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SOME GENERALIZATIONS OF NEVANLINNA'S FIVE-VALUE THEOREM

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Abstract

We generalize Nevanlinna's five-value theorem to the cases that two meromorphic functions partially sharing either five or more values, or five or more small functions. In each case, we formulate a way to measure how far these two meromorphic functions are from sharing either values or small functions, and use this measurement to get some uniqueness results.

1. Introduction

Nevanlinna's five-value theorem [3] says that if two meromorphic functions share five values ignoring multiplicity, then these two functions must be identical. More precisely, suppose $f(z)$ and $g(z)$ are meromorphic functions and $a_1, a_2, \dots, a_5$ are five distinct values. If

$$\overline{E}(a_j, f) = \overline{E}(a_j, g), \quad 1 \le j \le 5$$

where $\overline{E}(a, h) = \{z \mid h(z) - a = 0\}$ for a meromorphic function $h(z)$, then $f(z) \equiv g(z)$.

C. C. Yang [5] observed that one can weaken the assumption of sharing five values to "partially" sharing five values in Nevanlinna's five-value theorem. We say that a meromorphic function $f(z)$ partially shares a value a with a meromorphic function $g(z)$ if

$$\overline{E}(a, f) \subseteq \overline{E}(a, g).$$

Under this terminology, Yang [5] proved that if a meromorphic function $f(z)$ partially share five values $a_1, a_2, \dots, a_5$ with a meromorphic function $g(z)$, and

$$\varliminf_{r\to\infty} \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f - a_j}\right) \Big/ \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{g - a_j}\right) > \frac{1}{2},$$

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then $f(z)$ and $g(z)$ must be identical. In Nevanlinna's five-value theoerm, we have $\overline{E}(a_j, f) = \overline{E}(a_j, g)$ for all $1 \leq j \leq 5$. In this case,

$$\varliminf_{r\to\infty} \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f-a_j}\right) \Big/ \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{g-a_j}\right) = 1 > \frac{1}{2},$$

so $f(z) \equiv g(z)$. Hence, Yang's result is a generalization of Nevanlinna's five-value theorem.

For the cases of small functions, Li and Qiao [2] proved that if two meromorphic functions share five small functions, then they are identical.

In this paper, we generalize both Yang's result to the cases that two meromorphic functions partially share five or more values, and Li and Qiao's result to the cases that two meromorphic functions partially share five or more small functions. We will use Nevanlinna theory to prove our theorems, especially, the second fundamental theorem for small functions proved by Yamanoi [4] recently.

We will assume the reader is familiar with the basic notations and fundamental results of Nevanlinna's theory of meromorphic functions, as found in [1]. In particular, we use E to denote a subset of $(0, \infty)$ such that E is of finite linear measure, which may be varied in different places.

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2. Meromorphic functions partially share values

DEFINITION. *Let $h(z)$ be a non-constant meromorphic function and a be a value in the extended complex plane. We define*

$$\overline{E}(a, h) = \{z \mid h(z) - a = 0\}$$

in which each zero is counted only once.

In this section, we study two meromorphic functions partially share five or more values. Precisely speaking, we consider two meromorphic functions $f(z)$ and $g(z)$, and k distinct values $a_1, a_2, \ldots, a_k$, $k \geq 5$, such that

$$\overline{E}(a_j, f) \subseteq \overline{E}(a_j, g),$$

for all $1 \leq j \leq k$.

When $k = 5$, Yang [5] proved the following theorem.

THEOREM 1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and $a_1, a_2, \ldots, a_5$ be five distinct values. If*

$$\overline{E}(a_j, f) \subseteq \overline{E}(a_j, g),$$

for all $1 \leq j \leq 5$, and

$$\varliminf_{r\to\infty} \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f-a_j}\right) \Big/ \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{g-a_j}\right) > \frac{1}{2},$$

then $f(z) \equiv g(z)$.

