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KŌDAI MATH. SEM. REP.
28 (1977), 324—341

ON FOLIATIONS AND EXOTIC CHARACTERISTIC CLASSES

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§ 0. Introduction

The main aim of this paper is to formulate some basic notions in differential topology and differential geometry (vector bundles, connections, and characteristic classes, for example) for foliated manifolds.

Let M be a smooth manifold, $\mathcal{F}$ a smooth foliation on M. In § 3, we study a certain sort of fibre bundles over M, that is, bundles whose transition functions are locally constant on each leaf of $\mathcal{F}$. We will call them "$\mathcal{F}$-fibre bundles". The normal bundle of $\mathcal{F}$ is a typical example of $\mathcal{F}$-fibre bundle. In order to study the details, we give a finer classification of $\mathcal{F}$-fibre bundles than usual. We will call the resulting equivalence classes "$\mathcal{F}$-isomorphism classes". Relating to this finer classification, we formalize "$\mathcal{F}$-reducibility" of the structure group of an $\mathcal{F}$-fibre bundle. In the case of the normal bundle of $\mathcal{F}$, the structure group $GL(q, R)$ is $\mathcal{F}$-reducible to $O(q)$ if and only if M admits a bundle-like metric with respect to $\mathcal{F}$ (see [10] for the definition of the metric). § 2 is provided for § 3.

On an $\mathcal{F}$-vector bundle, we consider an "$\mathcal{F}$-flat connection", which is a generalization of the Bott's basic connection on the normal bundle of $\mathcal{F}$. In § 4, we deal with the existence of $\mathcal{F}$-flat connections and relations between $\mathcal{F}$-reducibility of the structure group and properties of $\mathcal{F}$-flat connections (holonomy group, for example).

Following the method of Bott, we generalize the notion of "exotic characteristic classes" in [1] to that of a complex $\mathcal{F}$-vector bundle. These cohomology classes of M depend only on the $\mathcal{F}$-isomorphism class of the $\mathcal{F}$-vector bundle. From the point of view in § 3, the exotic characteristic classes can be regarded as topological obstructions to the existence of $\mathcal{F}$-reduction of the structure group to the maximal compact subgroup. We show naturality of exotic characteristic classes with respect to maps between foliated manifolds which map each leaf of one foliation into a single leaf of another.

The last section is devoted to the study of exotic characteristic classes in the case of a complex analytic foliation $\mathcal{F}_c$. We prove a vanishing theorem on exotic characteristic classes of "$\mathcal{F}_c$-vector bundles". As a consequence of this

Received February 24, 1976

theorem, we find obstructions to integrability of the transversal almost complex structure of an almost complex foliation.

The author wishes to express hearty thanks to Professor T. Otsuki for his encouragement and valuable suggestions. He is also very thankful to Mr. M. Maeda.

§ 1. Basic definitions and notations

We assume that all objects are smooth and manifolds are paracompact throughout this paper. Let M be an n-dimensional manifold. We will take the following point of view toward foliations.

DEFINITION. A foliation of codimension q $(0\leqq q\leqq n)$ on M is given by a local coordinate system $\mathcal{F}=\{(U_\alpha, (x_\alpha{}^a, y_\alpha{}^i))\}(1\leqq a\leqq n-q, 1\leqq i\leqq q)$ of M such that

$$y_\alpha{}^i=f_{\alpha\beta}{}^i((y_\beta{}^j)) \qquad \text{on} \quad U_\alpha\cap U_\beta\neq\phi\,.$$

We call such a coordinate system to be flat. A local coordinate $(U, (x^a, y^i))$ of M is said to be flat with respect to $\mathcal{F}$, if the union $\mathcal{F}\cup\{(U, (x^a, y^i))\}$ is also a flat coordinate system.

Two flat coordinate systems $\mathcal{F}$ and $\mathcal{F}'$ define the same foliation if the union $\mathcal{F}\cup\mathcal{F}'$ is also a flat coordinate system.

For a foliation $\mathcal{F}$ on M, we denote the corresponding integrable subbundle of the tangent bundle T by F. For an open set $U\subset M$, $\mathcal{F}|U$ denotes the restriction of $\mathcal{F}$ on U. The trivial foliation of codimension 0 (resp. n) is denoted by $\mathcal{T}$ (resp. $\mathcal{O}$).

By analogy of the above definition, a complex analytic foliation of complex codimension q on M is defined by a local coordinate system $\mathcal{F}_c=\{(U_\alpha, (x^a{}_\alpha, y_\alpha{}^i))\}$ $(1\leqq a\leqq n-2q, 1\leqq i\leqq q$, and each $y_\alpha{}^i$ is a complex-valued function) of M such that

$$y_\alpha{}^i=f_{\alpha\beta}{}^i((y_\beta{}^j)) \qquad \text{on} \quad U_\alpha\cap U_\beta\neq\phi\,,$$

where $f_{\alpha\beta}{}^i$ is holomorphic in addition.

A complex analytic foliation of complex codimension q gives a foliation of codimension $2q$. We will call such a foliation the underlying smooth foliation of the complex analytic foliation.

DEFINITION. Given two foliations, $\mathcal{F}$ on M and $\mathcal{F}'$ on M', a smooth map $f\colon M\to M'$ is called an $(\mathcal{F}, \mathcal{F}')$-map if, for any flat coordinates $(U, (x^a, y^i))$ with respect to $\mathcal{F}$ and $(U', (x'^b, y'^j))$ with respect to $\mathcal{F}'$, f satisfies

$$y'^j\circ f=f^j((y^i)) \qquad \text{on} \quad f^{-1}(U')\cap U\neq\phi\,.$$

Let $\bar{\mathcal{F}}$ be the canonical foliation of codimension $q+1$ on $M\times R$. We define a map $j_s\colon M\to M\times R$ by $j_s(p):=(p, s)$ for $p\in M$ and $s\in R$. Then j_s is an $(\mathcal{F}, \bar{\mathcal{F}})$-map.

DEFINITION. Two $(\mathcal{F}, \mathcal{F}')$-maps $f_0, f_1\colon M\to M'$ are said to be $(\mathcal{F}, \mathcal{F}')$-homotopic if there exists an $(\bar{\mathcal{F}}, \mathcal{F}')$-map $H: M\times R\to M'$ such that $H\circ j_0=f_0$ and $H\circ j_1=f_1$.

For complex analytic foliations $\mathcal{F}_c$ on M and $\mathcal{F}_c'$ on M', we analogously define an $(\mathcal{F}_c, \mathcal{F}_c')$-map, demanding holomorphy of f^j in the above definition.

Especially an $(\mathcal{F}, \mathcal{O}')$-map (resp. $(\mathcal{F}_c, \mathcal{O}_c')$-map) is said to be $\mathcal{F}$-basic (resp. $\mathcal{F}_c$-basic).

§ 2. $\mathcal{F}$-cocycles

Let M be a manifold with a foliation $\mathcal{F}$, G a Lie group. The presheaf of all G-valued local $\mathcal{F}$-basic functions is denoted by $G_{\mathcal{F}}$. For the general background of § 2 and § 3, see Hirzebruch [6].

