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ON THE TRIDEGREE OF FORMS ON f-MANIFOLDS

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1. Introduction

The motivation for the material presented here is twofold. Firstly, the remarkable correspondence between Kaehlerian and Sasakian geometries has, up to now, been lacking a result analogous to the fact that a compact Kaehlerian manifold of complex dimension n and having strictly positive sectional curvatures cannot carry a non-trivial harmonic r-form of bidegree $(r, 0)$ for $1 \leqq r \leqq n$. This situation is examined and resolved in § 4 and § 5.

Secondly, several authors have used a definition of "tridegree" which is, in fact, not well-defined, yielding possible difficulties. A proper definition is made in § 3, some consequences discussed in § 6 and the above-mentioned problems clarified in § 7.

2. Preliminaries

Let M be a smooth manifold. An f-structure on M is a smooth linear transformation field $f \neq 0$ satisfying

$$f^3 + f = 0,$$

and having constant (necessarily even) rank $2n$([11, 12]). A smooth manifold carrying an f-structure of rank $2n$ will be termed a *$(2n+s)$-dimensional f-manifold.* If $\mathfrak{X}(M)$ is the space of smooth vector fields on M, complementary projection operators l and m are defined by

$$l = -f^2, \qquad m = f^2 + I,$$

yielding complementary distributions $L = l(\mathfrak{X}(M))$, $\mathcal{M} = m(\mathfrak{X}(M))$ where L has dimension $2n = \operatorname{rank} f$. An f-structure on M is *integrable* ([5]) if the Nijenhuis torsion of f vanishes (see § 6).

A $(2n+s)$-dimensional f-manifold M has a *complemented framing* if there exist s vector fields E_α and s 1-forms η^α, $\alpha = 1, \cdots, s$, satisfying

$$\eta^\alpha(E_\beta) = \delta^\alpha_\beta \quad \text{and} \quad m = \eta^\alpha \otimes E_\alpha$$

(δ^α_β is the "Kronecker delta" and here, as in the sequel, the summation convention is employed). With such a complemented framing, M becomes a *framed*

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f-manifold. A framed f-manifold is *normal* if

$$[f, f]+d\eta^{\alpha}\otimes E_{\alpha}=0$$

where $[f, f]$ if the Nijenhuis torsion of f (see § 6).

A framed f-manifold M becomes a *metric framed f-manifold* if there exists a Riemannian structure g satisfying

$$g(fX, fY)=g(X, Y)-\sum_{\alpha}\eta^{\alpha}(X)\eta^{\alpha}(Y)$$

for all X, Y in $\mathcal{X}(M)$. It readily follows that L and $\mathcal{M}$ are orthogonal distributions and $\{E_{\alpha}(x)\}$ is an orthonormal basis of $\mathcal{M}(x)$ for all x in M. The *fundamental 2-form* F of a metric framed f-manifold is given by

$$F(X, Y)=2g(X, fY).$$

Finally, an operator $\tilde{F}$ on the space $\Lambda(M)$ of forms on an f-manifold M is given by

$$(\tilde{F}\alpha)(X_{1}, \cdots, X_{p})=\sum_{i=1}^{p}\alpha(X_{1}, \cdots, fX_{i}, \cdots, X_{p}),$$

for α a p-form $(\alpha\varepsilon\Lambda_{p}(M))$ and $X_{1}, \cdots, X_{p}$ in $\mathcal{X}(M)$.

A *Sasakian manifold* is a normal metric framed $(2n+1)$-dimensional f-manifold with f-structure ϕ of rank $2n$ and fundamental 2-form Φ which satisfies

$$d\eta=\Phi$$

where (E, η) gives the complemented framing. If x is any point of a Sasakian manifold M, a *ϕ-frame* at x is an orthonormal basis $\{X_{1}, \cdots, X_{2n+1}\}$ of $T_{x}(M)$, the tangent space at x, where

$$X_{2n+1}=X_{\Delta}=E(x),$$
$$X_{i^*}=X_{i+n}=\phi X_{i}, \qquad 1\leqq i\leqq n.$$

In the sequel, when giving the components of tensors and forms with respect to a ϕ-frame, indices $A, B, C, \cdots$ take values in $\{1, 2, \cdots, 2n, 2n+1=\Delta\}$, $\alpha, \beta, \gamma, \cdots$ values in $\{1, 2, \cdots, 2n\}$ and $i, j, k, \cdots$ values in $\{1, 2, \cdots, n\}$.

In a later section, we shall need the following known relations ([7], [8]) satisfied by the components of the Riemannian curvature tensor of a Sasakian manifold with respect to an arbitrary ϕ-frame:

$$\begin{aligned}(2.1)\qquad & R_{ijk*l}=-R_{ijkl*}=R_{ij*k*l*},\\ & R_{i*jk*l*}=R_{ij*k*l*}=-R_{i*jkl*}+(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}),\\ & R_{ijkl}=R_{i*j*kl*}=R_{ijk*l*}+(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}),\\ & R_{\Delta\alpha\Delta\beta}=\delta_{\alpha\beta},\ 1\leqq i, j, k, l\leqq n,\ 1\leqq\alpha, \beta\leqq 2n,\ i*=i+n, \text{ etc}.\end{aligned}$$

The complexification of a real vector space V will be denoted by V^{c} and, if f is a linear operator on V, we denote by f the extension to V^{c}.

3. The trigraded structure in $\Lambda(M)^c$

Because the results of this section closely parallel a corresponding section in [3], few details will be given.

Let M be a $(2n+s)$-dimensional f-manifold. The linear transformation field f, acting as an operator on $\mathfrak{X}(M)^c$, has eigenvalues 0, i and $-i$ with corresponding eigenspaces $\mathfrak{X}_0(M)$, $\mathfrak{X}_i(M)$ and $\mathfrak{X}_{-i}(M)$, respectively, yielding the direct sum decomposition

$$\mathfrak{X}(M)^c=\mathfrak{X}_0(M)\oplus\mathfrak{X}_i(M)\oplus\mathfrak{X}_{-i}(M)\,.$$

Clearly, $\mathfrak{X}_0(M)=\mathcal{M}^c$ and $\mathfrak{X}_i(M)\oplus X_{-i}(M)=L^c$, where $\mathcal{M}$ and L are the distributions given is § 2. The projection operators corresponding to eigenvalues 0, i and $-i$ are P_0, P and $\bar{P}$, respectively, and are given by $P_0=m$, $P=\frac{1}{2}(l-if)$, $\bar{P}=\frac{1}{2}(l+if)$.

