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GAPS IN THE EXPONENT SPECTRUM OF SUBGROUPS OF DISCRETE QUASICONFORMAL GROUPS

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Abstract

Let G be a discrete quasiconformal group preserving $\mathbf{B}^3$ whose limit set $\Lambda(G)$ is purely conical and all of $\partial\mathbf{B}^3$. Let $\hat{G}$ be a non-elementary normal subgroup of G: we show that there exists a set $\mathscr{A}$ of full measure in $\Lambda(G)$ so that $\mathscr{A}$, regarded as a subset of $\Lambda(\hat{G})$, has "fat horospherical" dynamics relative to $\hat{G}$. As an application we will bound from below the exponent of convergence of $\hat{G}$ in terms of the Hausdorff dimension of $\mathscr{A}$.

1. Introduction

Recall that the definition of a convergence group allows one to broadly replicate the dynamics of a discrete isometric group action on a pinched Hadamard manifold. For example, discrete convergence groups acting on perfect metrizable compacta have the following properties. The action of the group divides the underlying space into two dynamically distinct sets: the set of discontinuity and its complement, the limit set. The classification of elements of the group into the categories of loxodromic, elliptic, or parabolic remains valid, and in the non-elementary case loxodromic fixed points are dense in the limit set. See Gehring and Martin [11].

We focus on a class of convergence groups called *discrete quasiconformal groups*. A group G is a discrete quasiconformal group if it consists of quasiconformal homeomorphisms, and there exists a uniform bound on the dilatation of all elements of the group. From this point of view Kleinian groups acting on $\mathbf{S}^n$ are the discrete 1-quasiconformal groups on $\mathbf{S}^n$. While a discrete quasiconformal group does not typically act isometrically, the theory of quasiconformal mappings contains enough analytic and geometric structure to produce an interesting dynamical action on $\mathbf{S}^n$.

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In this note our goals are two-fold: first we explore the dynamical degeneration of the conical limit set of a non-elementary discrete quasiconformal group as one passes to a non-elementary normal subgroup, and then we show that the geometric picture one receives of this degeneration in the isometric setting is in fact an asymptotic limit of the geometric picture in the quasiconformal setting.

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2. Statement of main results

Let Γ be a non-elementary Kleinian group, and let $\Lambda(\Gamma)$ be its limit set. It is easy to show that if $\hat{\Gamma}$ is a non-elementary normal subgroup of Γ then $\Lambda(\hat{\Gamma}) = \Lambda(\Gamma)$. Though the limit sets of Γ and $\hat{\Gamma}$ are the same, the orbital dynamics, relative to the orbits $\Gamma(0)$ and $\hat{\Gamma}(0)$ $(0 \in \mathbf{B}^n)$, may be very different. Here we are most interested in the conical limit set. The foundational result of Bishop and Jones ([4]) says that *the Hausdorff dimension of the conical limit set of a non-elementary Kleinian group* Γ *is the exponent of convergence of the Poincaré series of* Γ. Though the limit sets are the same for $\hat{\Gamma}$ and Γ, the size (as measured by Hausdorff dimension) of the conical limit set of $\hat{\Gamma}$ may be strictly less than the size of the conical limit set of Γ (for examples of this see e.g. [9], [15], and [10]). The key point in explaining the phenomenon, from a Kleinian group point of view, is the following fact established by K. Matsuzaki [14]: *If* $\hat{\Gamma}$ *is a non-elementary normal subgroup of* Γ *then the conical limit set of* Γ *is contained in the horospherical limit set of* $\hat{\Gamma}$.

We will demonstrate a generalization of this phenomenon in the setting of discrete quasiconformal groups, and thus realize Matsuzaki's result as a limit as $K \to 1$ in this generalization.

Definition 2.1. Let G be a discrete quasiconformal group acting on $\mathbf{B}^n$. A point $\zeta \in \mathbf{S}^{n-1}$ is a *conical limit point* of G if there exists a sequence $\{g_j\} \subset G$ such that $\{g_j(0)\}$ converges to ζ within a Euclidean non-tangential cone based at ζ. The *conical limit set* $\Lambda_c(G)$ is the set of all conical limit points.

A conical limit point ζ is a *strong conical limit point* of G if there exists a conical approach $\{g_j(0)\}$ that has the additional property that $\{g_j^{-1}(0)\}$ converges to a point $b \neq \zeta$. The *strong conical limit set* $\Lambda_c^s(G)$ is the set of all strong conical limit points.

A non-elementary quasiconformal group G acting on $\mathbf{B}^n$ always has $\Lambda_c^s(G) \neq \emptyset$ because the loxodromic fixed point set of such a G is contained in $\Lambda_c^s(G)$.