DEFINITION. A G-valued $\mathcal{F}$-cocycle over an open covering $\mathcal{U}=\{U_\alpha\}_{\alpha\in\Lambda}$ of M is an assignment of an $\mathcal{F}|U_\alpha\cap U_\beta$-basic function $g_{\alpha\beta}: U_\alpha\cap U_\beta\to G$ to each $\alpha, \beta\in\Lambda$ such that, for all $\alpha, \beta, \gamma\in\Lambda$,

$$g_{\alpha\beta}\cdot g_{\beta\gamma}=g_{\beta\gamma} \quad \text{in} \quad U_\alpha\cap U_\beta\cap U_\gamma\neq\phi\,.$$

The set of all G-valued $\mathcal{F}$-cocycles over $\mathcal{U}$ is denoted by $Z^1(\mathcal{U}, G_{\mathcal{F}})$.

For example, if $\{(U_\alpha, (x_\alpha{}^a, y_\alpha{}^i))\}$ is a flat coordinate system of M with respect to $\mathcal{F}$, then

$$\nu:=\{(\partial y_\alpha{}^i/\partial y_\beta{}^j): U_\alpha\cap U_\beta\to GL(q, R)\}$$

is a $GL(q, R)$-valued $\mathcal{F}$-cocycle. We can regard a G-valued $\mathcal{O}$-cocycle (resp. $\mathcal{F}$-cocycle) as transition functions of a usual G-bundle (resp. flat G-bundle).

DEFINITION. Let $\eta^0\in Z^1(\mathcal{U}^0, G_{\mathcal{F}})$ and $\eta^1\in Z^1(\mathcal{U}^1, G_{\mathcal{F}})$, where $\mathcal{U}^0$ and $\mathcal{U}^1$ are open coverings of M. Then η^0 and η^1 are said to be $\mathcal{F}$-equivalent if they extend to an $\mathcal{F}$-cocycle over the union $\mathcal{U}^0\cup\mathcal{U}^1$. We denote the $\mathcal{F}$-equivalence class of $\eta\in Z^1(\mathcal{U}, G_{\mathcal{F}})$ by $[\eta]_{\mathcal{F}}$ or simply η. The set of all $\mathcal{F}$-equivalence classes of G-valued $\mathcal{F}$-cocycles over M is denoted by $H^1(M, G_{\mathcal{F}})$.

Let M' be another manifold with a foliation $\mathcal{F}'$. By the definition, we have

PROPOSITION 2.1. *If η' is a G-valued $\mathcal{F}'$-cocycle over M' and $f: M\to M'$ is an $(\mathcal{F}, \mathcal{F}')$-map, then the induced cocycle $f^*\eta'$ is a G-valued $\mathcal{F}$-cocyle over M. And there is a natural map*

$$f^*: H^1(M', G_{\mathcal{F}'}) \longrightarrow H^1(M, G_{\mathcal{F}})\,.$$

When we regard the identity map on M as an $(\mathcal{O}, \mathcal{F})$-map, we denote this by $\iota_{\mathcal{F}}^{\mathcal{O}}$. The induced map

$$\iota_{\mathcal{F}}^{\mathcal{O}}{}^*: H^1(M, G_{\mathcal{F}}) \longrightarrow H^1(M, G_{\mathcal{O}})$$

is neither surjective nor injective in general, where $H^1(M, G_{\mathcal{O}})$ can be regarded as the set of all isomorphism classes of usual G-bundles over M, see [6]. Using the properties of characteristic classes and exotic characteristic classes, we will see this fact in § 5 and § 6. From these fact, we see that the $\mathcal{F}$-equivalence relation is finer than the $\mathcal{O}$-equivalence relation in the set of all G-valued $\mathcal{F}$-cocycles over M.

We now define the following equivalence relation, which is finer than the $\mathcal{O}$-equivalence relation and coaser than the $\mathcal{F}$-equivalence relation.

DEFINITION. Two G-valued $\mathcal{F}$-cocycles η^0 and η^1 over M are said to be smoothly $\mathcal{F}$-homotopic if there exists a G-valued $\bar{\mathcal{F}}$-cocycle $\bar{\eta}$ over $M\times R$ such that $j_s^*\bar{\eta}$ is $\mathcal{F}$-equivalent to η^s for $s=0, 1$. The equivalence relation in the set of all G-valued $\mathcal{F}$-cocycles over M generated by smooth $\mathcal{F}$-homotopy is called $\mathcal{F}$-homotopy.

The following fact is well-known, see [6] for example:

Fact 2.2. If maps $f_0, f_1: M\rightarrow M'$ are $(\mathcal{O}, \mathcal{O}')$-homotopic, then $f_0^*=f_1^*: H^1(M', G_{\mathcal{O}'})\rightarrow H^1(M, G_{\mathcal{O}})$. Especially two G-valued $\mathcal{O}$-cocycles over M are $\mathcal{O}$-equivalent if and only if they are $\mathcal{O}$-homotopic.

In general, however, we can only deduce the following proposition which is weaker than Fact 2.2.

PROPOSITION 2.3. *If η' is a G-valued $\mathcal{F}'$-cocycle over M', and $(\mathcal{F}, \mathcal{F}')$-maps $f_0, f_1: M\rightarrow M'$ are $(\mathcal{F}, \mathcal{F}')$-homotopic, then the induced $\mathcal{F}$-cocycles $f_0^*\eta', f_1^*\eta'$ are $\mathcal{F}$-homotopic.*

Let M be a manifold with a complex analytic foliation $\mathcal{F}_c$, G a complex Lie group. Replacing $\mathcal{F}$ by $\mathcal{F}_c$ in the above definitions, we define G-valued $\mathcal{F}_c$-cocycles and $\mathcal{F}_c$-equivalence relation. For example, if $\{(U_\alpha, (x_\alpha{}^a, y_\alpha{}^i))\}$ is a flat coordinate system of M with respect to $\mathcal{F}_c$, then

$$\nu_c := \{(\partial y_\alpha{}^i/\partial y_\beta{}^j) : U_\alpha \cap U_\beta \longrightarrow GL(q, C)\}$$

is a $GL(q, C)$-valued $\mathcal{F}_c$-cocycle.

§ 3. $\mathcal{F}$-fibre bundles and $\mathcal{F}$-reductions

Let M be a manifold with a foliation $\mathcal{F}$, N a manifold, G a Lie group which acts on N effectively. Let W be a fibre bundle over M with structure group G, fibre N and projection π.

DEFINITION. A fibre bundle W is called an $\mathcal{F}$-fibre bundle if there exists a bundle coordinate system $\{(U_\alpha, h_\alpha)\}$ of admissible charts [6] such that its transition functions $\{g_{\alpha\beta}\}$ can be regarded as a G-valued $\mathcal{F}$-cocycle over M, where $h_\alpha: \pi^{-1}(U_\alpha)\rightarrow U_\alpha\times G$ gives a local triviality of W on an open set U_α in M. Such a coordinate system is said to be flat.

Let W be an $\mathcal{F}$-fibre bundle. A diffeomorphism $h: \pi^{-1}(U)\rightarrow U\times G$, U open in M, is called an $\mathcal{F}$-admissible chart for the flat system $\{(U_\alpha, h_\alpha)\}$ if the union

$\{(U_\alpha, h_\alpha)\} \cup \{(U, h)\}$ is also flat.

DEFINITION. Two flat coordinate systems of W make W the same $\mathcal{F}$-fibre bundle if and only if every $\mathcal{F}$-admissible chart for one system is an $\mathcal{F}$-admissible chart for the other system.

DEFINITION. Let W and W' be $\mathcal{F}$-fibre bundles over M with structure group G and fibre N. A bundle isomorphism $f: W \to W'$ which covers identity map on M is called $\mathcal{F}$-isomorphism if the induced chart $(U, h' \circ f)$ is an $\mathcal{F}$-admissible chart of W for every $\mathcal{F}$-admissible chart (U, h') of W'.

