} \tag{3.1}$$

(each empty space in M_n represents the number 0), and the block matrices satisfy $M_L \geq M(g_L)$, $M_n \geq M(g_n)$ and $M_R \geq M(g_R)$. (In fact $M_L = M(g_L)$, although we will not be using this fact.)

Throughout this subsection, we assume that the trees $\mathscr{G}_L$ and $\mathscr{G}_R$ are not trivial. It is straightforward to see the following from the defining conditions of g_L, g_n and g_R.

LEMMA 3.1. *Let $m_{i,j}$ be the (i,j) entry of $M(q_n)$. We have*

(1) $m_{\ell+n,\ell+n+2} = 1$ *and* $m_{\ell+n+1,\ell+n+2} = 1$, *and*

(2) $m_{\ell,\ell+1} > 0$, $m_{\ell+1,\ell+1} > 0$ *and* $m_{\ell+n+1,\ell+1} > 1$. *Moreover*, $m_{\ell+1,j} = m_{\ell+n+1,j}$ *for each* j *with* $1 \le j \le \ell$ *and* $m_{\ell+1,j_0} > 0$ *for some* $1 \le j_0 \le \ell$, *and*
(3) $m_{\ell+n+2,\ell+1} > 0$.

An important feature is that the growth rate of q_n is always greater than 1 if $M(g_L)$ and $M(g_R)$ are irreducible, which will be shown in Proposition 3.3. We first show that $M(q_n)$ is irreducible in this case. Notice that $M(g_n)$ is always irreducible, and since $M_n \ge M(q_n)$ so M_n must be irreducible as well.

LEMMA 3.2. *Let* $q_n = g_R g_n g_L : \mathcal{Q}_{n,+/-} \to \mathcal{Q}_{n,+/-}$ *be the combined tree map. Assume that both* $M(g_L)$ *and* $M(g_R)$ *are irreducible. Then,* $M(q_n)$ *is irreducible.*

Proof. Note that M_L, M_R and M_n are irreducible. Let G_{q_n} be the directed graph of $M(q_n)$. We identify vertices of G_{q_n} with edges of $\mathcal{Q}_{n,+/-}$. Let V_L (resp. V_R, V_n) be the set of vertices of G_{q_n} coming from the set of edges of the subtree $\mathcal{G}_L$ (resp. $\mathcal{G}_R$, $\mathcal{G}_{n,+/-}$) of $\mathcal{Q}_{n,+/-}$. Lemma 3.1(2) shows that there exists an edge connecting the set V_L to the set V_n, and there exists an edge connecting the set V_n to the set V_L. This is also true between V_n and V_R by Lemma 3.1(1,3). Thus, one can find an edge path between any two vertices of G_{q_n}. □

PROPOSITION 3.3. *Under the assumptions of Lemma* 3.2, $M(q_n)$ *is PF.*

Proof. Lemma 3.1(2) says that the directed graph G_{q_n} has an edge from the vertex $v_{\ell+1}$ to itself, and we denote such edge by e. Since $M(q_n)$ is irreducible, for any vertex v of G_{q_n} there exists an edge path $E = e_1 e_2 \cdots e_{n(v)}$ from $v_{\ell+1}$ to v. Thus, for any $n \ge n(v)$ we have an edge path $e \cdots eE$ of length n from $v_{\ell+1}$ to v. Since the number of vertices is finite, there exists an integer $N > 0$ such that for any vertex w of G_{q_n} and any integer $n \ge N$ we have an edge path of length n from $v_{\ell+1}$ to w. Since there exists an edge path from any vertex x of G_{q_n} to $v_{\ell+1}$, we can find a sufficiently large integer N' such that for any pair of vertices x and w there exists an edge path of length N' from x to w. Thus, $M(q_n)$ is PF. □

The following property is crucial in proving Proposition 1.1 and Theorem 1.3.

PROPOSITION 3.4. *Under the assumptions of Lemma* 3.2, *we have* $\lambda(M(q_n)) > \lambda(M(q_{n+1})) > 1$.

Proof. To compare $M(q_{n+1})$ with $M(q_n)$ we introduce a new labeling of edges of $\mathcal{Q}_{n+1,+/-}$. The trees $\mathcal{G}_L$ and $\mathcal{G}_R$ are the common subtrees for both trees $\mathcal{Q}_{n,+/-}$ and $\mathcal{Q}_{n+1,+/-}$. Edges of the subtrees $\mathcal{G}_L$ and $\mathcal{G}_R$ of $\mathcal{Q}_{n+1,+/-}$ are numbered in the same manner as those of $\mathcal{Q}_{n,+/-}$, and edges of $\mathcal{G}_{n+1}$ are numbered

$$\ell+1, \ell+n+r+1, \ell+2, \ell+3, \ldots, \ell+n+1$$

in the clockwise/counterclockwise direction. Here the edge sharing a vertex with the ℓ^{th} edge is numbered $\ell+1$.