On $\Lambda_p(M)^c$ we defined operators $\Pi_{\lambda,\mu,\nu}$, $0\leqq\lambda$, $\mu\leqq n$, $0\leqq\nu\leqq s$, $\lambda+\mu+\nu=p$, by

$$\begin{aligned}&(\Pi_{\lambda,\mu,\nu}\alpha)(X_1,\cdots,X_p)\\&=\frac{1}{\lambda!\,\mu!\,\nu!}\sum_{\alpha\varepsilon s_p}(\operatorname{sgn}\sigma)\alpha(PX_{\sigma(1)},\cdots,PX_{\sigma(\lambda)},\bar{P}X_{\sigma(\lambda+1)},\cdots,\bar{P}X_{\sigma(\lambda+\mu)},\\&\qquad P_0X_{\sigma(\lambda+\mu+1)},\cdots,P_0X_{\sigma(p)})\end{aligned}$$

for arbitrary $\alpha\varepsilon\Lambda_p(M)^c$, $X_1,\cdots,X_p\in\mathfrak{X}(M)^c$, where S_p is the group of all permutations on $\{1,\cdots,p\}$. If deg $\alpha\neq\lambda+\mu+\nu$, we set $\Pi_{\lambda,\mu,\nu}\alpha=0$, thereby defining the operators $\Pi_{\lambda,\mu,\nu}$ on all of $\Lambda(M)^c$. It is easy to check that these operators are projections, that is,

$$\Pi_{\lambda,\mu,\nu}\circ\Pi_{\lambda',\mu',\nu'}=\delta_{\lambda\lambda'}\delta_{\mu\mu'}\delta_{\nu\nu'}\Pi_{\lambda,\mu,\nu}\,,$$

$$\sum_{\substack{0\leqq\lambda,\,\mu\leqq n\\0\leqq\nu\leqq s}}\Pi_{\lambda,\mu,\nu}=I\,.$$

Setting $\Lambda_{\lambda,\mu,\nu}(M)=\Pi_{\lambda,\mu,\nu}\Lambda(M)^c$, we get

$$\Lambda(M)^c=\sum_{\substack{0\leqq\lambda,\,\mu\leqq n\\0\leqq\nu\leqq s}}\Lambda_{\lambda,\mu,\nu}(M)\qquad\text{(direct sum)}\,.$$

Furthermore, if $\alpha\varepsilon\Lambda_{\lambda,\mu,\nu}(M)$ and $\beta\varepsilon\Lambda_{\lambda',\mu',\nu'}(M)$, it is easily checked that $\alpha\wedge\beta\in\Lambda_{\lambda+\lambda',\mu+\mu',\nu+\nu'}(M)$, where $\Lambda_{\lambda,\mu,\nu}(M)=\{0\}$ if $\lambda>n$, $\mu>n$ or $\nu>s$. Thus we have

THEOREM 3.1 *In a $(2n+s)$-dimensional f-manifold M, the algebra $\Lambda(M)^c$ of complex-valued forms on M carries a trigraded structure with two grades of degree n and one of degree s.*

Remarks. The case of $s=0$, an almost complex manifold, is well known (see, for example, [6, chapter 3]). The case of $s=1$, an almost contact manifold, was considered by Fujitani ([3]) who considered manifolds with more structure, but, in this context, only the almost contact structure is required. Finally, note that for a framed f-manifold we can write $P=\frac{1}{2}\{I-\eta^{\alpha}\otimes E_{\alpha}-if\}$, $\bar{P}=\frac{1}{2}\{I-\eta^{\alpha}\otimes E_{\alpha}+if\}$ and $P_0=\eta^{\alpha}\otimes E_{\alpha}$.

In the next section we will require the following

LEMMA 3.2 *If $\alpha\in\Lambda_{\lambda,\mu,\nu}(M)$, then $\tilde{F}\alpha=i(\lambda-\mu)\alpha$.*

Proof. Let $\alpha\in\Lambda_{\lambda,\mu,\nu}(M)$. Then $\Pi_{\lambda,\mu,\nu}\alpha=\alpha$. For arbitrary

$$X_1, \cdots, X_p\in\mathfrak{X}(M)^c, \qquad p=\lambda+\mu+\nu,$$

$$\begin{aligned}
&(\Pi_{\lambda,\mu,\nu}\tilde{F}\alpha)(X_1, \cdots, X_p)\\
&=\frac{1}{\lambda!\,\mu!\,\nu!}\sum_{\sigma\in S_p}(\operatorname{sgn}\sigma)\Big[\sum_{k=1}^{\lambda}\alpha(PX_{\sigma(1)}, \cdots, fPX_{\sigma(k)}, \cdots, PX_{\sigma(\lambda)},\\
&\qquad\qquad \bar{P}X_{\sigma(\lambda+1)}, \cdots, P_0X_{\sigma(p)})\\
&\quad+\sum_{k=\lambda+1}^{\lambda+\mu}\alpha(PX_{\sigma(1)}, \cdots, PX_{\sigma(\lambda)}, \bar{P}X_{\sigma(\lambda+1)}, \cdots, f\bar{P}X_{\sigma(k)}, \cdots,\\
&\qquad\qquad \bar{P}X_{\sigma(\lambda+\mu)}, P_0X_{\sigma(\lambda+\mu+1)}, \cdots, P_0X_{\sigma(p)})\\
&\quad+\sum_{k=\lambda+\mu+1}^{p}\alpha(PX_{\sigma(1)}, \cdots, \bar{P}X_{\sigma(\lambda+\mu)}, P_0X_{\sigma(\lambda+\mu+1)}, \cdots,\\
&\qquad\qquad fP_0X_{\sigma(k)}, \cdots, P_0X_{\sigma(p)})\Big]\\
&=\frac{1}{\lambda!\,\mu!\,\nu!}\sum_{\sigma\in S_p}(\operatorname{sgn}\sigma)\Big[\sum_{k=1}^{\lambda}\alpha(PX_{\sigma(1)}, \cdots, iPX_{\sigma(k)}, \cdots,\\
&\qquad\qquad PX_{\sigma(\lambda)}, \bar{P}X_{\sigma(\lambda+1)}, \cdots, P_0X_{\sigma(p)})\\
&\quad+\sum_{k=\lambda+1}^{\lambda+n}\alpha(PX_{\sigma(1)}, \cdots, PX_{\sigma(\lambda)}, \bar{P}X_{\sigma(\lambda+1)}, \cdots, -i\bar{P}X_{\sigma(k)}, \cdots,\\
&\qquad\qquad \bar{P}X_{\sigma(\lambda+\mu)}, P_0X_{\sigma(\lambda+\mu+1)}, \cdots, P_0X_{\sigma(p)})\Big]\\
&=i(\lambda-\mu)(\Pi_{\lambda,\mu,\nu}\alpha)(X_1, \cdots, X_p)\\
&=i(\lambda-\mu)\alpha(X_1, \cdots, X_p).
\end{aligned}$$

A similar calculation yields

$$(\Pi_{\lambda',\mu',\nu'}\tilde{F}\alpha)(X_1, \cdots, X_p)=i(\lambda'-\mu')(\Pi_{\lambda',\mu',\nu'}\alpha)(X_1, \cdots, X_p)=0$$

for $(\lambda', \mu', \nu')\neq(\lambda, \mu, \nu)$, $\lambda'+\mu'+\nu'=p$, proving the lemma.

4. The semidegree of a form on M

In order to get something corresponding to a trigrading in $\Lambda(M)$, the last section does not provide entirely satisfactory answers since, clearly, $\Pi_{\lambda,\mu,\nu}$ is a real operator only when $\lambda=\mu$. Partial answers are available, however. To begin, the projections l and m on $X(M)$ can yield a bigrading. Operators $\Pi_{\lambda;\mu}$.