Now we have the following

PROPOSITION 3.1. *The set $H^1(M, G_{\mathcal{F}})$ can be regarded as the set of all $\mathcal{F}$-isomorphism classes of $\mathcal{F}$-fibre bundles over M with structure group G and fibre N (with a given effective action).*

The proof is analogous to that of the corresponding theorem for $H^1(M, G_{\mathcal{O}})$ in [6]. $\mathcal{F}$-fibre bundles in the isomorphism class corresponding to $\eta \in H^1(M, G_{\mathcal{F}})$ are said to be associated to η.

When an $\mathcal{F}$-fibre bundle is a vector bundle in the same time, we call this bundle an $\mathcal{F}$-vector bundle and local frame fields corresponding to $\mathcal{F}$-admissible charts are said to be $\mathcal{F}$-admissible. For a vector bundle V, ΓV denotes the spaces of all smooth cross sections of V.

EXAMPLE 3.1. A usual vector bundle has the canonical $\mathcal{O}$-vector bundle structure.

EXAMPLE 3.2. A vector bundle with a flat connection has the canonical $\mathcal{F}$-vector bundle structure of which $\mathcal{F}$-admissible frame fields are parallel with respect to the flat connection.

EXAMPLE 3.3. The normal bundle T/F of a foliation $\mathcal{F}$ has the canonical $\mathcal{F}$-vector bundle structure associated to ν in § 2. Moreover we generalize this as follows:

PROPOSITION 3.2. *Let $\mathcal{F}$ be a foliation on M, F the corresponding integrable subbundle of the tangent bundle T of M. If E is a subbundle of T such that $E \supset F$ and $[\Gamma F, \Gamma E] \subset \Gamma E$, then T/E has an $\mathcal{F}$-vector bundle structure.*

Proof. Let $\pi: T \to T/E$ be the projection, and let n, $n-r$ and $n-q$ be fibre dimensions of T, E and F respectively. We can take a flat coordinate system $\{(U_\alpha, (x_\alpha^a, y_\alpha^i))\}$ of M with respect to $\mathcal{F}$ such that for each $p \in M$ there is a neighborhood $U_{\alpha,p} \subset U_\alpha$ where $\pi(\partial/\partial y_\alpha^1) \wedge \cdots \wedge \pi(\partial/\partial y_\alpha^r) \neq 0$. There exist functions $\underset{\alpha}{A}{}_u^t$ on $U_{\alpha,p}$ satisfying $\pi(\partial/\partial y_\alpha^u) = \underset{\alpha}{A}{}_u^t \pi(\partial/\partial y_\alpha^t)$ over $U_{\alpha,p}$, where $1 \leqq t \leqq r < u \leqq q$. On $U_{\alpha,p}$, we have

$$[\partial/\partial x_\alpha^a,\ \partial/\partial y_\alpha^u - \underset{\alpha}{A}{}_u^t(\partial/\partial y_\alpha^t)] = -((\partial/\partial x_\alpha^a)\underset{\alpha}{A}{}_u^t)(\partial/\partial y_\alpha^t).$$

The condition $[\Gamma F, \Gamma E]\subset\Gamma E$ implies $(\partial/\partial x_\alpha{}^a)\underset{\alpha}{A_u{}^t}=0$, that is, $(A_u{}^t)$ is $\mathcal{F}|U_{\alpha,p}$-basic. On $U_{\alpha,p}\cap U_{\beta,p}\neq\phi$, we have

$$\pi(\partial/\partial y_\beta{}^s)=((\partial y_\alpha{}^t/\partial y_\beta{}^s)+(\partial y_\alpha{}^u/\partial y_\beta{}^s)\underset{\alpha}{A_u{}^t})\pi(\partial/\partial y_\alpha{}^t),$$

where $1\leqq s\leqq r$. Thus we have an $\mathcal{F}$-cocycle. Q.E.D.

EXAMPLE 3.4. If V and V' are $\mathcal{F}$-vector bundles over M, then V^*, $V\oplus V'$ and $V\otimes V'$ have also canonical $\mathcal{F}$-vector bundle structures.

Let M be a manifold with a complex analytic foliation $\mathcal{F}_c$, N a complex manifold, G a complex Lie group which acts on N effectively and holomorphically. Replacing $\mathcal{F}$ by $\mathcal{F}_c$ in the above definitions, we define $\mathcal{F}_c$-fibre bundles, $\mathcal{F}_c$-admissible charts and $\mathcal{F}_c$-isomorphism.

EXAMPLE 3.5. Let $\{(U_\alpha, (x_\alpha{}^a, y_\alpha{}^i))\}$ be a flat coordinate system of M with respect to $\mathcal{F}_c$, where $y^i=u^i+\sqrt{-1}v^i$. The complex subbundle of $(T/F)\otimes C$ defined by local frame fields $((\pi(\partial/\partial u^i)-\sqrt{-1}\pi(\partial/\partial v^i))/2)$ is denoted by Q. This bundle Q is an $\mathcal{F}_c$-vector bundle associated to the $\mathcal{F}_c$-cocycle ν_c in § 2.

Hereafter in this section, we deal with certain reductions of structure groups of $\mathcal{F}$-fibre bundles over M with $\mathcal{F}$.

DEFINITION. Let W be an $\mathcal{F}$-fibre bundle over M with structure group G, H a Lie subgroup of G. The structure group of W is said to be $\mathcal{F}$-reducible to H if the corresponding element of $H^1(M, G_{\mathcal{F}})$ included in the image of the map $\iota_*: H^1(M, H_{\mathcal{F}})\to H^1(M, G_{\mathcal{F}})$ which is induced from the inclusion map $\iota: H\to G$.

We have the following fact for usual fibre bundles, see [6]:

Fact 3.3. If H is a closed Lie subgroup of G for which G/H is a cell, then the map

$$\iota_*: H^1(M, H_{\mathcal{O}})\longrightarrow H^1(M, G_{\mathcal{O}})$$

is bijective, that is, the structure group G is always $\mathcal{O}$-reducible to H.

The corresponding fact for $\mathcal{F}$-fibre bundle does not hold in general. In the case of $\mathcal{F}$-vector bundle, we can find counterexamples. Comparing with the definition in [10], we have

PROPOSITION 3.4. *The structure group $GL(q, R)$ of the normal bundle T/F is $\mathcal{F}$-reducible to $O(q)$ if and only if M admits a bundle-like metric with respect to $\mathcal{F}$.*

This is easily deduced from the following

PROPOSITION 3.5. *Let V be an $\mathcal{F}$-vector bundle over M with structure group $GL(k, R)$(resp. $GL(k, C)$). The structure group of V is $\mathcal{F}$-reducible to $O(k)$(resp. $U(k)$) if and only if V admits a metric whose components relative to $\mathcal{F}$-admissible frame fields are $\mathcal{F}$-basic functions.*

For the later use, we prepare the following

DEFINITION. A foliation $\mathcal{F}$ of codimen $2q$ is called an almost complex foliation if the structure group of T/F is $\mathcal{F}$-reducible to $GL(q, C)$.

The underlying smooth foliation of a complex analytic foliation is an almost complex foliation, where the $\mathcal{F}$-reduction is induced from the complex structure of Q in Example 3.5.