Let $M(q_n) = (m_{i,j})_{1 \le i,j \le \ell+n+r}$ be the matrix given in (3.1). Then, $M(q_{n+1}) = (m'_{i,j})_{1 \le i,j \le \ell+n+r+1}$ with new labeling has the following form:

$$M(q_{n+1}) = \left(\begin{array}{c|c|c|c} M_L & A & \mathbf{0} & \\ \hline B & \begin{matrix} * & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ * & & & \end{matrix} & C & \begin{matrix} 1 \\ \\ \\ \\ \\ \end{matrix} \\ \hline D & E & M_R & \\ \hline & 1 & & \end{array}\right).$$

Put $s = \lambda(M(q_{n+1})) > 1$ and let $\mathbf{y} = {}^t(y_1, \dots, y_{\ell+n+r+1})$ be the PF eigenvector for $M(q_{n+1})$. Then,

$$\sum_{j=1}^{\ell+n+r+1} m'_{i,j} y_j = s y_j \quad \text{for } i \text{ with } 1 \le i \le \ell+n+r+1. \tag{3.2}$$

For $i = \ell+1$ and $i = \ell+n+r+1$ of (3.2) we have

$$\sum_{j=1}^{\ell+n+r+1} m'_{\ell+1,j} y_j = \sum_{j=1}^{\ell+1} m_{\ell+1,j} y_j + y_{\ell+n+r+1} = s y_{\ell+1} \quad \text{and}$$

$$y_{\ell+2} = s y_{\ell+n+r+1}.$$

These two equalities together with $s > 1$ yield

$$\sum_{j=1}^{\ell+1} m_{\ell+1,j} y_j + y_{\ell+2} > s y_{\ell+1}. \tag{3.3}$$

The equalities (3.2) for all $i \neq \ell+1$, $\ell+n+r+1$ together with the inequality (3.3) imply

$$M(q_n)\hat{\mathbf{y}} \ge s\hat{\mathbf{y}}, \quad \text{where } \hat{\mathbf{y}} = {}^t(y_1, \dots, y_{\ell+n+r}).$$

By Lemma 2.2(1), we have $\lambda(M(q_n)) \ge s = \lambda(M(q_{n+1}))$. By Lemma 2.2(2) together with (3.3), we have $\lambda(M(q_n)) > s$. $\square$

3.2. Asymptotic behavior of growth rate

In this section we concentrate on the combined tree obtained from the triple $(\mathscr{G}_L, \mathscr{G}_{n,+/-}, \mathscr{T}_0)$. We assume that $g_L(\mathscr{G}_L) = \mathscr{G}_L$ and study the combined tree map $q_n = g_n g_L$.

Let $\mathscr{R}$ be the subtree of $\mathscr{Q}_{n,+/-}$ such that $\mathscr{R}$ is obtained from the subtree $\mathscr{G}_L$ together with the $(\ell+1)^{\text{st}}$ edge. (Hence $E(\mathscr{R}) = \{e_1, e_2, \dots, e_{\ell+1}\}$.) For an example of $\mathscr{R}$, see Figure 8.

By the assumption $g_L(\mathscr{G}_L) = \mathscr{G}_L$, we have $q_n(\mathscr{R}) \supset \mathscr{R}$ and hence the following tree map $\bar{r} : \mathscr{R} \to \mathscr{R}$ is well defined: for each $e \in E(\mathscr{R})$, the edge path $\bar{r}(e)$ is given by the edge path $q_n(e)$ by eliminating edges which do not belong to $E(\mathscr{R})$. The tree map $\bar{r}$ does not depend on the choice of n. The transition matrix $M(\bar{r})$ is given by the upper-left $(\ell+1) \times (\ell+1)$ submatrix of $M(q_n)$. We call $\mathscr{R}$ the *dominant tree* and $\bar{r}$ the *dominant tree map* for a family of combined tree maps $\{q_n\}_{n \geq 1}$.

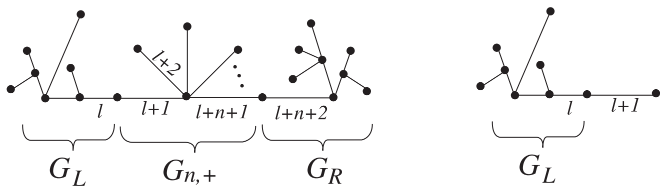

FIGURE 8. (left) $\mathscr{Q}_{n,+}$, (right) its subtree $\mathscr{R}$.