$0\leqq\lambda\leqq 2n$, $0\leqq\nu\leqq s$, $\lambda+\nu=p$, are defined on $\Lambda_p(M)$ by

$$(\Pi_{\lambda,\nu}\alpha)(X_1,\cdots,X_p)=\frac{1}{\lambda!\nu!}\sum_{\sigma\in S_p}(\operatorname{sgn}\sigma)\alpha(lX_{\sigma(1)},\cdots,lX_{\sigma(\lambda)},$$
$$mX_{\sigma(\lambda+1)},\cdots,mX_{\sigma(p)}),$$

for arbitrary $\alpha\in\Lambda_p(M)$, $X_1,\cdots,X_p\in\mathfrak{X}(M)$. If deg $\alpha\neq\lambda+\nu$, we set $\Pi_{\lambda;\nu}\alpha=0$, defining $\Pi_{\lambda;\nu}$ on all of $\Lambda(M)$. Again it is easily verified that these operators are projections. Thus, setting $\Lambda_{\lambda;\mu}(M)=\Pi_{\lambda,\mu}\Lambda(M)$, we get

$$\Lambda(M)=\sum_{\substack{0\leqq\lambda\leqq 2n\\0\leqq\nu\leqq s}}\Lambda_{\lambda,\mu}(M)\qquad\text{(direct sum)}.$$

Again, if $\alpha\in\Lambda_{\lambda;\nu}(M)$ and $\beta\in\Lambda_{\lambda';\nu'}(M)$, then $\alpha\wedge\beta\in\Lambda_{\lambda+\lambda';\nu+\nu'}(M)$. Thus we have

THEOREM 4.1 *On a $(2n+s)$-dimensional f-manifold M, the algebra $\Lambda(M)$ of forms on M carries a bigraded structure with one grade of degree $2n$ and one of degree s.*

As operators on $\mathfrak{X}(M)^c$, we clearly have $l=P+\bar{P}$, $m=P_0$, and hence $\Pi_{\lambda,\nu}=\sum_{\lambda'+\mu'=\nu}\Pi_{\lambda',\mu',\nu}$ and

$$\Lambda_{\lambda;\nu}(M)^c=\sum_{\lambda'+\mu'=\lambda}\Lambda_{\lambda',\mu',\nu}(M)\qquad\text{(direct sum)}.$$

To further decompose $\Lambda(M)$, we can use Lemma 3.2 and the fact that $\Lambda_{\lambda;\nu}(M)\subset\sum_{\lambda'+\mu'=\lambda}\Lambda_{\lambda',\mu',\nu}(M)$. Each $\Lambda_{\lambda',\mu',\nu'}(M)$ is a space of eigenvectors of $\tilde{F}$ with eigenvalue $i(\lambda'-\mu')$. Thus $\Lambda_{\lambda',\mu',\nu}(M)$ consists of eigenvectors of $\tilde{F}^2$ with real eigenvalues $-(\lambda'-\mu')^2$. Hence $\tilde{F}^2$ has only real eigenvalues when operating on $\Lambda_{\lambda;\mu}(M)$, namely, $-\mu^2$ where $\mu\in\{\lambda,\lambda-2,\cdots,\varepsilon(\lambda)\}$, where $\varepsilon(\lambda)=0$ or 1 and $\varepsilon(\lambda)\equiv\lambda(\text{mod }2)$. If $\alpha\in\Lambda_{\lambda;\nu}(M)$ and $\tilde{F}^2\alpha=-\mu^2\alpha$, we shall say that α has *semidegree* μ. If we denote by $\Lambda_{\lambda(\mu);\nu}(M)$ the space of all vectors in $\Lambda_{\lambda,\nu}(M)$ having semidegree μ, the above discussion yields

$$\Lambda_{\lambda,\nu}(M)=\sum_{k=0}^{(\lambda-\varepsilon(\lambda))/2}\Lambda_{\lambda(\varepsilon(\lambda)+k),\nu}(M)\qquad\text{(direct sum)}$$

Hence we have

THEOREM 4.2 *The algebra $\Lambda(M)$ of forms on an f-manifold M of dimension $2n+s$ can be decomposed into a direct sum $\sum_{\lambda,\mu,\nu}\Lambda_{\lambda(\mu),\nu}(M)$, where $0\leqq\lambda\leqq 2n$, $0\leqq\nu\leqq s$, $\mu\in\{\lambda,\lambda-2,\cdots,\varepsilon(\lambda)\}$. The subscripts λ and ν give the bigrading on $\Lambda(M)$ and $-\mu^2$ is an eigenvalue of the operator $\tilde{F}^2$.*

Remarks The simplest case where the notion of semidegree has any meaning is for forms in $\Lambda_{2;0}(M)$, where the semidegree could be 0 or 2. If $\alpha\in\Lambda_{2,0}(M)$, an easy calculation yields

$$(\tilde{F}^2\alpha)(X,Y)=-2\alpha(X,Y)+2\alpha(fX,fY).$$

(In this calculation we use $\alpha(mX, Y)=\alpha(X, mY)=0$, since $\nu=0$). If α is of semidegree 2, then $\tilde{F}^2\alpha=-4\alpha$, and hence $\alpha(X, Y)+\alpha(fX, fY)=0$, or α is *pure.* If α is of semidegree 0, then $\tilde{F}^2\alpha=0$ and hence $\alpha(X, Y)-\alpha(fX, fY)=0$, or α is *hybrid.* Thus semidegree generalizes the notions of purity and hybridicity of 2-forms. We note that it is known that the 2-form F is hybrid and hence that it is in $\Lambda_{2(0),0}(M)$.

In our application of the notion of semidegree we shall want a characterization of forms in $\Lambda_{r(r),0}(M)$. Let $\alpha\in\Lambda_{r,0}(M)$, $X, Y\in\mathcal{X}(M)$. We denote by $\alpha_{X,Y}$ the $(r-2)$-form $(\iota_X\iota_Y+\iota_{fX}\iota_{fY})\alpha$.

LEMMA 4.3 *Let $\alpha\in\Lambda_{r,0}(M)$, $r\geqq 2$, with $\tilde{F}^2\alpha=\lambda\alpha$, and let $X, Y\in\mathcal{X}(M)$. Then $\alpha_{X,Y}\in\Lambda_{r-2,0}(M)$, with $\tilde{F}^2\alpha_{X,Y}=\lambda\alpha_{X,Y}$*

Proof. We have $\alpha_{X,Y}\in\Lambda_{r-2}(M)$. To show $\alpha_{X,Y}\in\Lambda_{r-2,0}(M)$, we need only show that $\iota_E\alpha_{X,Y}=0$ for arbitrary $E\in\mathcal{M}$. But

$$\iota_E\alpha_{XY}=\iota_E(\iota_X\iota_Y+\iota_{fX}\iota_{fY})\alpha=(\iota_X\iota_Y+\iota_{fX}\iota_{fY})\iota_E\alpha=0\,,$$

since $\alpha\in\Lambda_{r;0}(M)$.