Remark 2.2. We will demonstrate (Lemma 2.10) that the strong conical limit set is of full measure in the limit set of a co-finite Kleinian group.

We remark that the condition of the sequence $\{g_j(0)\}$ converging to ζ within a Euclidean non-tangential cone can equivalently be described via the existence of a constant $C > 0$ such that

$$\frac{1 - |g_j(0)|}{|g_j(0) - \zeta|} \geq C \quad \text{for all } j.$$

A third, and equivalent, condition is the existence of a constant $\tilde{C}$ such that $\{g_j(0)\}$ converges to ζ within hyperbolic distance $\tilde{C}$ from the hyperbolic geodesic ray $[0, \zeta)$.

Definition 2.3. Let G be a discrete quasiconformal group acting on $\mathbf{B}^n$. A point $\zeta \in \mathbf{S}^{n-1}$ is a *K-fat-horospherical limit point* of G if there exists a sequence $\{g_j\} \subset G$ and a constant $C > 0$ such that

$$\frac{1 - |g_j(0)|}{|g_j(0) - \zeta|^{K+1}} \geq C \quad \text{for all } j.$$

The *K-fat-horospherical limit set* $\Lambda_K(G)$ is the set of all K-fat horospherical limit points.

Remark 2.4.
- The 1-fat horospherical limit points are just the points $\zeta \in \mathbf{S}^{n-1}$ for which there exists an orbit $\{g_j(0)\}$ that approaches ζ within a horoball based at ζ, see for example [17].
- If Γ is a geometrically finite Kleinian group acting on $\mathbf{B}^n$ then every point in the limit set is 1-fat horospherical ([3].)

Now let G be a discrete K-quasiconformal group, and let $\hat{G} \lhd G$ be a non-elementary normal subgroup of G. While $\Lambda(G)$ equals $\Lambda(\hat{G})$ under these assumptions, it may happen that $\Lambda_c(\hat{G})$ is properly contained in $\Lambda_c(G)$. An example where the conical limit set of the subgroup is smaller (even in the sense of dimension) than the conical limit set of the big group can be easily constructed, using a result of R. Brooks: see Example 3.3 at the end of Section 3.

Though the property of being conical may not be preserved in passing to a normal subgroup, the degeneration happens in a geometrically controlled fashion.

Theorem 2.5. *Let G be a discrete K-quasiconformal group acting on $\mathbf{B}^n$ with empty regular set, and let $\hat{G} \lhd G$ be a non-elementary normal subgroup of G. Then*

$$\Lambda_c^s(G) \subset \Lambda_K(\hat{G}).$$

Remark 2.6. This theorem partially generalizes Theorem 6 of K. Matsuzaki [14] to the discrete quasiconformal group setting.

Let G be a discrete quasiconformal group acting on $\mathbf{B}^n$. We denote by $\delta(G)$ the *exponent of convergence* of G, that is

$$\delta(G) = \inf\left\{ s > 0 \,\middle|\, \sum_{g \in G} e^{-sd(x, g(y))} < \infty \right\},$$

where d is the hyperbolic metric in $\mathbf{B}^n$ and $x, y \in \mathbf{B}^n$ are arbitrary points. See [17] for the basics concerning the exponent of convergence of Kleinian groups. Note that the exponent of convergence does not depend on the choice of points x, y. It is easy to realize $\delta(G)$ into a more Euclidean form, that is,

$$\delta(G) = \inf\left\{ s > 0 \,\middle|\, \sum_{g \in G} (1 - |g(0)|)^s < \infty \right\}.$$

For discrete quasiconformal groups preserving $\mathbf{B}^n$ the relationship between the exponent of convergence and the Hausdorff dimension of the conical limit set is:

Theorem 2.7 ([5], Theorem 2.7). *Let G be a discrete quasiconformal group preserving* $\mathbf{B}^n$. *Then* $\dim(\Lambda_c(G)) \leq \delta(G)$.

We note that the inequality cannot be promoted to equality. In [5] we provide an example of a discrete quasiconformal group G that is quasiconformally conjugate to a finitely generated Fuchsian group, having the property that $\dim \Lambda_c(G)$ is strictly smaller than $\delta(G)$. For the further development of Patterson-Sullivan theory in the setting of discrete quasiconformal groups see e.g. [5], [6], [2], and [7].

A standard argument will yield the following result.