We now define a polynomial $S(t)$ (resp. $U(t)$) as follows: Consider the matrix $tI - M(q_n)$ (resp. $I - tM(q_n)$) and replace the $(\ell+1)^{\text{st}}$ row by the last row. Take the upper-left $(\ell+1) \times (\ell+1)$ submatrix of the resulting matrix, denoted by S (resp. U), and then $S(t)$ (resp. $U(t)$) is defined equal to the determinant of S (resp. U). It is not hard to see that the matrices S and U do not depend on n.

The following statement, which will be crucial later, tells us that $M(\bar{r})(t)$ is the dominant polynomial and $S(t)$ is the recessive polynomial for a family of polynomials $\{M(q_n)(t)\}_{n \geq 1}$.

PROPOSITION 3.5. *We have*
(1) $M(q_n)(t) = t^n M(\bar{r})(t) + S(t)$, *and*
(2) $M(q_n)_*(t) = t^n U(t) + M(\bar{r})_*(t)$.

Proof. The transition matrix $M(q_n) = (m_{i,j})$ is of the form

$$M(q_n) = \left(\begin{array}{ccc|ccc} & M(\bar{r}) & & & & \\ & & & 1 & & \\ \hline & & & & \ddots & \\ & & & & & 1 \\ m_{\ell+n+1,1} & \cdots & m_{\ell+n+1,\ell+1} & & & \end{array}\right),$$

and it is easy to see that $m_{\ell+1,j} = m_{\ell+n+1,j}$ for $1 \leq j \leq \ell$. For the proof of (1) (resp. (2)), apply the determinant expansion with respect to the last row of $tI - M(q_n)$ (resp. $I - tM(q_n)$). $\square$

Proposition 3.6. *Suppose that $M(g_L)$ is irreducible. Then, we have*
(1) *$M(q_n)$ is PF for each n and $\lambda(M(q_n)) > \lambda(M(q_{n+1}))$, and*
(2) *$\lambda(M(\bar{r})) = \lim_{n\to\infty} \lambda(M(q_n))$.*

Proof. (1) The proof is parallel to the proofs of Propositions 3.3 and 3.4.
(2) Apply Lemma 2.5 with Proposition 3.5(1). □

4. Proof

This section is devoted to proving Proposition 1.1 and Theorems 1.2, 1.3, 1.6.

Proposition 4.1. *The braids $\beta_{(m_1,\dots,m_{k+1})}$ are pA.*

By a result of W. Menasco's [6, Corollary 2], if L is a non-split prime alternating link which is not a torus link, then $S^3\backslash L$ has a complete hyperbolic structure of finite volume. Since $\overline{\beta_{(m_1,\dots,m_{k+1})}}$ is a 2 bridge link as depicted in Figure 13, his result tells us that $\beta_{(m_1,\dots,m_{k+1})}$ is pA. Here we will show Proposition 4.1 by using Proposition 2.4. As a result, we will find the polynomial whose largest root equals the dilatation of $\beta_{(m_1,\dots,m_{k+1})}$.

Proof of Proposition 4.1. To begin with, we define a tree $\mathcal{Q}_{(m_1,\dots,m_{k+1})}$ and a tree map $q_{(m_1,\dots,m_{k+1})}$ on the tree $\mathcal{Q}_{(m_1,\dots,m_{k+1})}$ inductively.

For $k=1$ let $\mathcal{Q}_{(m_1,m_2)}$ be the combined tree obtained from the triple $(\mathcal{G}_{m_1,+},\mathcal{G}_{m_2,-},\mathcal{T}_0)$. Take the tree maps g_{m_1} and g_{m_2} with conditions **n1**, **n2**, **n3** and let us define $q_{(m_1,m_2)}$ as the combined tree map

$$q_{(m_1,m_2)} = g_{m_2}g_{m_1} : \mathcal{Q}_{(m_1,m_2)} \to \mathcal{Q}_{(m_1,m_2)} \quad \text{(Figure 9).}$$

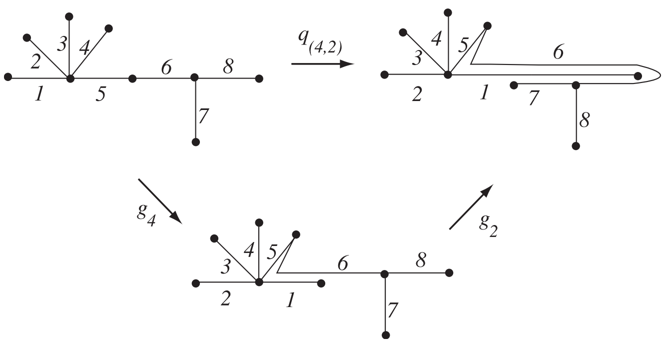

Figure 9. case $(m_1,m_2)=(4,2)$.