§ 4. Connections on $\mathcal{F}$-vector bundles

Let M be a manifold with a foliation $\mathcal{F}$, V and $\mathcal{F}$-vector bundle over M with structure group $GL(k, K)$, where K is the real or complex number field. T^* denotes the (complexified) cotangent bundle of M.

DEFINITION. A connection $D: \Gamma V \to \Gamma(T^* \otimes V)$ on an $\mathcal{F}$-vector bundle V is said to be $\mathcal{F}$-flat if $D_X s=0$, $\forall X \in \Gamma(F|U)$ for every $\mathcal{F}$-admissible local frame field s.

By this definition, we get

PROPOSITION 4.1. *If a connection D is $\mathcal{F}$-flat, then*

$$D_X D_Y - D_Y D_X - D_{[X,Y]} = 0, \qquad \forall X,\ Y \in \Gamma F.$$

Now we show some examples.

EXAMPLE 4.1. There exists the canonical $\mathcal{F}$-flat connection of T/E in Example 3.3 as follows:

Fix a decomposition $T=F\oplus F^{\perp}$. Let $\pi_1: T\to F$ and $\pi_2: T\to F^{\perp}$ be projections, D' a connection on T/E. For every $X \in \Gamma T$, $Y \in \Gamma(T/E)$, we define a connection on T/E by

$$D_X Y := \pi([\pi_1 X,\ \tilde{Y}]) + D'_{\pi_2 X} Y,$$

where $\tilde{Y} \in \Gamma T$ is such that $\pi(\tilde{Y})=Y$. This is independent of the choice of a lift $\tilde{Y}$.

When $E=F$, this connection is called a basic connection [1].

EXAMPLE 4.2. The flat connection on a $\mathcal{F}$-vector bundle in Example 3.2 is $\mathcal{F}$-flat.

We can prove a general theorem on existence of an $\mathcal{F}$-flat connection on an $\mathcal{F}$-vector bundle.

THEOREM 4.2. *On an $\mathcal{F}$-vector bundle V, there exists an $\mathcal{F}$-flat connection.*

Proof. Let $\{U_\alpha\}$ be an open covering of M with a family of $\mathcal{F}$-admissible frame fields $\{s_\alpha\}$. We can assume that this covering is locally finite and admits a partition of unity $\{f_\alpha\}$. Define a connection D^α on $V|U_\alpha$ by $D^\alpha s_\alpha=0$ for the frame s_α. And define a connection D on V by $\sum_\alpha f_\alpha D^\alpha$. Then we have

$$Ds_\alpha = (\sum_\beta f_\beta dg_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}) s_\alpha .$$

Since the transition function $g_{\alpha\beta}$ is $\mathcal{F}|U_\alpha \cap U_\beta$-basic, we get

$$D_X s_\alpha = 0,\ \forall X \in \Gamma(F|U). \qquad \text{Q. E. D.}$$

There is the following relation between two $\mathcal{F}$-flat connections on a vector bundle which has $\mathcal{F}$-vector bundle structures.

PROPOSITION 4.3. *Assume that a vector bundle has two structures of $\mathcal{F}$-vector bundle V and V'. Let D (resp. D') be an $\mathcal{F}$-flat connection on V (resp. V'). Then V and V' are the same $\mathcal{F}$-vector bundle if and only if $D_X = D'_X$, $\forall X \in \Gamma F$.*

Proof. The sufficiency is obvious. Let s (resp. s') be an $\mathcal{F}$-admissible frame field of V (resp. V') over U (resp. U'). When $U \cap U' \neq \phi$, there exists a map $g: U \cap U' \to GL(k, K)$ such that $s' = gs$. Then we have

$$D_X s' = (Xg)s + g(D_X s) = (Xg)s, \qquad \forall X \in \Gamma(F|U \cap U').$$

This formula and the assumption $D_X = D'_X$, $\forall X \in \Gamma F$ implies $Xg = 0$, $\forall X \in \Gamma(F|U \cap U')$, that is, g is $\mathcal{F}|U \cap U'$-basic. Q. E. D.

As for the properties of maps between two foliated manifolds M with $\mathcal{F}$ and M' with $\mathcal{F}'$, we easily get

PROPOSITION 4.4. *Let V' be an $\mathcal{F}'$-vector bundle over M', D' an $\mathcal{F}'$-flat connection on V'. If $f: M \to M'$ is an $(\mathcal{F}, \mathcal{F}')$-map, then the induced connection f^*D' on f^*V' is an $\mathcal{F}$-flat connection on the $\mathcal{F}$-vector bundle f^*V'.*

As a generalization of standard connection theory on vector bundles, we get some results which relate to $\mathcal{F}$-reducibility of structure groups of $\mathcal{F}$-vector bundles as follows (cf. [4]).

THEOREM 4.5. *If an $\mathcal{F}$-vector bundle V admits an $\mathcal{F}$-flat connection D whose holonomy group is contained in a Lie subgroup G of the structure group $GL(k, K)$, then the structure group of V is $\mathcal{F}$-reducible to G.*

Proof. Let B be the linear frame bundle of V, $\pi: B \to M$ the projection. We take a flat coordinate system $\{(U_\alpha, (x_\alpha^a, y_\alpha^i))\}$ such that $(x_\alpha^a, y_\alpha^i): U_\alpha \to R^{n-q} \times R^q$ is a diffeomorphism. Let $p_\alpha \in U_\alpha$ (resp. $A_\alpha \subset U_\alpha$) be the point (resp. subset) corresponding to the origin $(0, 0)$(resp. subset $\{0\} \times R^q$) in $R^{n-q} \times R^q$. Let π_α: $U_\alpha \to A_\alpha$ be the induced projection map, then each $\pi_\alpha^{-1}(p)$, $p \in A_\alpha$, is a leaf of $\mathcal{F}|U_\alpha$. We can assume that there exists an $\mathcal{F}$-admissible frame field s_α of V over U_α, for each α. For $u \in B|U_\alpha$, there is an element $g \in GL(k, K)$ such that $u = s_\alpha(\pi(u))g$, then we set $\tilde{\pi}_\alpha(u) := s_\alpha(\pi_\alpha(\pi(u)))g$. Thus we define a bundle map $\tilde{\pi}_\alpha: B|U_\alpha \to B|A_\alpha$ which covers π_α.

Fix $u_0 \in B$ as a reference point for the holonomy group of the $\mathcal{F}$-flat connection D. Given a connection on V, we denotes the parallel displacement along a piece-wise smooth curve c in M by $\tilde{c}$. Choose a family of piece-wise smooth curves $\{c_\alpha\}$ in M such that all start from $\pi(u_0)$ and each ends at p_α. Set $u_\alpha := \tilde{c}_\alpha u_0$ in $\pi^{-1}(p_\alpha)$. For any $u \in B | U_\alpha$, let r_α be the ray (with respect to the above coordinate) in A_α from p_α to $\pi_\alpha(\pi(u))$. The parallel displacement of u gives a point $\tilde{r}_\alpha^{-1}(\tilde{\pi}_\alpha(u)) = u_\alpha g$ for some $g \in GL(k, K)$ and then we define a map $\varphi_\alpha : B | U_\alpha \to GL(k, K)$ by $\varphi_\alpha(u) := g$. Define a new frame field over U_α by $\hat{s}_\alpha(p) := s_\alpha(p)(\varphi_\alpha(s_\alpha(p)))^{-1}$ for $p \in U_\alpha$. Then this is also an $\mathcal{F}$-admissible frame field of V because of the definition of φ_α. As to these new frame fields, we have, for $p \in U_\alpha \cap U_\beta \neq \phi$,

$$\hat{s}_\beta(p) = \hat{s}_\alpha(p) \varphi_\alpha(s_\beta(p))(\varphi_\beta(s_\beta(p)))^{-1} .$$

Set $\hat{g}_{\alpha\beta} := \varphi_\alpha(s_\beta(p))(\varphi_\beta(s_\beta(p)))^{-1}$ for $p \in U_\alpha \cap U_\beta$. Then by the well-known method in the usual holonomy reduction theorem, we can show that the transition functions $\{\hat{g}_{\alpha\beta}\}$ take values in G. Q. E. D.