Theorem 2.8. *Let G be a discrete quasiconformal group acting on* $\mathbf{B}^n$. *Then*

$$\dim \Lambda_K(G) \leq (K+1)\delta(G).$$

Combining Theorems 2.5 and 2.8 we obtain finer information about the conical limit set:

Theorem 2.9. *Let G be a discrete K-quasiconformal group acting on* $\mathbf{B}^n$ *with empty regular set, and let* $\hat{G} \lhd G$ *be a non-elementary normal subgroup of G. Then*

$$\dim \Lambda_c^s(G) \leq (K+1)\delta(\hat{G}).$$

If G is a discrete quasiconformal group acting on $\mathbf{B}^3$ with empty regular set and purely conical limit set then one can show:

Lemma 2.10. *Let G be a discrete quasiconformal group acting on* $\mathbf{B}^3$ *with empty regular set and purely conical limit set. Then the strong conical limit set of G has full 2-dimensional Lebesgue measure in* $\mathbf{S}^2$ *and thus* $\dim \Lambda_c^s(G) = \dim \Lambda_c(G) = \delta(G) = 2$.

Proof. First let us assume that the strong conical limit set of G is of full Lebesgue measure in $\mathbf{S}^2$: then $2 = \dim(\Lambda_c^s)(G)$. Note that since $\Lambda_c^s(G) \subseteq \Lambda_c(G)$ we have from Theorem 2.7 that $\delta(G) \geq 2$. A result of Gehring and Martin [12] asserts that in this setting $\delta(G) \leq 2$, and so under the assumption we conclude $\dim(\Lambda_c^s(G)) = \delta(G) = 2$.

To show that the strong conical limit set of G *has* full measure we will need some preliminary observations. Let $\tilde{G}$ be the discrete quasiconformal group, induced by G, acting on $\partial\mathbf{B}^3 = \mathbf{S}^2$. From a result of Sullivan [18], and separately Tukia [19], we can find a Kleinian group Γ, and a quasiconformal mapping $\phi : \mathbf{S}^2 \to \mathbf{S}^2$, so that $\tilde{G} = \phi \circ \Gamma \circ \phi^{-1}$. Both the conjugating map ϕ and the Kleinian group Γ extend to $\mathbf{B}^3$, and so one easily sees that on $\mathbf{B}^3$ the actions of the quasiconformal groups $\phi \circ \Gamma \circ \phi^{-1}$ and G differ at most by a uniformly bounded hyperbolic distance. Thus in particular the strong conical limit set of $\phi \circ \Gamma \circ \phi^{-1}$ and that of G must agree. If we can show that Γ has a strong conical limit set that is of full measure in $\mathbf{S}^2$, then because a quasiconformal conjugacy maps strong conical limit sets to strong conical limit sets, and because a quasiconformal mapping of $\mathbf{S}^2$ of a set of full measure has image with full measure (Theorem 33.2 in [22]), then G has a strong conical limit set of full measure.

We will thus need to observe that, under the assumptions on G, any Kleinian group that is quasiconformally conjugate to $\tilde{G}$ has the property that its strong conical limit set is of full measure in $\mathbf{S}^2$. But this is a well known observation: first any Kleinian group Γ quasiconformally conjugate to such a G has a limit set that is purely conical and is the whole sphere at infinity. Such a Γ must uniformize, up to finite index, a closed hyperbolic 3-manifold (since any Dirichlet polyhedron for the action of Γ on $\mathbf{B}^3$ has compact closure, see Chapter 6 of [13]) and thus Γ is geometrically finite and therefore of divergence type. Next observe that the strong conical limit set contains the Myberg limit set ([16]), which is known to be of full measure for Kleinian groups of divergence type (Agard [1], Tukia [21]). □

Using Lemma 2.10 and Theorem 2.9 we show in dimension 3 under certain assumptions that a drop in the exponent of convergence for a non-elementary normal subgroup both can't be too large and is bounded in terms of K (compare Matsuzaki [15]; see also Falk and Stratmann [10]):

THEOREM 2.11. *Let G be a discrete K-quasiconformal group acting on $\mathbf{B}^3$, with empty regular set and purely conical limit set. Let $\hat{G} \lhd G$ be a non-elementary normal subgroup of G. Then*

$$\delta(\hat{G}) \geq \frac{\delta(G)}{K+1} = \frac{2}{K+1}.$$

Remark 2.12.

- Lemma 2.10 can be proved using the geometry of hyperbolic quasigeodesics as in [8].