Next, suppose that these are defined up to k. Let $\mathcal{Q}_{(m_1,\dots,m_{k+1})}$ be the combined tree obtained from the triple $(\mathcal{Q}_{(m_1,\dots,m_k)},\mathcal{G}_{m_{k+1},+/-},\mathcal{T}_0)$ in case $k+1$ odd/even. We extend $q_{(m_1,\dots,m_k)} : \mathcal{Q}_{(m_1,\dots,m_k)} \to \mathcal{Q}_{(m_1,\dots,m_k)}$ to a tree map

$$\hat{q} : \mathscr{Q}_{(m_1,\dots,m_{k+1})} \to \mathscr{Q}_{(m_1,\dots,m_{k+1})}$$

satisfying **L1**, **L2**, **L3** so that for the edge e of $\mathscr{G}_{m_{k+1},+/-}$ sharing a vertex with the edge of $\mathscr{Q}_{(m_1,\dots,m_k)}$, the length of the edge path $\hat{q}(e)$ is 3. Let us define the combined tree map

$$q_{(m_1,\dots,m_{k+1})} = g_{m_{k+1}} \hat{q} : \mathscr{Q}_{(m_1,\dots,m_{k+1})} \to \mathscr{Q}_{(m_1,\dots,m_{k+1})}.$$

By Proposition 3.6(1), the transition matrix $M(q_{(m_1,\dots,m_{k+1})})$ is PF.

We now deform $\mathscr{Q} = \mathscr{Q}_{(m_1,\dots,m_{k+1})}$ into a train track $\tau_{(m_1,\dots,m_{k+1})}$ (as in Figure 10):

1. Puncture a disk near each valence 1 vertex of $\mathscr{Q}$, and connect a 1-gon at the vertex which contains the puncture.

2. Deform a neighborhood of a valence $m_i + 1$ vertex of the subtree $\mathscr{G}_{m_i,+/-}$ of $\mathscr{Q}$ into an $(m_i + 1)$-gon for each i.

3. For each $i + 1$ odd/even, puncture above/below the vertex of valence 2 which connects the two subtrees $\mathscr{G}_{m_i,+/-}$ and $\mathscr{G}_{m_{i+1},-/+}$. Deform a neighborhood of the vertex and connect a 1-gon which contains the puncture.

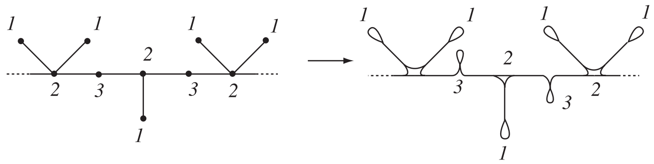

FIGURE 10. numbers 1, 2 and 3 correspond to deformations 1, 2 and 3.

Then, $q = q_{(m_1,\dots,m_{k+1})}$ induces the graph map $\hat{q}$ on $\tau = \tau_{(m_1,\dots,m_{k+1})}$ into itself. Since $\hat{q}$ rotates a part of the train track smoothly (Figure 11), it turns out that $\hat{q}$ is a smooth map. It is easy to show the existence of a representative homeomorphism f of $\phi = \Gamma(\beta_{(m_1,\dots,m_{k+1})})$ such that τ is invariant under f and such that $\hat{q}$ is the train track map induced by f. The transition matrix of $\hat{q}$ with respect to real edges of τ is exactly equal to the PF matrix $M(q_{(m_1,\dots,m_{k+1})})$. By Proposition 2.4, the braid $\beta_{(m_1,\dots,m_{k+1})}$ is pA. This completes the proof of Proposition 4.1. $\square$

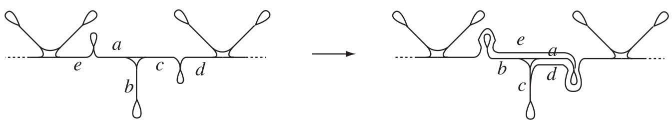

FIGURE 11. $\hat{q}$ rotates a part of the train track smoothly.