Now let $X_1, \cdots, X_{r-2}\in\mathcal{X}(M)$. Then

$$\begin{aligned}
&(\tilde{F}^2\alpha_{X,Y})(X_1, \cdots, X_{r-2})\\
&=\sum_{i=1}^{r-2}\alpha_{X,Y}(X_1, \cdots, f^2X_i, \cdots, X_{r-2})+2\sum_{1\leqq i\leqq j\leqq r-2}\alpha_{X,Y}(X_1, \cdots,\\
&\qquad\qquad fX_i, \cdots, fX_j, \cdots, X_{r-2})\\
&=-(r-2)\alpha_{X,Y}(X_1, \cdots, X_{r-2})+2\sum_{1\leqq i<j\leqq r-2}\alpha_{X,Y}(X_1, \cdots,\\
&\qquad\qquad fX_i, \cdots, fX_j, \cdots, X_{r-2})\\
&=-(r-2)\alpha(Y, X, X_1, \cdots, X_{r-2})-(r-2)\alpha(fY, fX, X_1, \cdots, X_{r-2})\\
&\quad+2\sum_{1\leqq i<j\leqq r-2}\alpha(Y, X, X_1, \cdots, fX_i, \cdots, fX_j, \cdots, X_{r-2})\\
&\quad+2\sum_{1\leqq i<j\leqq r-2}\alpha(fY, fX, X_1, \cdots, fX_i, \cdots, fX_j, \cdots, X_{r-2})\\
&=(\tilde{F}^2\alpha)(Y, X, X_1, \cdots, X_{r-2})+(\tilde{F}^2\alpha)(fY, fX, X_1, \cdots, X_{r-2})\\
&\quad+2\alpha(Y, X, X_1, \cdots, X_{r-2})+2\alpha(fY, fX, X_1, \cdots, X_{r-2})\\
&\quad-2\sum_{i=1}^{r-2}\alpha(fY, X, X_1, \cdots, fX_i, \cdots, X_{r-2})-2\sum_{i=1}^{r-2}\alpha(Y, fX, X_1, \cdots, fX_i, \cdots, X_{r-2})\\
&\quad-2\alpha(fY, fX, X_1, \cdots, X_{r-2})-2\sum_{i=1}^{r-2}\alpha(f^2Y, fX, X_1, \cdots, fX_i, \cdots, X_{r-2})\\
&\quad-2\sum_{i=1}^{r-2}\alpha(fY, f^2X, X_1, \cdots, fX_i, \cdots, X_{r-2})-2\alpha(f^2Y, f^2X, X_1, \cdots, X_{r-2})\\
&=(\tilde{F}^2\alpha)(Y, X, X_1, \cdots, X_{r-2})+(\tilde{F}^2\alpha)(fY, fX, X_1, \cdots, X_{r-2})
\end{aligned}$$

$$=\lambda\alpha(Y, X, X_1, \cdots, X_{r-2})+\lambda\alpha(fY, fX, X_1, \cdots, X_{r-2})$$

$$=\lambda\alpha_{X,Y}(X_1, \cdots, X_{r-2}),$$

where we have used the facts that $f^2=-I+m$ and $\alpha(Y_1, \cdots, Y_r)=0$ if any of the $Y_i\in\mathcal{M}$.

COROLLARY *Let* $\alpha\in\Lambda_{r(r),0}(M)$, $X, Y\in\mathfrak{X}(M)$. *Then* $\alpha_{X,Y}=0$.

Proof. From the lemma, $\tilde{F}^2\alpha=-r^2\alpha$ implies $\tilde{F}^2\alpha_{X,Y}=-r^2\alpha_{X,Y}$. But since $r\notin\{r-2, r-4, \cdots, \varepsilon(r-2)\}$, we must have $\alpha_{X,Y}=0$.

Note that $\alpha_{X,Y}=0$ says that α is pure in its first two variables and, since α is skew-symmetric, it is pure in any two variables.

5. Harmonic forms on compact Sasakian manifolds

Let M be a compact $(2n+1)$-dimensional Sasakian manifold with f-structure ϕ, complemented framing (E, η) and metric g. In studying harmonic forms on M, we have the following useful lemmas.

LEMMA 5.1 (Tachibana [9]). *Let α be a harmonic r-form, $r\leqq n$, on a compact $(2n+1)$-dimensional Sasakian manifold M. Then* $\iota_E\alpha=0$.

LEMMA 5.2 (Tachibana [9]). *Let α be a harmonic r-form, $r\leqq n$, on a compact $(2n+1)$-dimensional Sasakian manifold M. Then the r-form $\tilde{\Phi}\alpha$ is also harmonic.*

If now α is a harmonic r-form on M, $r\leqq n$, then $\alpha\in\Lambda_{r,0}(M)$. From the decomposition of $\Lambda_{r,0}(M)$ given by Theorem 4.2, we can uniquely write

$$\alpha=\alpha_r+\alpha_{r-2}+\cdots+\alpha_{\varepsilon(r)}$$

where $\alpha_s\in\Lambda_{r(s),0}(M)$, i.e., $\tilde{\Phi}^2\alpha_s=-s^2\alpha_s$. By uniqueness, the equations

$$\alpha=\alpha_r+\alpha_{r-2}+\cdots+\alpha_{\varepsilon(r)+2}+\alpha_{\varepsilon(r)}$$

$$\tilde{\Phi}^2\alpha=-r^2\alpha_r-(r-2)^2\alpha_{r-2}-\cdots-(\varepsilon(r)+2)^2\alpha_{\varepsilon(r)+2}-(\varepsilon(r))^2\alpha_{\varepsilon(r)}$$

$$\tilde{\Phi}^4\alpha=r^4\alpha_r+(r-2)^4\alpha_{r-2}+\cdots+(\varepsilon(r)+2)^4\alpha_{\varepsilon(r)+2}+(\varepsilon(r))^4\alpha_{\varepsilon(r)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(-1)^{(r-\varepsilon(r))/2}\tilde{\Phi}^{r-\varepsilon(r)}\alpha=r^{r-\varepsilon(r)}\alpha_r+(r-2)^{r-\varepsilon(r)}\alpha_{r-2}+\cdots+(\varepsilon(r))^{r-\varepsilon(r)}\alpha_{\varepsilon(r)}$$

can be solved to express each α_s as a linear combination of $\alpha, \tilde{\Phi}^2\alpha, \tilde{\Phi}^4\alpha, \cdots, \tilde{\Phi}^{r-\varepsilon(r)}\alpha$, each of which is harmonic, by Lemma 5.2. If we denote by $H_{r(s)}(M)$ the space of all harmonic forms of degree r and semidegree s on M, we then get the

PROPOSITION 5.3 *The space $H_r(M)$ of all harmonic r-forms, $r\leqq n$, on a compact Sasakian manifold M of dimension $2n+1$ can be decomposed as*

$$H_r(M)=\sum_{S\varepsilon\{r,r-2,\cdots,\varepsilon(r)\}}H_{r(s)}(M) \qquad \text{(direct sum)}$$

where $H_{r(s)}(M)$ is the space of harmonic r-forms of semidegree s.

We note that this generalizes the well-known decomposition of harmonic 2-forms on a Sasakian manifold ([7], [10]) and is the analogue of the decomposition of harmonic forms on Kaehler manifolds into harmonic forms of various bidegrees.