As an inverse version of this theorem, we get the following

THEOREM 4.6. *If G is a reductive Lie subgroup of $GL(k, K)$ and the structure group of an $\mathcal{F}$-vector bundle V is $\mathcal{F}$-reducible to G, then V admits an $\mathcal{F}$-flat connection of which holonomy group is contained in G.*

Proof. Let B be the linear frame bundle of V, $\mathfrak{gl}(k, K)$ the Lie algebra of $GL(k, K)$, $\omega : TB \to \mathfrak{gl}(k, K)$ a connection form which corresponds to an $\mathcal{F}$-flat connection D on V in Theorem 4.2. If a local section $s : U \to B$ is an $\mathcal{F}$-admissible frame field, then $i_X(s^*\omega) = 0$, $\forall X \in \Gamma(F | U)$, and vice versa.

As G is reductive, there is a linear subspace $\mathfrak{m}$ of $\mathfrak{gl}(k, K)$ such that $\mathfrak{gl}(k, K) = \mathfrak{m} \oplus \mathfrak{g}$, $\mathrm{AD}(G)\mathfrak{m} \subset \mathfrak{m}$, where $\mathfrak{g}$ is the Lie algebra of G. We denote the $\mathfrak{g}$-component of ω by ω'. Let B' be the subbundle of B which gives the $\mathcal{F}$-reduction to G, $\iota : B' \to B$ the inclusion map. Then we can prove that the induced form $\iota^*\omega' : TB' \to \mathfrak{g}$ is a connection from and that $i_X(s^*(\iota^*\omega')) = 0$, $\forall X \in \Gamma(F | U)$ for every $\mathcal{F}$-admissible frame field s of V which takes value in B'. The linear connection D' on V which corresponds to $\iota^*\omega'$ is an $\mathcal{F}$-flat connection with holonomy group contained in G. Q. E. D.

Next we consider reductions which are given by tensor fields on V. From Proposition 3.5, we easily get

PROPOSITION 4.7. *Let D be an $\mathcal{F}$-flat connection on a real (resp. complex) $\mathcal{F}$-vector bundle V. Then a metric tensor field g on V gives an $\mathcal{F}$-reduction to $O(k)$ (resp. $U(k)$) if and only if $D_X g = 0$, $\forall X \in \Gamma F$.*

For an $\mathcal{F}$-vector bundle V and a metric g on V, an $\mathcal{F}$-flat connection D satisfying $D_X g = 0$, $\forall X \in \Gamma F$, is not a metric connection in general. However we can find an $\mathcal{F}$-flat connection D' on V which coincides with D with respect to F (see Proposition 4.3) and is a metric connection (i. e. $D'_X g = 0$, $\forall X \in \Gamma T$).

Proposition 4.7 is true for other tensor fields on V. Relating to an almost complex foliation, if a tensor field J on V gives an $\mathcal{F}$-reduction of the structure group $GL(2k, R)$ to the subgroup $GL(k, C)$, then $D_X J=0$, $\forall X \in \Gamma F$ for an $\mathcal{F}$-flat connection D on V.

Hereafter in this section, we deal with the complex analytic foliation. Let M be a manifold with a complex analytic foliation $\mathcal{F}_c$, $\mathcal{F}$ the underlying smooth foliation of $\mathcal{F}_c$, and V an $\mathcal{F}_c$-vector bundle over M.

DEFINITION. An $\mathcal{F}$-flat connection $D: \Gamma V \to \Gamma(T^* \otimes V)$ is said to be $\mathcal{F}_c$-flat, if $D_X s=0$, $\forall X \in \Gamma(\bar{Q}|U)$, for every $\mathcal{F}_c$-admissible local frame field s, where $\bar{Q}$ is the complex conjugate of Q in Example 3.5.

For the existence of $\mathcal{F}_c$-flat connections on $\mathcal{F}_c$-vector bundles, we have

THEOREM 4.8. *On an $\mathcal{F}_c$-vector bundle V, there exists an $\mathcal{F}_c$-flat connection.*

The proof is analogous to that of Theorem 4.2.

§ 5. Connections and characteristic classes

We will use the Chern-Weil theory of characteristic classes in the form as described in [3]. Let V be a complex vector bundle over a manifold M, T^* the complexified cotangent bundle of M, $A^*(M)$ the graded algebra of all complex-valued grobal forms on M. For a connection $D: \Gamma V \to \Gamma(T^* \otimes V)$ and a local frame field s of V, we denote the connection (resp. curvature) matrix relative to s by $\theta(s)$ (resp. $\Theta(s)$) or simply θ (resp. Θ).

Let $I^*(GL(k, C))$ denote the graded algebra of complex-valued adjoint-invariant symmetric multilinear functions on $\mathfrak{gl}(k, C)$. For $\varphi \in I^r(GL(k, C))$ and $A \in \mathfrak{gl}(k, C)$, we denote $\varphi(A, \cdots, A)$ by $\varphi(A)$ for simplicity. Then adjoint-invariancy of φ is equivalent to

$$\varphi(gAg^{-1}) = \varphi(A), \qquad \forall g \in GL(k, C), \qquad \forall A \in \mathfrak{gl}(k, C).$$

Let D^0 and D^1 be connections on V.

DEFINITION. Homomorphisms of C-modules

$$\lambda(D^1): I^r(GL(k, C)) \longrightarrow A^{2r}(M) \qquad \text{and}$$

$$\lambda(D^0, D^1): I^r(GL(k, C)) \longrightarrow A^{2r-1}(M)$$

are defined as follows: for $\varphi \in I^r(GL(k, C))$, locally

$$\lambda(D^1)\varphi := \varphi(\Theta^1),$$

$$\lambda(D^0, D^1)\varphi := r\int_0^1 \varphi(\theta^1 - \theta^0, \Theta^t, \cdots, \Theta^t)dt,$$

where Θ^t is the curvature matrix of the connection $tD^1 + (1-t)D^0$.

From the Chern-Weil theory, we have

Fact 5.1. $d(\lambda(D^1)\varphi)=0$ and

$$d(\lambda(D^0, D^1)\varphi)=\lambda(D^1)\varphi-\lambda(D^0)\varphi .$$

In other words, the closed form $\lambda(D^1)\varphi$ represents a complex de Rham cohomology class $[\lambda(D^1)\varphi]\in H^{2r}(M)$, and the induced homomorphism of graded algebras

$$\lambda(D^1)^*: I^*(GL(k, C)) \longrightarrow H^*(M)$$

does not depend on the choice of connections on V. We can prove that $\lambda(D^1)^*$ depends only on the $\mathcal{O}$-isomorphism class $[V]_{\mathcal{O}}$. Denote the subalgebra $\lambda(D^1)^*(I^*(GL(k, C)))$ by Chern*(V), Chern*$(V)\cap H^r(M)$ by Chern$^r(V)$.