EXAMPLE 4.2. Let us express the formula to compute the dilatation of the braids $\beta_{(4,m)}$ for $m \ge 1$. We know that the dilatation of the braid $\beta_{(4,m)}$ is the largest root of the polynomial $M(q_{(4,m)})(t)$ for the combined tree map $q_{(4,m)}$. By using $q_{(4,2)}$ shown in Figure 9, the transition matrix for $q_{(4,2)}$ is

$$M(q_{(4,2)}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix}.$$

The transition matrix $M(\bar{r})$ of the dominant tree map $\bar{r}$ (Figure 12) for a family of combined tree maps $\{q_{(4,m)}\}_{m\geq 1}$ is the upper-left 6×6 matrix. Hence the dominant polynomial $M(\bar{r})(t)$ for $\{q_{(4,m)}\}_{m\geq 1}$ equals $t^6 - t^5 - 2t$ with the largest root ≈ 1.45109. In this case the recessive polynomial $S(t)$ is

$$S(t) = \begin{vmatrix} t & -1 & 0 & 0 & 0 & 0 \\ 0 & t & -1 & 0 & 0 & 0 \\ 0 & 0 & t & -1 & 0 & 0 \\ 0 & 0 & 0 & t & -1 & -1 \\ -1 & 0 & 0 & 0 & t & -1 \\ -1 & 0 & 0 & 0 & 0 & -2 \end{vmatrix} = -2t^5 - t + 1.$$

By Proposition 3.5(1), the dilatation of $\beta_{(4,m)}$ is the largest root of

$$M(q_{(4,m)})(t) = t^m M(\bar{r})(t) + S(t) = t^m(t^6 - t^5 - 2t) + (-2t^5 - t + 1),$$

and Proposition 3.6(2) says that $\lim_{m\to\infty} \lambda(\beta_{(4,m)}) \approx 1.45109$.

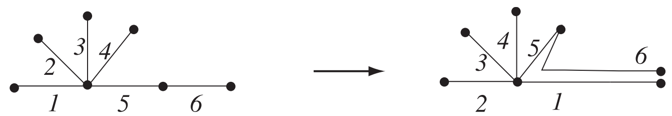

FIGURE 12. dominant tree map $\bar{r}: \mathscr{R} \to \mathscr{R}$ for $\{q_{(4,m)}\}_{m\geq 1}$.

We are now ready to show Proposition 1.1.

Proof of Proposition 1.1. Recall the tree $\mathscr{Q}_{(m_1,\ldots,m_{k+1})}$ and the tree map $q_{(m_1,\ldots,m_{k+1})}$ used in the proof of Proposition 4.1. For i even, $\mathscr{Q}_{(m_1,\ldots,m_{k+1})}$ is also the combined tree obtained from the triple $(\mathscr{Q}_{(m_1,\ldots,m_{i-1})}, \mathscr{G}_{m_i,-}, \mathscr{Q}_{(m_{i+1},\ldots,m_{k+1})})$. Then, $q_{(m_1,\ldots,m_{k+1})}$ is also the combined tree map given by

$$\hat{q}_{(m_1,\ldots,m_{i-1})} g_{m_i} \hat{q}_{(m_{i+1},\ldots,m_{k+1})} : \mathscr{Q}_{(m_1,\ldots,m_{k+1})} \to \mathscr{Q}_{(m_1,\ldots,m_{k+1})},$$

where $\hat{q}_{(m_1,\ldots,m_{i-1})}$ and $\hat{q}_{(m_{i+1},\ldots,m_{k+1})}$ are suitable extensions of $q_{(m_1,\ldots,m_{i-1})}$ and $q_{(m_{i+1},\ldots,m_{k+1})}$ respectively. By Proposition 3.4, the claim holds.

For i odd, $\mathscr{Q}_{(m_1,\ldots,m_{k+1})}$ is the combined tree obtained from $(\mathscr{Q}_{(m_1,\ldots,m_{i-1})}, \mathscr{G}_{m_i,+}, \mathscr{Q}'_{(m_{i+1},\ldots,m_{k+1})})$, where $\mathscr{Q}'_{(m_{i+1},\ldots,m_{k+1})}$ is the tree obtained from $\mathscr{Q}_{(m_{i+1},\ldots,m_{k+1})}$ by the horizontal reflection. Then, the proof is similar to that for the even case. □