- A more general statement can be proved: *Let Γ be a convex co-compact Kleinian group acting on $\mathbf{H}^n$, possibly with non-empty regular set. Then the strong conical limit set is of full measure in the measure class of the Patterson-Sullivan measure of dimension $\delta(\Gamma)$ based at* 0. We cannot use this formulation in part because we will not be able to conclude that the strong conical limit set of G (as a quasiconformal conjugate of Γ) is of full measure in $\Lambda(G)$. In particular, it is not the case that a quasiconformal conjugation will necessarily preserve sets of full measure in sets of Hausdorff dimension less than $n-1$, even in the Kleinian setting. E.g. see [20].
- If Theorem 2.5 is established for purely conical quasiconformal groups having non-empty regular set, then we can obtain a more general version of Theorem 2.11. In particular, using techniques from our work in [7] we can show that a discrete K-quasiconformal group G acting on $\mathbf{B}^3$ with non-empty regular set has the property that

$$\delta(G) \leq \frac{2K \dim \Lambda_c^s(G)}{2 + (K-1) \dim \Lambda_c^s(G)}. \tag{2.1}$$

Together with Theorem 2.9 this would imply:

CONJECTURE 2.13. *Let G be a discrete K-quasiconformal group acting on $\mathbf{B}^3$ with purely conical limit set and non-empty regular set. Let $\hat{G} \lhd G$ be a non-elementary normal subgroup of G. Then*

$$\delta(\hat{G}) \geq \frac{2\delta(G)}{2K^2 + 2K - \delta(G)(K^2-1)}.$$

We note that as K approaches 1 in this conjecture we recover the analogous theorem for the Kleinian case which was proved by Matsuzaki in [15].

3. Proofs

We begin by recalling that quasiconformal maps of $\mathbf{B}^n$ are $(K, K\log 4)$-quasiisometries (in the hyperbolic metric), see, for example Thm. 11.2 in [23].

LEMMA 3.1. *Let $f : \mathbf{B}^n \to \mathbf{B}^n$ be a K-quasiconformal homeomorphism. Then*

$$\frac{1}{K}\, d(x,y) - \log 4 \leq d(f(x), f(y)) \leq K\, d(x,y) + K \log 4$$

for all $x, y \in \mathbf{B}^n$.

It is crucial to our arguments that the bi-Lipschitz distortion constant is K here. Using this fact we will show that the degeneration from the conformal case is controlled by the quasiconformal dilatation of the group.

In order to prepare for the proof of Theorem 2.5, recall the definition of the K-fat horospherical limit set (Definition 2.3); the proposition below is a useful characterization of the set of fat-horospherical limit points in terms of the geometry of covering orbits.

PROPOSITION 3.2. *Let G be a discrete quasiconformal group acting on $\mathbf{B}^n$, and let $\zeta \in \mathbf{S}^{n-1}$ and $K \geq 1$. Then ζ belongs to $\Lambda_K(G)$ if and only if there exists a constant C and elements $g_j \in G$ such that*

$$d(q_j, g_j(0)) \leq \frac{K}{K+1} d(0, g_j(0)) + C \tag{3.1}$$

for all j. Here, q_j denotes the (hyperbolic) projection of $g_j(0)$ onto the geodesic ray from 0 to ζ.

Proof of Proposition 3.2. We first note that (3.1) implies that $\{g_j(0)\}$ converges to ζ as otherwise the distance between $g_j(0)$ and the geodesic ray from 0 to ζ would increase exponentially in $d(0, g_j(0))$. Hence the angle θ_j at 0 formed by the geodesic ray $[0, \zeta)$ and the geodesic segment $[0, g_j(0)]$ tends to zero as $j \to \infty$, see Figure 1. Furthermore we note that if $\{g_j(0)\}$ tends to ζ in a Euclidean non-tangential cone based at ζ then $\zeta \in \Lambda_K(G)$ for any $K \geq 1$ (in fact, $\zeta \in \Lambda_K(G)$ for $K = 0$, but here we only consider values of K that are ≥ 1), and also (3.1) is true for any $K \geq 1$. Thus we may assume from now on that $d(g_j(0), q_j) \to \infty$ as $j \to \infty$.

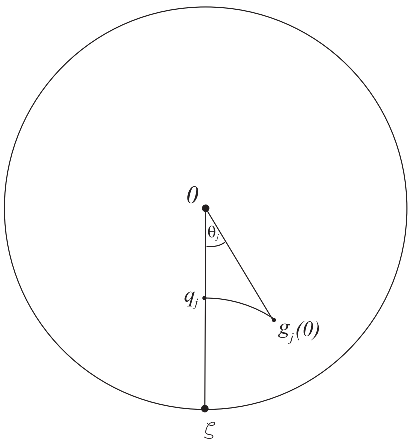

FIGURE 1

An application of the hyperbolic sine law in the right triangle 0, q_j, $g_j(0)$ implies that