If we set $b_{r(s)}(M)=\dim H_{r(s)}(M)$ and $b_r(M)=\dim H_r(M)$(the r^{th} Betti number of M), we have the

COROLLARY 5.4 *With the same conditions on r and M as in Proposition* 5.3,

$$b_r(M)=b_{r(r)}(M)+b_{r(r-2)}(M)+\cdots+b_{r(\varepsilon(r))}(M).$$

For the rest of this section we shall concern ourselves with $H_{r(r)}(M)$ and $b_{r(r)}(M)$ for $2\leqq r\leqq n$, $\dim M=2n+1$. First we consider the familiar Bochner-Lichnerowicz form F_r on $\Lambda_r(M)$,

$$F_r(\alpha)_x=\sum_{\substack{A,B\\ C_1,\cdots,C_{r-1}}} R_{AB}\alpha_{AC_1\cdots C_{r-1}}\alpha_{BC_1\cdots C_{r-1}}$$
$$-\frac{1}{2}(r-1)\sum_{\substack{A,B,C,D\\ E_1,\cdots,E_{r-2}}} R_{ABCD}\alpha_{ABE_1\cdots E_{r-2}}\alpha_{CDE_1\cdots E_{r-2}},$$

where x is an arbitrary point of M and the components of the r-form α and of the Riemannian and Ricci curvature tensors are given with respect to an arbitrary orthonormal basis of $T_x(M)$.

LEMMA 5.5 *If $\alpha\in\Lambda_{r(r),0}(M)$, $r\geqq 2$, and x is an arbitrary point of M, there exists a ϕ-frame at x with respect to which*

$$F_r(\alpha)_x=(2-r)\sum_{\sigma_k}(\alpha_{\sigma_1\cdots\sigma_r})^2+2\sum_{i,\tau_k}(K_{ii^*}+\sum_{j\neq i}(K_{ij}+K_{ij^*}))(\alpha_{i\tau_1\cdots\tau_{r-1}})^2$$
$$=(2-r)\sum_{(\sigma)}\alpha_{(\sigma)}{}^2+2\sum_{i,(\tau)}(K_{ii^*}+\sum_{j\neq i}(K_{ij}+K_{ij^*}))\alpha_i{}^2{}_{(\tau)},$$

where K_{AB} denotes the sectional curvature of the section spanned by $\{X_A, X_B\}\subset\{X_\sigma, X_\Delta\}$, the required ϕ-frame at x.

Proof. With respect to an arbitrary ϕ-frame at x,

$$\sum_{\substack{A,B,\\ C_k}} R_{AB}\alpha_{AC_1\cdots C_{r-1}}\alpha_{BC_1\cdots C_{r-1}}=\sum_{\substack{\lambda,\mu,\\ (\sigma)}} R_{\lambda\mu}\alpha_{\lambda(\sigma)}\alpha_{\mu(\sigma)},$$

and

$$\sum_{\substack{A,B,C,D,\\ E_k}} R_{ABCD}\alpha_{ABE_1\cdots E_{r-2}}\alpha_{CDE_1\cdots E_{r-2}}=\sum_{\substack{\lambda,\mu,\nu,\sigma\\ (\tau)}} R_{\lambda\mu\nu\sigma}\alpha_{\lambda\mu(\tau)}\alpha_{\nu\sigma(\tau)} \tag{5.1}$$
$$=\sum_{\substack{i,j,k,l\\ (\tau)}}[R_{ijkl}\alpha_{ij(\tau)}\alpha_{kl(\tau)}+R_{ijkl^*}\alpha_{ij(\tau)}\alpha_{kl^*(\tau)}+R_{ijk^*l}\alpha_{ij(\tau)}\alpha_{k^*l(\tau)}$$
$$+\cdots+R_{i^*j^*k^*l^*}\alpha_{i^*j^*(\tau)}\alpha_{k^*l^*(\tau)}].$$

If $\alpha\in\Lambda_{r(r),0}(M)$, Corollary 4.4 implies that the components of α with respect to a ϕ-frame satisfy

$$\alpha_{i*j*(\tau)}=-\alpha_{ij(\tau)},\qquad \alpha_{i*j(\tau)}=\alpha_{ij*(\tau)}. \tag{5.2}$$

Using relations (2.1), (5.2), the symmetries of R_{ABCD} and the skew-symmetry of α, the right-hand side of (5.1) becomes

$$2\sum_{\substack{i,j,k,l,\\(\tau)}}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})(\alpha_{ij(\tau)}\alpha_{kl(\tau)}+\alpha_{ij*(\tau)}\alpha_{kl*(\tau)})=2\sum_{(\sigma)}\alpha_{(\sigma)}{}^{2}.$$

If R_1 is the Ricci mean curvature transformation of M, it is known that $R_1\phi X=\phi R_1X$. Hence there exists a ϕ-frame with respect to which the only non-zero components of R_1 are $R_{ii}=R_{i*i*}$ and $R_{\Delta\Delta}=2n$. Furthermore,

$$\begin{aligned}R_{ii}&=\sum_{A}R_{iAiA}\\&=K_{i\Delta}+K_{ii*}+\sum_{j\neq i}(K_{ij}+K_{ij*})\\&=1+K_{ii*}+\sum_{j\neq i}(K_{ij}+K_{ij*}).\end{aligned}$$

With respect to such at ϕ-frame at x, we get

$$\begin{aligned}F_r(\alpha)_x&=\sum_{i,(\tau)}(1+K_{ii*}+\sum_{j\neq i}(K_{ij}+K_{ij*}))(\alpha_{i}{}^{2}{}_{(\tau)}+\alpha^{2}_{i*(\tau)})-(r-1)\sum_{(\sigma)}\alpha^{2}_{(\sigma)}\\&=2\sum_{i,(\tau)}(1+K_{ii*}+\sum_{j\neq i}(K_{ij}+K_{ij*}))\alpha^{2}_{i(\tau)}-(r-1)\sum_{(\sigma)}\alpha^{2}_{(\sigma)}\\&=(2-r)\sum_{(\sigma)}\alpha_{(\sigma)}{}^{2}+2\sum_{i,(\tau)}(K_{ii*}+\sum_{j\neq i}(K_{ij}+K_{ij*}))\alpha^{2}_{i(\tau)},\end{aligned}$$

proving the lemma.

Now consider the effects of an L-homothety ([10]) $g\mapsto\bar g$ on M $\bar g=cg+(c^2-c)\eta\otimes\eta$, $\bar\phi=\phi$, $\bar\eta=c\eta$, $\bar E=c^{-1}E$, c a constant >0.

LEMMA 5.6 (Tanno [10]). *If α is a harmonic r-form on M, $r\leqq n$, and $g\mapsto\bar g$ is an L-homothety, then α is harmonic with respect to $\bar g$.*

LEMMA 5.7 *If $\alpha\in\Lambda_{r(s),q}(M)$ and $g\mapsto\bar g$ is an L-homothety, then α remains in $\Lambda_{r(s),q}(M)$.*

Proof. Under an L-homothety, ϕ remains unchanged, and hence $\tilde\Phi$ remains unchanged. Also the distributions L and M are invariant.