We turn to the proof of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. By Proposition 2.4 and by the proof of Proposition 4.1, the dilatation of $\beta_{(m_1,\dots,m_{k+1})}$ is the largest root of $M(q_{(m_1,\dots,m_{k+1})})(t)$. Fixing $m_1,\dots,m_k \geq 1$, let $\mathscr{R}_{(m_1,\dots,m_k)}$ and $\bar{r}_{(m_1,\dots,m_k)}$ be the dominant tree and the dominant tree map for $\{q_{(m_1,\dots,m_{k+1})}\}_{m_{k+1}\geq 1}$, and we set

$$R_{(m_1,\dots,m_k)}(t) = M(\bar{r}_{(m_1,\dots,m_k)})(t).$$

By Proposition 3.5,

$$M(q_{(m_1,\dots,m_{k+1})})(t) = t^{m_{k+1}} R_{(m_1,\dots,m_k)}(t) + S_{(m_1,\dots,m_k)}(t) \quad \text{and} \tag{4.1}$$

$$M(q_{(m_1,\dots,m_{k+1})})_*(t) = t^{m_k+1} U_{(m_1,\dots,m_k)}(t) + R_{(m_1,\dots,m_k)_*}(t), \tag{4.2}$$

where $S_{(m_1,\dots,m_k)}(t)$ is the recessive polynomial for $\{M(q_{(m_1,\dots,m_{k+1})})(t)\}_{m_{k+1}\geq 1}$ and $U_{(m_1,\dots,m_k)}(t)$ is the dominant polynomial for $\{M(q_{(m_1,\dots,m_{k+1})})_*(t)\}_{m_{k+1}\geq 1}$.

Claim 4.3. *We have*
(1) $S_{(m_1,\dots,m_k)}(t) = (-1)^{k+1} R_{(m_1,\dots,m_k)_*}(t)$ *and*
(2) $U_{(m_1,\dots,m_k)}(t) = (-1)^{k+1} R_{(m_1,\dots,m_k)}(t)$.

Proof. It is enough to show Claim 4.3(1). For if (1) holds, by (4.1) we have

$$M(q_{(m_1,\dots,m_{k+1})})(t) = (-1)^{k+1} M(q_{(m_1,\dots,m_{k+1})})_*(t).$$

This together with (4.1), (4.2) implies Claim 4.3(2).

We prove Claim 4.3(1) by an induction on k. For $k=1$, this holds [4, Theorem 3.20(1)]. We assume Claim 4.3(1) up to $k-1$. Then, we have $S_{(m_1,\dots,m_{k-1})}(t) = (-1)^k R_{(m_1,\dots,m_{k-1})_*}(t)$,

$$M(q_{(m_1,\dots,m_k)})(t) = (-1)^k M(q_{(m_1,\dots,m_k)})_*(t) \quad \text{and} \tag{4.3}$$

$$U_{(m_1,\dots,m_{k-1})}(t) = (-1)^k R_{(m_1,\dots,m_{k-1})}(t).$$

For $k \geq 2$, the transition matrix $M(q_{(m_1,\dots,m_{k+1})})$ has the block form:

$$
\begin{array}{l}
M(q_{(m_1,\dots,m_{k+1})}) \\
= \begin{array}{cc}
 & \begin{array}{ccccccccc} 1 & n_1+1 & n_2+1 & \cdots & n_{k-1}+1 & n_k+1 & & & n_{k+1} \end{array} \\
\begin{array}{c} \\ \\ \\ n_k+1 \\ \\ \\ n_{k+1} \end{array} &
\begin{pmatrix}
 & & M(q_{(m_1,\dots,m_k)}) & & & & & & \\
 & & & & & 1 & & & \\
 & & & & & 1 & & & \\
1 & 2 & 2 & \cdots & 2 & 1 & 1 & & \\
 & & & & & & & \ddots & \\
 & & & & & & & & 1 \\
1 & 2 & 2 & \cdots & 2 & 2 & & &
\end{pmatrix}
\end{array},
\end{array}
$$

where $n_j = m_1 + \cdots + m_j + j$ for $1 \le j \le k+1$. Note that the last edge of the tree $\mathscr{R}_{(m_1,\dots,m_j)}$ is numbered n_j. In this case the polynomial $S_{(m_1,\dots,m_k)}(t)$ is the determinant of a matrix:

$$\left(\begin{array}{cccccc|cccc}
 & tI - M(\bar{r}_{(m_1,\dots,m_{k-1})}) & & & & & & & & \\
 & & & & & & -1 & & & \\
\hline
 & & & & & & t & \ddots & & \\
 & & & & & & & \ddots & -1 & -1 \\
-1 & -2 & & -2 & \cdots & -2 & & & t & -1 \\
-1 & -2 & & -2 & \cdots & -2 & & & & -2
\end{array}\right).$$