Let $\mathcal{F}$ be a foliation of codimension q on M, V *a* complex $\mathcal{F}$-vector bundle over M.

PROPOSITION 5.2. *If D^1 is an $\mathcal{F}$-flat connection on V, then*

$$\lambda(D^1)\varphi=0 \qquad \textit{for } \varphi \textit{ such that } \deg \varphi>q .$$

The proof is analogous to the corresponding theorem in [1] because of Proposition 4.1. From Theorem 4.2, we have

COROLLARY. *If a vector bundle V has a structure of an $\mathcal{F}$-vector bundle, then*

$$\text{Chern}^r(V)=0 \qquad \textit{for} \quad r>2q=2\cdot\text{codim}_R\mathcal{F} .$$

Let $\mathcal{F}_c$ be a complex analytic foliation of complex codimension q on M, V an $\mathcal{F}_c$-vector bundle over M. By analogy with Proposition 5.2, we can prove

PROPOSITION 5.3. *If D^1 is an $\mathcal{F}_c$-flat connection on V, then*

$$\lambda(D^1)\varphi=0 \qquad \textit{for } \varphi \textit{ such that } \deg\varphi>q .$$

And from Theorem 4.8, we have

COROLLARY. *If a vector bundle V has a structure of an $\mathcal{F}_c$-vector bundle, then*

$$\text{Chern}^{\,r}(V)=0 \qquad \textit{for} \quad r>2q=\text{codim}_{\,R}\mathcal{F}_c .$$

Remark. With certain geometrical conditions, the elements of Chern*(V) whose cohomology dimensions are greater than just the codimension of the foliation would vanish as in the above corollary. The following fact is an interpretation of one of the results in [9].

Fact 5.4. If the structure group $GL(q, C)$ of the complexified normal bundle $(T/F)\otimes C$ of a foliation $\mathcal{F}$ is $\mathcal{F}$-reducible to $U(q)$, then

$$\text{Chern}^r((T/F)\otimes C)=0 \qquad \text{for} \quad r>q=\text{codim}_R\mathcal{F} .$$

We can prove an analogous result.

PROPOSITION 5.5. *Let V be an $\mathcal{F}$-vector buncle over M. If an $\mathcal{F}$-admissible* *covering of M for V admits a partition of unity consisting of $\mathcal{F}$-basic functions, then*

$$\mathrm{Chern}^r(V)=0 \qquad \text{for} \quad r>q=\mathrm{codim}_R\mathcal{F}\,,$$

where an $\mathcal{F}$-admissible covering for V is an open covering whose elements admit $\mathcal{F}$-admissible frame fields of V.

§ 6. Exotic characteristic classes of $\mathcal{F}$-vector bundles

In this section, we generalize the notion of exotic characteristic classes in [1] to that of complex $\mathcal{F}$-vector bundles. These classes depend only on the $\mathcal{F}$-isomorphism class of an $\mathcal{F}$-vector bundle. For this purpose, we construct a cochain complex $WU(k)_q$ at first.

DEFINITION. Let $C[c_1, \bar{c}_1, \cdots, c_k, \bar{c}_k]$ be the polynomial ring over the complex number field C in variables $c_1, \bar{c}_1, \cdots, c_k, \bar{c}_k$ with dimensions $\dim c_i=\dim \bar{c}_i=2i$, $i=1, 2, \cdots, k$, I_{2q} the ideal generated by monomials whose dimensions are greater than $2q$. Denote the quotient ring $C[c_1, \cdots, \bar{c}_k]/I_{2q}$ by $C_q[c_1, \cdots, \bar{c}_k]$. Let $E_c(h_1, \cdots, h_k)$ be the exterior algebra over C generated by $h_1, \cdots, h_k$ with $\dim h_i=2i-1$, $i=1, 2, \cdots, k$. $WU(k)_q$ is a cochain complex

$$E_c(h_1, \cdots, h_k)\otimes C_q[c_1, \bar{c}_1, \cdots, c_k, \bar{c}_k]$$

with differential d_w given by

$$d_w c_i = d_w \bar{c}_i = 0 \qquad \text{and} \quad d_w h_i = (c_i - \bar{c}_i)/2\sqrt{-1}\,.$$

We denote the cohomology class of $c\in WU(k)_q$ by $[c]$.

Let $\mathcal{F}$ be a foliation of codimension q on M, V a complex $\mathcal{F}$-vector bundle over M with structure group $GL(k, C)$.

DEFINITION. Let D^1 be an $\mathcal{F}$-flat connection on V, D^0 a metric connection on V with respect to some Hermitian metric on V. A graded algebra homomorphism

$$\lambda_V : WU(k)_q \longrightarrow A^*(M)$$

is defined by requiring

$$\lambda_V(c_i) := \lambda(D^1)\tilde{c}_i\,, \qquad \lambda_V(\bar{c}_i) := \overline{\lambda(D^1)\tilde{c}_i}\,,$$

$$\lambda_V(h_i) := (\lambda(D^0, D^1)\tilde{c}_i - \overline{\lambda(D^0, D^1)\tilde{c}_i})/2\sqrt{-1}\,, \qquad 1\leqq i\leqq k\,,$$

where the invariant polynomials $\tilde{c}_i$ are given by

$$\det\left(I+\frac{\sqrt{-1}t}{2\pi}A\right)=1+\sum_{i=1}^{k} t^i\tilde{c}_i(A) \qquad \text{for} \quad A\in \mathfrak{gl}(k, C)\,.$$

LEMMA. $$d\lambda_V=\lambda_V d_w\,.$$

Proof. By Fact 5.1, the assertion is obvious for c_i and $\bar{c}_i$. By Fact 5.1 and the fact that

$$\overline{\tilde{c}_i(A)}=\tilde{c}_i(A) \qquad \text{for} \quad A\in\mathfrak{u}(k),$$

where $\mathfrak{u}(k)$ is the Lie algebra of $U(k)$, we get

$$\begin{aligned} d(\lambda_V(h_i)) &= (d(\lambda(D^0, D^1)\tilde{c}_i) - d\overline{(\lambda(D^0, D^1)\tilde{c}_i)})/2\sqrt{-1} \\ &= (\lambda(D^1)\tilde{c}_i - \overline{\lambda(D^1)\tilde{c}_i})/2\sqrt{-1} - (\lambda(D^0)\tilde{c}_i - \overline{\lambda(D^0)\tilde{c}_i})/2\sqrt{-1} \\ &= (\lambda_V(c_i) - \lambda_V(\bar{c}_i))/2\sqrt{-1} - 0 \\ &= \lambda_V((c_i - \bar{c}_i)/2\sqrt{-1}) = \lambda_V(d_w h_i). \end{aligned}$$

Q. E. D.

Then λ_V induces a homomorphism $\lambda_V{}^*$ from the cohomology of $WU(k)_q$ to the complex de Rham cohomology $H^*(M)$.