Under an L-homothety we have the following changes in sectional curvatures ([7], [10]). If $K_{\lambda\mu}$ denotes the sectional curvatures with respect to g of sections determined by vectors in a ϕ-frame $\{E(x), X_i, X_{i*}\}$ at x and, similarly, $\bar K_{\lambda\mu}$ with respect to $\bar g$ (and $\{E(x)=c^{-1}E(x), \bar X_i=c^{-1/2}X_i, X_{i*}=c^{-1/2}X_{i*}\}$), then

$$\bar K_{ii*}=c^{-1}\{K_{ii*}+3(1-c)\},\qquad i=1,\cdots,n,$$

$$\bar K_{ij}=c^{-1}K_{ij},\qquad i,j=1,\cdots,n,\qquad i\neq j,$$

$$\bar{K}_{ij*}=c^{-1}K_{ij*}, \qquad i, j=1, \cdots, n, \qquad i\neq j$$

With respect to $\bar{g}$, the Bochner-Lichnerowicz form $\bar{F}_r$, operating on α at x and using a ϕ-frame as in Lemma 5.5, thus takes the form

$$\begin{aligned}\bar{F}_r(\alpha)_x&=(2-r)\sum_{(\sigma)}\bar{\alpha}^2_{(\sigma)}+\sum_{i,(\tau)}(K_{ii*}+\sum_{j\neq i}(\bar{K}_{ij}+\bar{K}_{ij*}))\bar{\alpha}^2_{i(\tau)}\\&=(2-r)\sum_{(\sigma)}\bar{\alpha}^2_{(\sigma)}+2c^{-1}\sum_{i,(\tau)}(K_{ii*}+3(1-c)+\sum_{j\neq i}(K_{ij}+K_{ij*}))\bar{\alpha}^2_{i(\tau)}\\&=c^{-1}\{3-(1+r)c\}\sum_{(\sigma)}\bar{\alpha}^2_{(\sigma)}+2c^{-1}\sum_{i,(\tau)}\{K_{ii*}+\sum_{j\neq i}(K_{ij}+K_{ij*})\}\bar{\alpha}^2_{i(\tau)}.\end{aligned}$$

If there is a constant C such that $K_{ii*}+\sum_{j\neq i}(K_{ij}+K_{ij*})\geqq C$ for all i and all x, we get

$$\bar{F}_r(\alpha)_x\geqq c^{-1}\{3+(1-r)c+C\}\sum_{(\sigma)}\bar{\alpha}^2_{(\sigma)}$$

and, if $C>-3$, we can choose c to satisfy $0<c<\frac{3+C}{r+1}$, yielding $\bar{F}_r(\alpha)_x\geqq 0$, with equality only if $\alpha=0$. From this it follows that there are no harmonic r-forms of semidegree r on M. Thus we have

THEOREM 5.8 *Let M be a compact Sasakian manifold of dimension $2n+1\geqq 5$, and let $2\leqq r\leqq n$. Then under any of the following conditions on sectional curvature K, there are no harmonic r-forms of semidegree r on M:*

(a) *$K(X, Y)>\frac{-3}{2n-1}$ for any orthonormal vectors X, Y at any point,*

(b) *$K(X, \phi X)>-3$ for any unit vector $X\in L(x)$ at any point x and $K(X, Y)+K(X, \phi Y)>0$ for any orthonormal triple $\{X, Y, \phi Y\}$ at any point x*

(c) *$K(X, \phi X)>\frac{-3}{2n-1}$ for any unit vector $X\in L(x)$ at any point x and $K(X, Y)+K(X, \phi Y)>\frac{-6}{2n-1}$ for any orthonormal triple $\{X, Y, \phi Y\}$ at x,*

(d) *$R_1(X, X)=g(R_1X, X)>-2$ for any unit vector X.*

In particular, if M has strictly positive sectional curvatures, then $b_{r(r)}(M)=0$.

Proof. It is easily checked that under all conditions, if we take any ϕ-frame at any point of M, then $K_{ii*}+\sum_{j\neq i}(K_{ij}+K_{ij*})>-3$. The theorem then follows from the preceding discussion.

Remarks. The case $r=2$ was done by Tanno ([10]). Note that the theorem is the Sasakian analogue of the well-known result that a compact Kachlerian manifold can carry no harmonic r-forms of bidegree $(r, 0)$ if the manifold has strictly positive sectional curvatures and $0<r\leqq\dim_c M$.

6. Integrability and Normality of f-structures

Let M be a $(2n+s)$-dimensional framed f-manifold with framing $\{(E_\alpha, \eta^\alpha)|\alpha=1, \cdots, s\}$. The structure f is integrable if and only if $[f, f]=0$ ([5]), where

$$[f, f](X, Y)=[fX, fY]+f^2[X, Y]-f[fX, Y]-f[X, fY],$$

and is normal if and only if $[f, f]+d\eta^\alpha\otimes E_\alpha=0$. In this section we consider the consequences of these concepts on the decompositions of $\mathcal{X}(M)^c$ and $\Lambda(M)^c$.

THEOREM 6.1 *Let M be a $(2n+s)$-dimensional f-manifold. Then f is integrable if and only if the distributions $\mathcal{X}_i$, $\mathcal{X}_i\oplus\mathcal{X}_0$, $\mathcal{X}_i\oplus\mathcal{X}_{-i}$, $\mathcal{X}_0$ are all involutive.*

Proof. Suppose the f-structure is integrable. It is known ([5]) that L and $\mathcal{M}$ are both involutive, then. Hence $\mathcal{X}_i\oplus\mathcal{X}_{-i}=L^c$ and $\mathcal{X}_0=\mathcal{M}^c$ are involutive. If $X, Y\in\mathcal{X}_i$, then $fX=iX$, $fY=iY$, and

$$\begin{aligned}[f, f](X, Y)&=-[X, Y]-l[X, Y]-if[X, Y]-if[X, Y]\\ &=-2i\{f[X, Y]-i[X, Y]\},\end{aligned}$$

where we have used $m[X, Y]=0$ and hence $l[X, Y]=[X, Y]$. From $[f, f]=0$ it follows that $[X, Y]\in\mathcal{X}_i$. Similarly, if $X\in\mathcal{X}_i$, $Y\in\mathcal{X}_0$, then

$$fl[X, Y]=il[X, Y]$$

and $[X, Y]\in\mathcal{X}_i\oplus\mathcal{X}_0$. From this and the above, $\mathcal{X}_i\oplus\mathcal{X}_0$ is involutive.

Conversely, suppose the given distributions are involutive. Trivially, then, so are $\mathcal{X}_{-i}$ and $\mathcal{X}_{-i}\oplus\mathcal{X}_0$. If X, Y are arbitrary in $\mathcal{X}(M)^c$, we can write $X=X_i+X_{-i}+X_0$, $Y=Y_i+Y_{-i}+Y_0$ where $X_i, Y_i\in\mathcal{X}_i$, $X_{-i}, Y_{-i}\in\mathcal{X}_{-i}$, $X_0, Y_0\in\mathcal{X}_0$. Substituting into $[f, f](X, Y)$ and using $fX_i=iX_i$, etc., yields $[f, f](X, Y)=0$.