Subtract the second last row from the last row of this matrix, and let $A = (a_{i,j})$ be the resulting matrix. Applying the determinant expansion with respect to the last row of A, we have

$$S_{(m_1,\dots,m_k)}(t) = |A| = \sum_{j=1}^{n_k+1} (-1)^{n_k+1+j} a_{n_k+1,j} |A_{n_{k+1},j}|$$

where $A_{i,j}$ is the matrix obtained by A with row i and column j removed. Since $a_{n_k+1,j} = 0$ for $j \ne n_k, n_k+1$ and $a_{n_k+1,n_k} = -t$, $a_{n_k+1,n_k+1} = -1$, we have

$$|A| = t|A_{n_k+1,n_k}| - |A_{n_k+1,n_k+1}|.$$

We note that $|A_{n_k+1,n_k+1}| = M(q_{(m_1,\dots,m_k)})(t)$. For the computation of $|A_{n_k+1,n_k}|$, subtract the second last row of $A_{n_{k+1},n_{k+1}-1}$ from the last row, and for the resulting matrix, apply the determinant expansion of the last column successively. Then, we obtain

$$|A_{n_k+1,n_k}| = -t^{m_k-1} R_{(m_1,\dots,m_{k-1})}(t) + S_{(m_1,\dots,m_{k-1})}(t).$$

Thus,

$$S_{(m_1,\dots,m_k)}(t) = -M(q_{(m_1,\dots,m_k)})(t) - t^{m_k} R_{(m_1,\dots,m_{k-1})}(t) + tS_{(m_1,\dots,m_{k-1})}(t).$$

In the same manner we have

$$R_{(m_1,\dots,m_k)_*}(t) = M(q_{(m_1,\dots,m_k)})_*(t) + t^{m_k} U_{(m_1,\dots,m_{k-1})}(t) - tR_{(m_1,\dots,m_{k-1})_*}(t).$$

By using (4.3), these two equalities imply Claim 4.3(1). This completes the proof.

We now turn to proving Theorem 1.2. We will prove an inductive formula for $R_{(m_1,\dots,m_k)}(t)$. It is not hard to show that $R_{(m_1)}(t) = t^{m_1+1}(t-1) - 2t$.

For $k \ge 2$, one can verify

$$\text{(4.4)} \quad R_{(m_1,\dots,m_k)}(t) = tM(q_{(m_1,\dots,m_k)})(t) - t^{m_k} R_{(m_1,\dots,m_{k-1})}(t) + tS_{(m_1,\dots,m_{k-1})}(t).$$

Substitute the two equalities

$$M(q_{(m_1,\dots,m_k)})(t) = t^{m_k} R_{(m_1,\dots,m_{k-1})}(t) + (-1)^k R_{(m_1,\dots,m_{k-1})_*}(t) \quad \text{and}$$
$$S_{(m_1,\dots,m_{k-1})}(t) = (-1)^k R_{(m_1,\dots,m_{k-1})_*}(t)$$

into (4.4), then we find the inductive formula

$$R_{(m_1,\dots,m_k)}(t) = t^{m_k}(t-1)R_{(m_1,\dots,m_{k-1})}(t) + (-1)^k 2tR_{(m_1,\dots,m_{k-1})_*}(t).$$

This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. To begin with, we show

Claim 4.4. *We have*
(1) $\lim_{m_i\to\infty}\lim_{m_{i+1}\to\infty}\cdots\lim_{m_{k+1}\to\infty}\lambda(\beta_{(m_1,\dots,m_{k+1})}) = 1$ *for* $i = 1$ *and*
(2) $\lim_{m_i\to\infty}\lim_{m_{i+1}\to\infty}\cdots\lim_{m_{k+1}\to\infty}\lambda(\beta_{(m_1,\dots,m_{k+1})}) = \lambda(R_{(m_1,\dots,m_{i-1})}(t)) > 1$ *for* $i \geq 2$.

Proof. (1) By Theorem 1.2 and by Lemma 2.5

$$\lim_{m_{k+1}\to\infty}\lambda(\beta_{(m_1,\dots,m_{k+1})}) = \lambda(R_{(m_1,\dots,m_k)}(t)).$$

Recall that $R_{(m_1,\dots,m_i)}(t) = M(\bar{r}_{(m_1,\dots,m_i)})(t)$. It is not hard to see that the matrix $M(\bar{r}_{(m_1,\dots,m_i)})$ for $i \geq 1$ is PF, and hence the largest root of $R_{(m_1,\dots,m_i)}(t)$ is greater than 1. Then, by using the inductive formula of $R_{(m_1,\dots,m_i)}(t)$ in Theorem 1.2 together with Lemma 2.5, we have

$$\lim_{m_i\to\infty}\lambda(R_{(m_1,\dots,m_i)}(t)) = \lambda(R_{(m_1,\dots,m_{i-1})}(t))$$

for $i \geq 2$. Since $R_{(m_1)}(t) = t^{m_1+1}(t-1) - 2t$, we have $\lim_{m_1\to\infty}\lambda(R_{(m_1)}(t)) = 1$ by Lemma 2.6. This completes the proof of (1).