PROPOSITION 6.1. *$\lambda_V{}^*$: $H^*(WU(k)_q)\to H^*(M)$ is independent of the choices of metrics, metric connections and $\mathcal{F}$-flat connections on the $\mathcal{F}$-vector bundle V.*

Proof. Let g_0 (resp. g_1) be a metric on V, $D_0{}^0$ (resp. $D_1{}^0$) a metric connection with respect to g_0 (resp. g_1), and $D_0{}^1$, $D_1{}^1$ $\mathcal{F}$-flat connections on V. Let $p: M\times R\to M$ be the projection, $\hat{\mathcal{F}}$ the canonical foliation of codimension q on $M\times R$. In this situation, there exist the following objects:

$\hat{D}^1$: the $\hat{\mathcal{F}}$-flat connection on the induced $\hat{\mathcal{F}}$-vector bundle p^*V such that $j_s{}^*\hat{D}^1=sD_1{}^1+(1-s)D_0{}^1$,

$\hat{g}$: a metric on p^*V such that $j_0{}^*\hat{g}=g_0$ and $j_1{}^*\hat{g}=g_1$,

$\hat{D}^0$: a metric connection on p^*V with respect to $\hat{g}$ such that $j_0{}^*\hat{D}^0=D_0{}^0$ and $j_1{}^*\hat{D}^0=D_1{}^0$ (see [8]).

It is sufficient that we show the assertion on the indicated h's. For $s=0, 1$, we have

$$\begin{aligned} j_s{}^*(\lambda(\hat{D}^0, \hat{D}^1)\tilde{c}_i) &= i\int_0^1 \tilde{c}_i(j_s{}^*\theta^1 - j_s{}^*\theta^0, j_s{}^*\Theta^t, \cdots, j_s{}^*\Theta^t)dt \\ &= i\int_0^1 \tilde{c}_i(\theta_s{}^1-\theta_s{}^0, \Theta_s{}^t, \cdots, \Theta_s{}^t)dt = \lambda(D_s{}^0, D_s{}^1)\tilde{c}_i. \end{aligned}$$

Together with the fact that $j_0{}^*=j_1{}^*$: $H^*(M\times R)\to H^*(M)$, the proof completes.

Q. E. D.

We will call the elements of $\lambda_V{}^*(H^*(WU(k)_q)-[C_q[c_1, \cdots, \bar{c}_k]])$ the exotic characteristic classes of V by analogy with [1]. By Theorem 4.6, we get

PROPOSITION 6.2. *If the structure group $GL(k, C)$ of an $\mathcal{F}$-vector bundle V is $\mathcal{F}$-reducible to $U(k)$, then all of the exotic characteristic classes of V vanish, that is,* Im $(\lambda_V{}^*)=$Chern$^*(V)$.

In other words, the elements of the exotic characteristic classes are obstructions to existence of $\mathcal{F}$-reduction to the maximal compact subgroup $U(k)$.

Following to the method of Vey in [5], we get

PROPOSITION 6.3. *Putting* $u_i^- := (c_i - \bar{c}_i)/2\sqrt{-1}$ *and* $u_i^+ := (c_i + \bar{c}_i)/2$ *in* $C[c_1, \cdots, \bar{c}_k]$, *we have* $C_q[c_1, \cdots, \bar{c}_k] = C_q[u_1^-, u_1^+, \cdots, u_k^-, u_k^+]$ *and* $d_w u_i^- = d_w u_i^+ = 0$, $d_w h_i = u_i^-$. *A basis for* $H^*(WU(k)_q)$ *is given by the following two types of classes of cocycles.*

(A):
$$h_{i_1} \wedge \cdots \wedge h_{i_l} \otimes u_{j_1}^- \cdots u_{j_m}^- \cdot u_{s_1}^+ \cdots u_{s_t}^+,$$

where $1 \leqq i_1 < \cdots < i_l \leqq k$, $1 \leqq j_1 \leqq \cdots \leqq j_m \leqq \min(q, k)$ *and*
$1 \leqq s_1 \leqq \cdots \leqq s_t \leqq \min(q, k)$ *satisfying*

1) $i_1 + j_1 + \cdots + j_m + s_1 + \cdots + s_t > q$ *(cocycle) and*
2) $i_1 \leqq j_1$ *(independency)*.

(B):
$$u_{s_1}^+ \cdots u_{s_t}^+,$$

where $1 \leqq s_1 \leqq \cdots \leqq s_t \leqq \min(q, k)$.

The classes of type (A) (resp. (B)) correspond to exotic (resp. Chern) characteristic classes.

For $q' \geqq q$, the canonical projection induces the cochain homomorphism

$$\rho_q^{q'}: WU(k)_{q'} \longrightarrow WU(k)_q.$$

Let $\mathcal{F}'$ be a foliation of codimension q' on M', V' be an $\mathcal{F}'$-vector bundle over M' with structure group $GL(k, C)$. By Proposition 4.4, we have

PROPOSITION 6.4. *If* $f: M \to M'$ *is an* $(\mathcal{F}, \mathcal{F}')$*-map, then the following diagram is commutative*:

$$\begin{array}{ccc} H^*(WU(k)_q) & \longleftarrow & H^*(WU(k)_{q'}) \\ (\lambda_{f^* V'})^* \downarrow & (\rho_q^{q'})^* & \downarrow (\lambda_{V'})^* \\ H^*(M) & \underset{f^*}{\longleftarrow} & H^*(M') \end{array}$$

COROLLARY. *If two* $\mathcal{F}$*-vector bundles* V *and* V' *over* M *with structure group* $GL(k, C)$ *are* $\mathcal{F}$*-isomorphic, then*

$$\lambda_V^* = \lambda_{V'}^* \quad on \quad H^*(WU(k)_q).$$

By an analogous proof to that of Proposition 6.1, we have

PROPOSITION 6.5. *If two* $\mathcal{F}$*-vector bundles* V *and* V' *over* M *with structure group* $GL(k, C)$ *are* $\mathcal{F}$*-homotopic, then*

$$\lambda_V^* = \lambda_{V'}^* \quad on \quad \mathrm{Im}\,(\rho_q^{q+1})^*.$$

This proposition and Proposition 2.3 imply

COROLLARY. *If two* $(\mathcal{F}, \mathcal{F}')$*-maps* $f_0, f_1: M\to M'$ *are* $(\mathcal{F}, \mathcal{F}')$*-homotopic and* V' *is an* $\mathcal{F}'$*-vector bundle over* M' *with structure group* $GL(k, C)$, *then*

$$(\lambda_{f_0{}^*V'})^*=(\lambda_{f_1{}^*V'})^* \qquad on \quad \operatorname{Im}(\rho_q{}^{q+1})^* .$$

We will call the elements of $\lambda_V{}^*\operatorname{Im}(\rho_q{}^{q+1})^*$ the rigid classes of V by analogy with [5]. It holds that

$$\operatorname{Chern}^*(V)\subset\lambda_V{}^*\operatorname{Im}(\rho_q{}^{q+1})^*\subset\lambda_V{}^*H^*(WU(k)_q),$$

where they are invariants of V relative to $\mathcal{O}$-isomorphism, $\mathcal{F}$-homotopy and $\mathcal{F}$-isomorphism respectively. We will call the elements of $\lambda_V{}^*(H^*(WU(k)_q)-\operatorname{Im}(\rho_q{}^{q+1})^*)$ the non-rigid exotic characteristic classes of V. We get easily

PROPOSITION 6.6. *The elements of* $H^*(WU(k)_q)-\operatorname{Im}(\rho_p{}^{q+1})^*$ *are given by the classes of the cocycles of type* (A) *in Proposition* 6.3 *with*

$$i_1+j_1+\cdots+j_m+s_1+\cdots+s_t=q+1 .$$

Let V be a real $\mathcal{F}$-vector bundle. We denote $\lambda_{V\otimes C}$ by λ_V for simplicity, where $V\otimes C$ is the complexification of V. By direct calculations, we have