In the proof of this theorem, Yang gave an argument to show that if $f(z) \not\equiv g(z)$, then

$$\varliminf_{r\to\infty} \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f-a_j}\right) \Big/ \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{g-a_j}\right) \le \frac{1}{2}, \tag{1}$$

and hence the theorem is true. The inequality (1) is the crucial part of this theorem. It is a natural question to ask: if $f(z)$ and $g(z)$ partially share more than five values, what the corresponding inequality becomes? In this paper, we answer this question completely by the following theorem.

Theorem A. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and $a_1, a_2, \ldots, a_k$ be k distinct values, where $k \ge 5$, and $\overline{E}(a_j, f) \subseteq \overline{E}(a_j, g)$ for all $1 \le j \le k$. If $f(z) \not\equiv g(z)$, then*

$$\varliminf_{r\to\infty} \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) \Big/ \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g-a_j}\right) \le \frac{1}{k-3}.$$

Equivalently, if $f(z)$ and $g(z)$ partially share five or more values, as stated above, and

$$\varliminf_{r\to\infty} \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) \Big/ \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g-a_j}\right) > \frac{1}{k-3},$$

then $f(z) \equiv g(z)$.

Proof. Without loss of generality, we may assume that all a_j are finite. By Nevanlinna's second fundamental theorem, we have

$$(k-2)T(r,f) < \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f)$$

and

$$(k-2)T(r,g) < \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g-a_j}\right) + S(r,g)$$

By the hypothesis $f(z) \not\equiv g(z)$, and $\overline{E}(a_j, f) \subseteq \overline{E}(a_j, g)$, $1 \le j \le k$, we have

$$\sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) \le \overline{N}\left(r, \frac{1}{f-g}\right) \le T(r,f) + T(r,g) + O(1).$$

Hence,

$$\sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) \le \left(\frac{1}{k-2} + o(1)\right) \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) + \left(\frac{1}{k-2} + o(1)\right) \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g-a_j}\right)$$

for $r \notin E$, which implies

$$\left(\frac{k-3}{k-2} + o(1)\right) \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) \le \left(\frac{1}{k-2} + o(1)\right) \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g-a_j}\right)$$

for $r \notin E$. Therefore, we obtain

$$\varliminf_{r\to\infty} \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) \Big/ \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g-a_j}\right) \le \frac{1}{k-3},$$

which completes the proof. □

When $k = 5$, our theorem is exactly Yang's result, so Theorem A is a generalization to both Yang's result and Nevanlinna's five-value theorem.

Remark 1. There are non-identitical meromorphic functions satisfying the assumptions in Theorem A. For example, let $f(z) = e^z$, $g(z) = e^{kz}$ and $a_1, a_2, \ldots, a_k$ be the distinct roots of $x^k - x = 0$, where $k \ge 5$ is an integer. Then it is easy to see that

$$\overline{E}(a_j, f) \subseteq \overline{E}(a_j, g),$$

for all $1 \le j \le k$, and

$$\varliminf_{r\to\infty} \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) \Big/ \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g-a_j}\right) = \frac{1}{k} < \frac{1}{k-3}.$$

3. Meromorphic functions partially share small functions

We say that two non-constant meromorphic functions share a function $a(z)$ if we have $f(z) - a(z) = 0$ if and only if $g(z) - a(z) = 0$. For meromorphic functions sharing small functions, Zhang [6] proved the following theorem.

THEOREM 2. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and $a_1(z), a_2(z), \ldots, a_6(z)$ be six distinct small functions of $f(z)$ and $g(z)$. If $f(z)$ and $g(z)$ share $a_1(z), a_2(z), \ldots, a_6(z)$, then $f(z) \equiv g(z)$.*

Li and Qiao [2] proved a small function version of Nevanlinna's five-value theorem, which says that if two meromorphic functions share five small functions, then these two functions are identical.

Theorem 3. *Let* $f(z)$ *and* $g(z)$ *be two non-constant meromorphic functions, and* $a_1(z), a_2(z), \dots, a_5(z)$ *be five distinct small functions of* $f(z)$ *and* $g(z)$. *If* $f(z)$ *and* $g(z)$ *share* $a_1(z), a_2(z), \dots, a_5(z)$, *then* $f(z) \equiv g(z)$.