THEOREM 6.2 *Let M be an integrable $(2n+s)$-dimensional f-manifold. Then for all λ, μ, ν with $0\leqq\lambda$, $\mu\leqq n$, $0\leqq\nu\leqq s$,*

$$d\Lambda_{\lambda,\mu,\nu}(M)\subset\Lambda_{\lambda+1,\mu,\nu}(M)\oplus\Lambda_{\lambda,\mu+1,\nu}(M)\oplus\Lambda_{\lambda,\mu,\nu+1}(M)$$

Proof. Consider first the case of $\lambda=1$, $\mu=\nu=0$. Let $\alpha\in\Lambda_{1,0,0}(M)$. If $X, Y\in\mathcal{X}_{-i}$, then $d\alpha(X, Y)=X\alpha(Y)-Y\alpha(X)-\alpha([X, Y])=0$, since $\alpha(X)=\alpha(Y)=\alpha([X, Y])=0$ (using the proof of theorem 6.1). Hence $\Pi_{0,2,0}d\alpha=0$. Similarly, $\Pi_{0,0,2}d\alpha=\Pi_{0,1,1}d\alpha=0$ and $d\alpha\in\Lambda_{2,0,0}(M)\oplus\Lambda_{1,1,0}(M)\oplus\Lambda_{1,0,1}(M)$. Similarly, $d\Lambda_{0,1,0}(M)\subset\Lambda_{1,1,0}(M)\oplus\Lambda_{0,2,0}(M)\oplus\Lambda_{0,1,1}(M)$ and $d\Lambda_{0,0,1}(M)\subset\Lambda_{1,0,1}(M)\oplus\Lambda_{0,1,1}(M)\oplus\Lambda_{0,0,2}(M)$

If $\alpha\in\Lambda_{\lambda,\mu,\nu}(M)$, then α can be written locally as

$$\alpha=\alpha_{i1}\wedge\cdots\wedge\alpha_{i\lambda}\wedge\alpha_{-i1}\wedge\cdots\wedge\alpha_{-i\mu}\wedge\alpha_{01}\wedge\cdots\wedge\alpha_{0\nu}$$

where $\alpha_{ik}\in\Lambda_{1,0,0}(M)$, $1\leqq k\leqq\lambda$, $\alpha_{-ik}\in\Lambda_{0,1,0}(M)$, $1\leqq k\leqq\mu$, $\alpha_{0k}\in\Lambda_{0,0,1}(M)$, $1\leqq k\leqq\nu$, and hence, locally,

$$\begin{aligned}(6.1)\qquad d\alpha=&\sum_{k=1}^{\lambda}(-1)^{k-1}\alpha_{i1}\wedge\cdots\wedge d\alpha_{ik}\wedge\cdots\wedge\alpha_{i\lambda}\wedge\alpha_{-i1}\wedge\cdots\wedge\alpha_{0\nu}\\ &+\sum_{k=1}^{\mu}(-1)^{\lambda+k-1}\alpha_{i1}\wedge\cdots\wedge\alpha_{i\lambda}\wedge\alpha_{-i1}\wedge\cdots\wedge d\alpha_{-ik}\wedge\cdots\wedge\alpha_{-i\mu}\wedge\alpha_{01}\wedge\cdots\wedge\alpha_{0\nu}\\ &+\sum_{k=1}^{\nu}(-1)^{\lambda+\mu+k-1}\alpha_{i1}\wedge\cdots\wedge\alpha_{-i\mu}\wedge\alpha_{01}\wedge\cdots\wedge d\alpha_{0k}\wedge\cdots\wedge\alpha_{0\nu}\end{aligned}$$

If $\gamma\in\Lambda_{1,0,0}(M)$, $\beta\in\Lambda_{\lambda-1,\mu,\nu}(M)$, then

$$\begin{aligned} d\gamma\wedge\beta &\in d\Lambda_{1,0,0}(M)\wedge\Lambda_{\lambda-1,\mu,\nu}(M) \\ &\subset\{\Lambda_{2,0,0}(M)\oplus\Lambda_{1,1,0}(M)\oplus\Lambda_{1,0,1}(M)\}\wedge\Lambda_{\lambda-1,\mu,\nu}(M) \\ &\subset\Lambda_{\lambda+1,\mu,\nu}(M)\oplus\Lambda_{\lambda,\mu+1,\nu}(M)\oplus\Lambda_{\lambda,\mu,\nu+1}(M)\,. \end{aligned}$$

Similarly, we show that all terms in (6.1) are in the required space, completing the proof.

THEOREM 6.3 *Let M be a framed f-manifold of dimension $2n+s$ with complemented framing $\{(E_\alpha, \eta^\alpha)\,|\,\alpha=1,\cdots,s\}$. Then the f-structure is normal if and only if the following conditions hold:*

(i) *for all* $X, Y\in\mathcal{X}_i$, $[X, Y]\in\mathcal{X}_i$.

(ii) *for all* $X\in\mathcal{X}_i$, $\alpha\in\{1,\cdots,s\}$, $[X, E_\alpha]\in\mathcal{X}_i$

(iii) *for all* $\alpha, \beta\in\{1,\cdots,s\}$, $[E_\alpha, E_\beta]=0$

Proof. Suppose f is normal. If $X, Y\in\mathcal{X}_i$, then $fX=iX$, $fY=iY$, $\eta^\alpha(X)=\eta^\alpha(Y)=0$,

$$\begin{aligned} 0&=[fX, fY]+f^2[X, Y]-f[fX, Y]-f[X, fY]+d\eta^\alpha(X, Y)E_\alpha \\ &=-[X, Y]-l[X, Y]-2if[X, Y]-m[X, Y] \\ &=-2i\{f[X, Y]-i[X, Y]\}\,. \end{aligned}$$

Hence $f[X, Y]=i[X, Y]$, proving (i). Similarly, if $X\in\mathcal{X}_i$, then

$$\begin{aligned} 0&=[fX, fE_\alpha]+f^2[X, E_\alpha]-f[fX, E_\alpha]-f[X, fE_\alpha]+d\eta^\beta(X, E_\alpha)E_\beta \\ &=-l[X, E_\alpha]-if[X, E_\alpha]-m[X, E_\alpha] \\ &=-i\{f[X, E_\alpha]-i[X, E_\alpha]\}\,, \end{aligned}$$

proving (ii). If $\alpha, \beta\in\{1,\cdots,s\}$, then

$$\begin{aligned} 0&=[fE_\alpha, fE_\beta]+f^2[E_\alpha, E_\beta]-f[fE_\alpha, E_\beta]-f[E_\alpha, fE_\beta]+d\eta^\gamma(E_\alpha, E_\beta)E_\gamma \\ &=-[E_\alpha, E_\beta]\,, \end{aligned}$$

proving (iii).

Conversely, if (i), (ii) and (iii) hold, then by taking complex conjugates, we also get

(iv) *for all* $X, Y\in\mathcal{X}_{-i}$, $[X, Y]\in\mathcal{X}_{-i}$,

(v) *for all* $X\in\mathcal{X}_{-i}$, $\alpha\in\{1,\cdots,s\}$, $[X, E_\alpha]\in\mathcal{X}_{-i}$.