$$\sin \theta_j = \frac{\sinh d(g_j(0), q_j)}{\sinh d(0, g_j(0))}.$$

Euclidean trigonometry in the triangle 0, ζ, $g_j(0)$ implies that

$$\sin\theta_j = |g_j(0) - \zeta| \cdot c_j,$$

where $c_j \to 1$ as $j \to \infty$ (here we used that $\{g_j(0)\}$ does not converge to ζ within a non-tangential Euclidean cone based at ζ). Combining the previous two equations we obtain

$$|g_j(0) - \zeta| = \frac{e^{d(g_j(0), q_j)}}{e^{d(0, g_j(0))}} \cdot d_j,$$

where $d_j \to 1$ as $j \to \infty$. Hence

$$\frac{1 - |g_j(0)|}{|g_j(0) - \zeta|^{K+1}} = e^{-d(0, g_j(0))} \left(\frac{e^{d(0, g_j(0))}}{e^{d(g_j(0), q_j)}}\right)^{K+1} \cdot \tilde{d}_j$$
$$= \left(e^{(K/(K+1))\, d(0, g_j(0)) - d(g_j(0), q_j)}\right)^{K+1} \cdot \tilde{d}_j,$$

where $\tilde{d}_j \to 1$ as $j \to \infty$. Thus $\dfrac{1 - |g_j(0)|}{|g_j(0) - \zeta|^{K+1}}$ is bounded away from zero if and only if $\dfrac{K}{K+1}\, d(0, g_j(0)) - d(g_j(0), q_j)$ is bounded away from minus infinity, and this is equivalent to (3.1). □

We are now ready for the proof of the fact that the strong conical limit set of a discrete K-quasiconformal group G acting on $\mathbf{B}^n$ with empty regular set is contained in the K-fat-horospherical limit set of any non-elementary normal subgroup $\hat{G}$.

Proof of Theorem 2.5. Let G be a discrete K-quasiconformal group acting on $\mathbf{B}^n$, and let $\hat{G}$ be a non-elementary normal subgroup of G. Let $\zeta \in \Lambda_c^s(G)$. Hence there are elements $g_j \in G$ such that $g_j(0) \to \zeta$ in a Euclidean non-tangential cone based at ζ, and so that $\{g_j^{-1}(0)\}$ converges to a point $s \neq \zeta$. By passing to a subsequence if necessary we can assume that the sequence $\{g_j\}$ converges to ζ locally uniformly in $\overline{\mathbf{B}^n} \backslash \{s\}$, and $\{g_j^{-1}\}$ converges to s locally uniformly in $\overline{\mathbf{B}^n} \backslash \{\zeta\}$. By conjugation with a Möbius transformation ϕ (which maps $\Lambda_c^s(G)$ to $\Lambda_c^s(\phi \circ G \circ \phi^{-1})$ and which maps $\Lambda_K(\hat{G})$ to $\Lambda_K(\phi \circ \hat{G} \circ \phi^{-1})$ by Proposition 3.2) we may assume that $\zeta = -e_n$ and $s = e_n = (0, \dots, 0, 1)$.

In what follows, for $a, b \in \mathbf{S}^{n-1}$, the symbol $C_{a,b}$ denotes the hyperbolic geodesic whose endpoints are a and b. Since every element of G is K-quasiconformal, there exists a constant C (only depending on K) such that any hyperbolic geodesic $C_{a,b}$ and any element $g \in G$ satisfy that $g(C_{a,b})$ is contained in a hyperbolic C-neighborhood of $C_{g(a), g(b)}$.

Since $\hat{G}$ is a non-elementary normal subgroup of G, and G has empty regular set, we have that $\Lambda(\hat{G}) = \Lambda(G) = \mathbf{S}^{n-1}$, and hence there exists a loxodromic element $\hat{g} \in \hat{G}$ whose fixed points a, b are close to $e_1 = (1, 0, \dots, 0)$ and $-e_1$, respectively. In particular, the hyperbolic geodesics $C_{a,b}$ can be chosen to have an arbitrarily small hyperbolic distance to the origin. Thus there exists a