Now, we consider the case that two meromorphic functions partially share small functions.

Definition. Let $h(z)$ be a non-constant meromorphic function and $a(z)$ be a small function of $h(z)$. We define

$$\overline{E}(a,h) = \{z \mid h(z) - a(z) = 0\}$$

in which each zero is counted only once.

In the same way, we can extend the result of Theorem A to the case of small functions by using the second fundamental theorem for small functions proved by Yamanoi [4].

Theorem 4. *Let* $f(z)$ *be a non-constant meromorphic function and* $a_1(z)$, $a_2(z), \dots, a_k(z)$ *be* k *distinct small functions of* $f(z)$. *Then, for all* $\varepsilon > 0$

$$(k - 2 - \varepsilon)T(r, f) \le \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f - a_j}\right)$$

as $r \to \infty$, $r \notin E$.

Now, we can state and prove our theorem for small functions as follows.

Theorem B. *Let* $f(z)$ *and* $g(z)$ *be two non-constant meromorphic functions and* $a_1(z), a_2(z), \dots, a_k(z)$ *be* k *distinct small functions of* $f(z)$ *and* $g(z)$, *where* $k \ge 5$, *and* $\overline{E}(a_j, f) \subseteq \overline{E}(a_j, g)$ *for all* $1 \le j \le k$. *If* $f(z) \not\equiv g(z)$, *then*

$$\varliminf_{r\to\infty} \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f - a_j}\right) \Big/ \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g - a_j}\right) \le \frac{1}{k-3}.$$

Equivalently, *if*

$$\varliminf_{r\to\infty} \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f - a_j}\right) \Big/ \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g - a_j}\right) > \frac{1}{k-3},$$

then $f(z) \equiv g(z)$.

Proof. It follows from Theorem 4 that, given $\varepsilon > 0$, we have

$$(k-2-\varepsilon)T(r,f) \le \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right)$$

and

$$(k-2-\varepsilon)T(r,g) \le \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g-a_j}\right)$$

By the hypothesis $f(z) \not\equiv g(z)$, and $\overline{E}(a_j, f) \subseteq \overline{E}(a_j, g)$, $1 \le j \le k$, we have

$$\sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) \le \overline{N}\left(r, \frac{1}{f-g}\right) \le T(r,f) + T(r,g) + O(1).$$

Hence,

$$\sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) \le \left(\frac{1}{k-2-\varepsilon}\right) \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) + \left(\frac{1}{k-2-\varepsilon}\right) \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g-a_j}\right)$$

for $r \notin E$, which implies

$$\left(\frac{k-3-\varepsilon}{k-2-\varepsilon}\right) \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) \le \left(\frac{1}{k-2-\varepsilon}\right) \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g-a_j}\right)$$

for $r \notin E$. Therefore, we obtain

$$\underline{\lim}_{r\to\infty} \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) \Big/ \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g-a_j}\right) \le \frac{1}{k-3-\varepsilon},$$

which is true for all $\varepsilon > 0$. Hence,

$$\underline{\lim}_{r\to\infty} \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) \Big/ \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g-a_j}\right) \le \frac{1}{k-3},$$

and the proof is complete. □

Remark 2. In Remark 1, if we let $a_0 = \infty$, then $f(z)$ partially shares $k+1$ values $a_0, a_1, \dots, a_k$ with $g(z)$, and we still have

$$\underline{\lim}_{r\to\infty} \sum_{j=0}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) \Big/ \sum_{j=0}^{k} \overline{N}\left(r, \frac{1}{g-a_j}\right) = \frac{1}{k} = \frac{1}{(k+1)-1},$$

which shows that the results in Theorem A and B are almost sharp, at least one cannot replace $1/(k-3)$ by $1/(k-1+\varepsilon)$ for any $\varepsilon > 0$.

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