(2) The proof is identical to that of (1). This completes the proof of Claim 4.4.

Claim 4.4(1) says that for any $\lambda > 1$ there exists an integer $m_i(\lambda)$ for each i with $1 \leq i \leq k+1$ such that $\lambda(\beta_{m_1(\lambda),\dots,m_{k+1}(\lambda)}) < \lambda$. Set $m = \max\{m_i(\lambda) \mid i = 1,\dots,k+1\}$. By Proposition 1.1 $\lambda(\beta_{(m_1,\dots,m_{k+1})}) < \lambda$ whenever $m_i > m$. This completes the proof of Theorem 1.3(1).

The proof of Theorem 1.3(2) is identical to that of (1), but using Claim 4.4(2) instead of Claim 4.4(1). □

We show the existence of two kinds of families of pA mapping classes with arbitrarily small dilatation and with arbitrarily large volume.

Proof of Proposition 1.4. There exists a family of pseudo-Anosov mapping classes ψ_n of $\mathcal{M}(D_n)$ such that

$$\lim_{n\to\infty}\lambda(\psi_n) = 1.$$

It suffices to show that for any pA mapping class $\phi \in \mathcal{M}(\Sigma_{g,p})$, there exists a family of pA mapping classes $\hat{\phi}_n \in \mathcal{M}(\Sigma_{g,p(n)})$ such that the dilatation of $\hat{\phi}_n$ is same as ϕ and the volume of $\hat{\phi}_n$ goes to ∞ as n goes to ∞.

Let $\Phi \in \phi$ be a pA homeomorphism. Since the set of periodic orbits of Φ is dense on $\Sigma_{g,p}$, one can find a periodic orbit of Φ, say $Q = \{q_1, \dots, q_s\}$. Now puncture each point of Q, then the pA mapping class $\phi' \in \mathcal{M}(\Sigma_{g,p+s})$ induced by ϕ satisfies $\lambda(\phi') = \lambda(\phi)$. On the other hand, $\mathrm{vol}(\phi') > \mathrm{vol}(\phi)$ since $\mathbf{T}(\phi)$ is a complete hyperbolic manifold obtained topologically by filling a cusp of $\mathbf{T}(\phi')$ with a solid torus [9, Section 6]. The volume of any cusp is bounded below uniformly. Thus, if we puncture periodic orbits of $\Phi \in \phi$ successively, we obtain a family of pA mapping class with the desired property. □

Finally, we show Theorem 1.6.

Proof of Theorem 1.6. By Theorem 1.3(1), for each integer $k \geq 1$, the dilatation of $\beta_{(m_1,\dots,m_{k+1})}$ goes to 1 as $m_1, \dots, m_{k+1}$ all go to ∞. Thus, it suffices to show that the volume of $\beta_{(m_1,\dots,m_{k+1})}$ goes to ∞ as k goes to ∞.

One verifies that $\overline{\beta_{(m_1,\dots,m_{k+1})}}$ is a 2 bridge link as in Figure 13. In particular it is an alternating link with twist number $k+1$. Theorem 1 in [5] tells us that for each $m_1, \dots, m_{k+1} \geq 1$,

$$\mathrm{vol}(\beta_{(m_1,\dots,m_{k+1})}) > \frac{1}{2}(k-1)v_3,$$

where v_3 is the volume of a regular ideal tetrahedron. This completes the proof. □

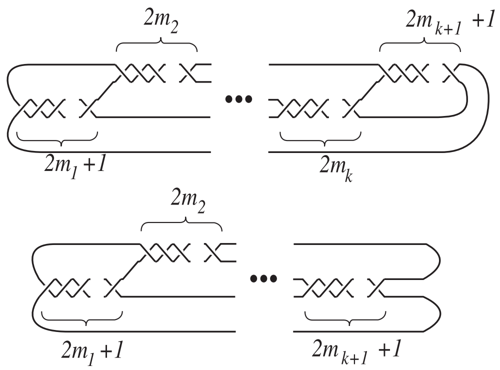

FIGURE 13. $\overline{\beta_{(m_1,\dots,m_{k+1})}}$: (top) k odd, (bottom) even.

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