If $X, Y\in\mathcal{X}(M)\subset\mathcal{X}(M)^c$, then $X=X_i+X_{-i}+\eta^\alpha(X)E_\alpha$, $Y=Y_i+Y_{-i}+\eta^\alpha(Y)E_\alpha$, where $X_i, Y_i\in\mathcal{X}_i$, $X_{-i}, Y_{-i}\in\mathcal{X}_{-i}$. Putting these decompositions of X and Y into ([f, f]

$+d\eta^\alpha \otimes E)(X, Y)$, using (i)-(v) and performing a straightforward, but tedious, computation yields $([f, f]+d\eta^\alpha\otimes E)(X, Y)=0$. Hence f is a normal structure.

THEOREM 6.4 *Let M be a $(2n+s)$-dimensional framed f-manifold with complemented framing $\{(E_\alpha, \eta^\alpha) \mid \alpha=1, \cdots, s\}$. If f is normal, then for all λ, μ, ν with $0 \leqq \lambda \leqq n$, $0 \leqq \mu \leqq n$, $0 \leqq \nu \leqq s$, we have*

$$dA_{\lambda,\mu,\nu}(M) \subset A_{\lambda+1,\mu,\nu}(M) \oplus A_{\lambda,\mu+1,\nu}(M) \oplus A_{\lambda,\mu,\nu+1}(M) \oplus A_{\lambda+1,\mu+1,\nu-1}(M) .$$

Proof. As in the proof of Theorem 6.2, it is sufficient to consider $d\omega$ where $\omega \in A_{1,0,0}(M)$ or $A_{0,1,0}(M)$ or $A_{0,0,1}(M)$. Let $\omega \in A_{1,0,0}(M)$. If $X, Y \in \mathcal{X}_{-i}$, then by the proof of Theorem 6.3, $[X, Y] \in \mathcal{X}_{-i}$ and hence $\omega(X)=\omega(Y)=\omega([X, Y])=0$, from which it follows that $d\omega(X, Y)=0$ and $\Pi_{0,2,0}d\omega=0$. Similarly, we can show that for any $X \in \mathcal{X}_{-i}$, $\alpha, \beta \in \{1, \cdots, s\}$, $d\omega(X, E_\alpha)=0$ and $d\omega(E_\alpha, E_\beta)=0$. Since $\{E_\alpha\}$ spans the distribution $\mathcal{X}_0$, it follows that $\Pi_{0,1,1}d\omega=0=\Pi_{0,0,2}d\omega$. Hence $d\omega \in A_{2,0,0}(M) \oplus A_{1,1,0}(M) \oplus A_{1,0,1}(M)$.

Similarly, if $\omega \in A_{0,1,0}(M)$, then $d\omega \in A_{1,1,0}(M) \oplus A_{0,2,0}(M) \oplus A_{0,1,1}(M)$.

If $\omega \in A_{0,0,1}(M)$, we can similarly show that $\Pi_{2,0,0}=d\omega=0=\Pi_{0,2,0}d\omega$, and hence $d\omega \in A_{1,1,0}(M) \oplus A_{1,0,1}(M) \oplus A_{0,1,1}(M) \oplus A_{0,0,2}(M)$. (Note that we cannot show $\Pi_{1,1,0}d\omega=0$ because normality will permit $P_0[X, Y] \neq 0$ for $X \in \mathcal{X}_i$, $Y \in \mathcal{X}_{-i}$.)

The rest of the proof proceeds as in Theorem 6.2.

Remark. D. E. Blair has constructed a space H^{2n+s} ([1]) which carries a normal framed metric f-structure on which each of the forms $\eta^\alpha \in A_{0,0,1}(M)$, $\alpha=1, \cdots, s$, satisfies $d\eta^\alpha=F$. Hence $d\eta^\alpha \in A_{1,1,0}(M)$.

7. Some remarks

Both D. E. Blair and S. I. Goldberg ([2], [4]) have decomposed the differential operator d on an f-manifold M as

$$d=d'+d''+d^0 \tag{7.1}$$

where it is clear from the context that if $\alpha \in A_{\lambda,\mu,\nu}(M)$, then $d'\alpha \in A_{\nu+1,\mu,\nu}(M)$, $d''\alpha \in A_{\lambda,\mu+1,\nu}(M)$, $d^0\alpha \in A_{\lambda,\mu,\nu+1}(M)$. Two problems that arise concern whether or not the operators are well-defined and the validity of (7.1).

The operators d', d'' and d^0 can be defined precisely, using the results of § 3, by

$$d'=\sum_{\lambda,\mu,\nu} \Pi_{\lambda+1,\mu,\nu} \circ d \circ \Pi_{\lambda,\mu,\nu}, \qquad d''=\sum_{\lambda,\mu,\nu} \Pi_{\lambda,\mu+1,\nu} \circ d \circ \Pi_{\lambda,\mu,\nu} ,$$

$$d^0=\sum_{\lambda,\mu,\nu} \Pi_{\lambda,\mu,\nu+1} \circ d \circ \Pi_{\lambda,\mu,\nu} .$$

The validity of the relation (7.1) is more difficult. Theorem 6.2 yields the fact that (7.1) is true if the f-structure is integrable, but Theorem 6.4 and the remark following show the falsity of (7.1) in general.

The paper [2] uses (7.1) when dealing with a cosympletic manifold M, i.e., a $(2n+1)$-dimensional metric f-manifold where f has rank $2n$ and there is a framing $\{E, \eta\}$. Among the conditions assumed are the normality of f, $[f, f]+d\eta\otimes E=0$, and the closure of η, $d\eta=0$. It follows that $[f, f]=0$, f is integrable, and (7.1) is valid.

In [4], S. I. Goldberg assumes the integrability of f in all of his major theorems; however, some of the lemmas are false as stated. The additional assumption of the integrability of f removes all of these problems.

REFERENCES

[1] D. E. BLAIR, Geometry of manifolds with structure group $u(n)\times O(s)$, J. Differential Geometry, **4** (1970), 155-167.

[2] D. E. BLAIR AND S. I. GOLDBERG, Topology of almost contact manifolds, J. Differential Geometry, **1** (1967), 347-354.

[3] T. FUJITANI, Complex-valued differential forms on normal contact Riemannian manifolds, Tôhoku Math. J., **18** (1966), 349-361.

[4] S. I. GOLDBERG, A generalization of Kaehler geometry, J. Differential Geometry, **6** (1972), 343-355.

[5] S. ISHIHARA AND K. YANO, On integrability of a structure f satisfying $f^3+f=0$. Quart. J. Math. Oxford (2), **15** (1964), 217-222.

[6] Y. MATSUSHIMA, *Differentiable Manifolds.* Decker (Pure and Applied Mathematics), New York, 1972.

[7] E. M. MOSKAL, Contact manifolds of positive curvature, Thesis, University of Illinois, 1966.

[8] S. SASAKI, *Almost Contact Manifolds* (Part III), Lecture Notes, Math. Institute, Tôhoku Univ. (1968).

[9] S. TACHIBANA, On harmonic tensors in compact Sasakian spaces, Tôhoku Math. J., **17** (1965), 271-284.

[10] S. TANNO, The topology of contact Riemannian manifolds, Illinois J. Math., **12** (1968), 700-717.

[11] K. YANO, On structure f satisfying $f^3+f=0$, Tech. Rep. No. 2, June 20 (1961), Univ. of Washington.

[12] K. YANO, On a structure defined by tensor field of type (1, 1) satisfying $f^3+f=0$, Tensor N. S., **14** (1963), 99-109.

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