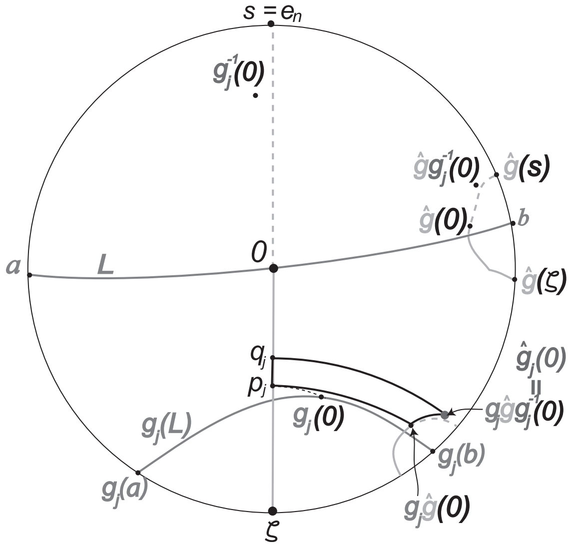

FIGURE 2

quasiline L equidistant (in the hyperbolic metric) from $C_{a,b}$ that intersects $C_{\zeta,s}$ at the origin and at an angle arbitrarily close to $\pi/2$. As before, any element $g \in G$ satisfies that $g(L)$ lies within bounded hyperbolic distance of $C_{g(a),g(b)}$, where the bound is independent of g. Furthermore, we can choose $\hat{g}$ such that $d(0, \hat{g}(0))$ is as large as desired (we will see later in the proof how large we need this quantity to be). Fix such an element $\hat{g} \in \hat{G}$, and define $\hat{g}_j = g_j \circ \hat{g} \circ g_j^{-1}$. Since $\hat{G}$ is normal in G this implies that $\hat{g}_j \in \hat{G}$. We will show that $\{\hat{g}_j(0)\}$ converges to ζ within a K-fat-horosphere. Note that clearly $\{\hat{g}_j(0)\}$ converges to ζ since the sequence $\{g_j\}$ converges to ζ locally uniformly in $\overline{\mathbf{B}^n}\backslash\{e_n\}$ and $\{g_j^{-1}\}$ converges to e_n locally uniformly in $\overline{\mathbf{B}^n}\backslash\{\zeta\}$. Thus $g_j^{-1}(0) \to e_n$, so $\hat{g}g_j^{-1}(0) \to \hat{g}(e_n) \neq e_n$, and so $g_j(\hat{g}(g_j^{-1}(0))) \to \zeta$. Let p_j be the hyperbolic projection of $g_j(0)$ onto the radial segment $[0, \zeta)$, and let q_j be the hyperbolic projection of $\hat{g}_j(0)$ onto the same hyperbolic segment (since $\hat{g}_j(0) \to \zeta$ we can assume that $\hat{g}_j(0)$ is contained in the lower half of $\mathbf{B}^n$ for all j). We wish to show that $d(p_j, q_j)$ is bounded above independently of j. To do so, consider the quadrilateral whose four sides are the hyperbolic segments (q_j, p_j), $(p_j, g_j\hat{g}(0))$, $(g_j\hat{g}(0), \hat{g}_j(0))$, and $(\hat{g}_j(0), q_j)$; see Figure 2. We make several observations:

(1) Clearly, the angle formed at q_j is a right angle by construction.

(2) The angle at p_j is bounded away from zero in terms of K only. This can be seen as follows. The segment $(p_j, g_j\hat{g}(0))$ has bounded hyperbolic distance from the segment $(g_j(0), g_j\hat{g}(0))$ since $d(g_j(0), p_j)$ is bounded above independently of j by choice of g_j. Furthermore, the segment $(g_j(0), g_j\hat{g}(0))$ has bounded hyperbolic distance from $g_j(L)$ since $\hat{g}(0)$ and L have bounded distance from each other. Hence the side $(p_j, g_j\hat{g}(0))$

has bounded hyperbolic distance from $C_{g_j(a),g_j(b)}$. Furthermore, (q_j, p_j) has an arbitrarily small (Euclidean) distance from $g_j([g_j^{-1}(0), \zeta))$ for large enough j, since (q_j, p_j) lies on the ray $[0, \zeta)$, and $g_j([g_j^{-1}(0), \zeta)) = [0, g_j^{-1}(\zeta))$ converges to $[0, \zeta)$ as $j \to \infty$. Here, $[g_j^{-1}(0), \zeta)$ denotes the hyperbolic geodesic ray from $g_j^{-1}(0)$ to ζ. Since $g_j^{-1}(0) \to e_n$, we have that L and $[g_j^{-1}(0), \zeta)$ intersect at an angle close to $\pi/2$, and since all g_j have uniformly bounded quasiconformal dilatation, the angle of the quadrilateral at p_j is bounded below only in terms of K.

(3) The length of the side $(p_j, g_j\hat{g}(0))$ is

$$d(p_j, g_j\hat{g}(0)) \geq d(g_j(0), g_j\hat{g}(0)) - d(g_j(0), p_j) \geq \frac{1}{K}\, d(0, \hat{g}(0)) - C,$$

(see Lemma 3.1 for the last inequality) and this is as large as desired (yet bounded independently of j) if $\hat{g}$ has been chosen to have large enough translation length.