PROPOSITION 6.7. *If* D^0 *(resp.* D^1*) is the connection on* $V\otimes C$ *which is extended from a metric (resp.* $\mathcal{F}$*-flat) connection on* V *by linearlity, then*

$$\lambda_V u_i{}^-=(-1)^{(i+1)/2}\lambda(D^1)\tilde{c}_i{}^R \qquad for \quad i=odd,$$

$$\lambda_V h_i=(-1)^{(i+1)/2}\lambda(D^0, D^1)\tilde{c}_i{}^R \qquad for \quad i=odd,$$

$$\lambda_V u_j{}^+=(-1)^{j/2}\lambda(D^1)\tilde{c}_j{}^R \qquad for \quad j=even, \ and$$

$$\lambda_V=0 \qquad for\ other\ generators \quad u_j{}^-,\ h_j,\ u_i{}^+,$$

where invariant polynomials $\tilde{c}_i{}^R$ *are given by*

$$\det\left(I-\frac{t}{2\pi}A\right)=1+\sum_{i=1}^{k}t^i\tilde{c}_i{}^R(A) \qquad for \quad A\in\mathfrak{gl}(k, C).$$

This proposition suggests that we might deal with only the following cochain complex for real $\mathcal{F}$-vector bundles.

DEFINITION. Let l be the largest odd integer $\leqq k$. $WO(k)_q$ is a cochain complex $E_c(h_1, h_3, \cdots, h_l)\otimes C_q[u_1{}^-, u_2{}^+, u_3{}^-, u_4{}^+, \cdots, u_l{}^-, (u_{l+1}{}^+)]$ with d_w given by $d_w u_j{}^+=d_w u_i{}^-=0$ and $d_w h_i=u_i{}^-$.

If $k=q$, then the cochain complex $WO(q)_q$ coincides with $WO_q\otimes C$ which is the complexification of WO_q in [1].

EXAMPLE 6.1. For $q\geqq1$ and $k\geqq1$, we have

$$d_W(h_1\otimes(u_1{}^-)^q)=(u_1{}^-)^{q+1}=0 \qquad \text{in} \quad WO(k)_q .$$

Let $\mathcal{F}$ be a foliation of codimension q on M, V a real $\mathcal{F}$-vector bundle over M with structure group $GL(k, R)$. The resulting cohomology class $\lambda_V^*[h_1\otimes(u_1^-)^q]$ $\in H^{2q+1}(M)$ is called the generalized Godbillon-Vey invariant of V (cf. [7]). If $\mathcal{F}$ is a foliation of codimension 1, then $\lambda_{T/F}^*[h_1\otimes u_1^-]$ coincides with the Godbillon-Vey invariant [1]. Proposition 6.6 shows that generalized Godbillon-Vey invariants are non-rigid exotic characteristic classes.

EXAMPLE 6.2. A vector bundle V with a flat connection D has the canonical $\mathcal{I}$-vector bundle structure. Moreover Proposition 4.3 guarantees the existence of one-one correspondence between flat connections on V and $\mathcal{I}$-vector bundle structures on V. Denote by V_D the corresponding $\mathcal{I}$-vector bundle to a flat connection D. As $\mathcal{I}$ is the foliation of codimension 0, we may choose the following:

$$WU(k)_0 = E_c(h_1, \cdots, h_k) \qquad \text{with} \quad d_w h_i = 0\,,$$

then we have an isomorphism $H^*(WU(k)_0)\cong WU(K)_0$. If D and D' are flat connections on V such that V_D and $V_{D'}$ are $\mathcal{I}$-homotopic, then

$$\lambda_{V_D}{}^*[h_i] = \lambda_{V_{D'}}{}^*[h_i] \qquad \text{for} \quad i \geqq 2\,,$$

that is, h_i is rigid for $i\geqq 2$. This fact follows from Proposition 6.6.

§ 7. Exotic characteristic classes of $\mathcal{F}_c$-vector bundles

Let $\mathcal{F}_c$ be a complex analytic foliation of complex codimension q on M, V_c an $\mathcal{F}_c$-vector bundle over M with structure group $GL(k, C)$. We denote $\mathcal{F}$ the underlying smooth foliation of codimension $2q$, and by V the underlying $\mathcal{F}$-vector bundle of V_c.

DEFINITION. Let $C[c_1, \bar{c}_1, \cdots, c_k, \bar{c}_k]$ and $E_c(h_1, \cdots, h_k)$ be the same as in the definition of $WU(k)_q$ in § 6. And let $I_{2q,2q}$ be the ideal in $C[c_1, \cdots, \bar{c}_k]$ generated by monomials in variables c's and monomials in variables $\bar{c}$'s whose dimensions are greater than $2q$. $WU(k)_q'$ is a cochain complex

$$E_c(h_1, \cdots, h_k)\otimes(C[c_1, \bar{c}_1, \cdots, c_k, \bar{c}_k]/I_{2q,2q})$$

with the same differential d_w as that of $WU(k)_q$.

Especially $WU(q)_q'$ coincides with WU_q in [2]. Replacing the $\mathcal{F}$-flat connection D^1 in the definition of λ_V in § 6 by an $\mathcal{F}_c$-flat connection, we define the graded algebra homomorphism

$$\lambda_{V_c}': WU(k)_q' \longrightarrow A^*(M)\,,$$

which induces the homomorphism $(\lambda_{V_c}')^*: H^*(WU(k)_q')\to H^*(M)$.

PROPOSITION 7.1. *$(\lambda_{V_c}')^*$ is independent of the choices of metrics, metic connections and $\mathcal{F}_c$-flat connections on V_c.*

The proof is analogous to that of Proposition 6.1. Moreover we can prove that $(\lambda_{V_c}')^*$ depends only on the $\mathcal{F}_c$-isomorphism class of V_c, using the complex analytic version of Propositions 4.4 and 6.4.

As the ideal I_{4q} is contained in $I_{2q,2q}$, there is the canonical projection

$$\rho : WU(k)_{2q} \longrightarrow WU(k)_q' ,$$

which induces the homomorphism $\rho^* : H^*(WU(k)_{2q}) \rightarrow H^*(WU(k)_q')$.

PROPOSITION 7.2. *The following diagram is commutative*:

$$\begin{array}{ccc} H^*(WU(k)_{2q}) & \xrightarrow{\lambda_V^*} & H^*(M) \\ \downarrow{\rho^*} & \nearrow{(\lambda_{V_c}')^*} & \\ H^*(WU(k)_q') & & \end{array}$$

COROLLARY. *If an $\mathcal{F}$-vector bundle V over M has a structure of an $\mathcal{F}_c$-vector bundle, then*

$$\lambda_V^* u = 0 \qquad \textit{for} \quad u \in \operatorname{Ker}(\rho^*)(\subset H^*(WU(k)_{2q})) .$$

This corollary is a generalization of the corollary of Proposition 5.3. Because we have

$$\lambda_V^*(\operatorname{Ker} \rho^*) \supset \bigcup_{4q \geqq r > 2q} \operatorname{Chern}^r(V)$$

Let $\mathcal{F}$ be an almost complex foliation of real codimension $2q$, T/F the normal bundle witn structure group $GL(q, C)$, then the elements of $\lambda_{T/F}^*(\operatorname{Ker} \rho^*)$ $(\subset H^*(M))$ are regarded as obstructions to integrability of the transversal almost complex structure of $\mathcal{F}$.

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