(4) Similarly we see that the angle at $g_j\hat{g}(0)$ is bounded away from zero in terms of K only; see Figure 2.

Altogether this implies that if $d(0, \hat{g}(0))$ was chosen large enough initially (depending on K only), then $d(p_j, q_j)$ is bounded above independently of j.

We can now draw several conclusions. In what follows, we use the symbol "$A \sim B$" to mean that the two quantities A and B differ only by an additive amount that is independent of j. The symbol "$A \lesssim B$" is used to mean that there exists a constant C independent of j such that $A \leq B + C$. We first observe:

$$d(0, g_j(0)) \sim d(0, q_j) \sim d(0, \hat{g}_j(0)) - d(\hat{g}_j(0), q_j). \tag{3.2}$$

The first relation here follows from the facts that $d(g_j(0), p_j)$ and $d(p_j, q_j)$ are bounded, the second relation follows from hyperbolic geometry in the right triangle with corners 0, q_j, $\hat{g}_j(0)$. Next we note:

$$d(\hat{g}_j(0), g_j(0)) \sim d(q_j, \hat{g}_j(0)). \tag{3.3}$$

This equality comes from the facts that $d(g_j(0), p_j)$ and $d(p_j, q_j)$ are bounded. Finally we have:

$$d(g_j(0), \hat{g}_j(0)) \leq d(g_j(0), g_j\hat{g}(0)) + d(\hat{g}_j g_j(0), \hat{g}_j(0)) \lesssim K\, d(g_j(0), 0). \tag{3.4}$$

where in the last inequality we have used Lemma 3.1 to see that $d(\hat{g}_j g_j(0), \hat{g}_j(0)) \lesssim K\, d(g_j(0), 0)$ and that $d(g_j(0), g_j\hat{g}(0)) \lesssim K\, d(0, \hat{g}(0))$ and is thus bounded above independently of j.

Equations (3.2), (3.3) and (3.4) now imply

$$\begin{aligned} d(q_j, \hat{g}_j(0)) &\sim d(g_j(0), \hat{g}_j(0)) \\ &\lesssim K\, d(g_j(0), 0) \\ &\sim K\, d(0, \hat{g}_j(0)) - K\, d(\hat{g}_j(0), q_j). \end{aligned}$$

This implies that

$$d(q_j, \hat{g}_j(0)) \lesssim \frac{K}{K+1}\, d(0, \hat{g}_j(0)).$$

Using Proposition 3.2, this last inequality implies that ζ is a K-fat-horospherical limit point of $\hat{G}$. □

The following proof of Theorem 2.8 uses a standard argument which we include for the reader's convenience.

Proof of Theorem 2.8. Let G be a discrete quasiconformal group acting on $\mathbf{B}^n$, and fix $K \geq 1$. For each $M \in \mathbf{N}$ and each $g \in G$ with $g(0) \neq 0$ let $B_{g,M,K}$ be the Euclidean ball centered at the projection $\dfrac{g(0)}{|g(0)|}$ of $g(0)$ onto $\mathbf{S}^{n-1}$ and of radius $M(1-|g(0)|)^{1/(K+1)}$.

Then a point $\zeta \in \mathbf{S}^{n-1}$ belongs to $\Lambda_K(G)$ if and only if there exists $M \in \mathbf{N}$ such that ζ belongs to $B_{g,M,K}$ for infinitely many $g \in G$. Let

$$E_{M,K} = \{x \in \mathbf{S}^{n-1} : x \in B_{g,M,K} \text{ for infinitely many } g \in G\}.$$

Then

$$\Lambda_K(G) = \bigcup_{M=1}^{\infty} E_{M,K}.$$

Let now $\delta = \delta(G)$, and let $\varepsilon > 0$ be arbitrary. Then

$$\begin{aligned}\sum_{g \in G : g(0) \neq 0} (\operatorname{diam} B_{g,M,K})^{(K+1)(\delta+\varepsilon)} &\leq \sum_{g \in G} (2M(1-|g(0)|)^{1/(K+1)})^{(K+1)(\delta+\varepsilon)} \\ &= (2M)^{(K+1)(\delta+\varepsilon)} \sum_{g \in G} (1-|g(0)|)^{\delta+\varepsilon} < \infty.\end{aligned}$$

Let now $r > 0$ be arbitrary. Then, since G is discrete, there are only finitely many $g \in G$ for which $\operatorname{diam} B_{g,M,K} \geq r$, and so $\{B_{g,M,K} : \operatorname{diam} B_{g,M,K} < r\}$ is a cover for $E_{M,K}$. We set some notation. For a set E let $\mathscr{H}_s^r(E) = \inf\{\sum_j \operatorname{diam}(U_j)^s\}$, where the infimum is taken over all covers $\{U_j\}$ of E so that $\operatorname{diam} U_j < r$ for all j. Note that as $r \to 0$ we recover the s-dimensional Hausdorff measure of E. Hence

$$\begin{aligned}\mathscr{H}^r_{(K+1)(\delta+\varepsilon)}(E_{M,K}) &= \inf\left\{ \sum_{U \in \mathscr{U}} (\operatorname{diam} U)^{(K+1)(\delta+\varepsilon)} \mid \mathscr{U} \text{ is a diameter} < r \text{ cover of } E_{M,K} \right\} \\ &\leq \sum_{g \in G : \operatorname{diam} B_{g,M,K} < r} (\operatorname{diam} B_{g,M,K})^{(K+1)(\delta+\varepsilon)} \\ &\leq (2M)^{(K+1)(\delta+\varepsilon)} \sum_{g \in G} (1-|g(0)|)^{\delta+\varepsilon}.\end{aligned}$$

Thus

$$\mathscr{H}_{(K+1)(\delta+\varepsilon)}(E_{M,K}) = \lim_{r\to 0} \mathscr{H}^r_{(K+1)(\delta+\varepsilon)}(E_{M,K}) < \infty,$$

and this shows that

$$\dim E_{M,K} \le (K+1)\delta(G)$$

for all $M \in \mathbf{N}$. Since $E_{M,K} \subset E_{(M+1),K}$ for all $M \in \mathbf{N}$, and since $\Lambda_K(G) = \bigcup_{M=1}^{\infty} E_{M,K}$, we have that

$$\dim \Lambda_K(G) \le \limsup_{M\to\infty} E_{M,K} \le (K+1)\delta(G). \qquad \square$$

We end by providing an example of a non-elementary normal subgroup $\hat{G}$ of a discrete quasiconformal group G so that $\delta(\hat{G}) < \delta(G)$.

Example 3.3. Let S be a closed Riemann surface of genus $g \ge 2$, and let F be a Fuchsian group of the first kind that uniformizes S. The Retrosection Theorem guarantees that we can find a Schottky group H that uniformizes the Riemann surface S; in particular the regular set $\Omega_S = \Omega(H)$ is a planar Riemann surface (with infinitely generated fundamental group) that is a regular cover of S (with covering group H). Let Γ be a Fuchsian group that uniformizes Ω_S; since Ω_S is a non-amenable cover of S (the covering group H is a free group on $g \ge 2$ generators, and is thus non-amenable) then by a result of Brooks [9] we have that $\delta(\Gamma) < \delta(F) = 1$. Let $\phi : \mathbf{B}^2 \to \mathbf{B}^2$ be a K-quasiconformal mapping; by Lemma 3.1 we know that ϕ acts as a $(K, K\log 4)$-quasiisometry with respect to the hyperbolic metric. If we let $G = \phi F \phi^{-1}$ and $\hat{G} = \phi\Gamma\phi^{-1}$, then we observe that $\frac{1}{K}\delta(\Gamma) \le \delta(\hat{G}) \le K\delta(\Gamma)$. This implies for all $\epsilon > 0$ sufficiently small that if ϕ is $(1+\epsilon)$-quasiconformal then we have that $\delta(\hat{G}) < 1$. Because a quasiconformal mapping preserves the conical limit set and because S is closed, we have that the Hausdorff dimension of the conical limit set of G is one, from which we conclude that $\delta(G) = 1$. In particular, $\hat{G}$ is a non-trivial normal subgroup of the discrete quasiconformal group G so that $\delta(\hat{G}) < \delta(G)$ and so that there are K-fat horospherical limit points of $\hat{G}$ (by Theorem 2.9 and Theorem 2.7) that are not conical limit points.

The question below is true in the conformal setting via the Bishop and Jones result that the exponent of convergence is the Hausdorff dimension of the conical limit set of a non-elementary Kleinian group. However, because the exponent of convergence can be strictly greater than the Hausdorff dimension of the conical limit set of a discrete and non-elementary quasiconformal group (Example 4.1 in [5]), we are motivated to ask:

Question: Let G be a non-elementary discrete K-quasiconformal group $(K > 1)$ that preserves $\mathbf{B}^n$, and let $\hat{G}$ be a non-trivial normal subgroup so that $\delta(\hat{G}) < \delta(G)$. Then is the conical limit set of $\hat{G}$ *properly contained in* the conical limit set of G?

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