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DISTRIBUTION OF VALUES OF ENTIRE FUNCTIONS OF LOWER ORDER LESS THAN ONE

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1. Introduction. Quite recently, Tsuzuki [3] has proved the following result:

Let $f(z)$ be an entire function of order less than one and let $\{w_n\}$ be an unbounded sequence. Assume that there exists β such that $0<\beta<\pi/2$ and all the roots of equations

$$f(z)=w_n \qquad (n=1, 2, \cdots),$$

belong to the sector

$$\{z \mid |\arg z-\pi| \leqq \beta\}.$$

Then $f(z)$ is a linear function.

The purpose of this paper is to generalize the above result by an elementary argument. The proof given here is quite different from that of Tsuzuki and, I hope, somewhat simpler.

THEOREM. *Let $f(z)$ be an entire function and let $T(r, f)$ be its characteristic function. Assume that there exists an unbounded sequence $\{w_n\}$ such that all the roots of equations*

$$f(z)=w_n \qquad (n=1, 2, \cdots),$$

lie in the half plane

$$\left\{z \,\middle|\, |\arg z-\pi| \leqq \frac{\pi}{2}\right\}.$$

Assume further that

$$(*) \qquad \liminf_{r\to\infty} \frac{T(r, f)}{r}=0.$$

Then $f(z)$ is a polynomial of degree not greater than two.

Considering Mittag-Leffler's function, we can easily assure that this theorem is no longer true when the opening of the sector is greater than π.

Further the assumption (*) cannot be improved, in general. This is easily seen on an example such that

$$f(z)=\exp(-z).$$

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2. Preliminaries. Before proceeding with the proof of Theorem, we need some preliminary facts.

LEMMA 1. *Let $f(z)$ be a nonconstant entire function satisfying the assumption* (*). *If all the zeros $\{a_n\}$ of $f(z)$ are*

$$\operatorname{Re} a_n \leqq 0 \qquad (n=1, 2, \cdots), \tag{1}$$

then

$$\operatorname{Re} \frac{f'(z)}{f(z)} > 0 \tag{2}$$

in the right half plane

$$R = \{z \mid \operatorname{Re} z > 0\}.$$

Proof. For each point c in R, set

$$f_c(z) = f(z+c).$$

Then it follows from an elementary formula [2] that

$$r \operatorname{Re} \frac{f_c'(0)}{f_c(0)} = \frac{1}{\pi} \int_0^{2\pi} \log |f_c(re^{it})| \cos t \, dt + \sum_{|a_n - c| < r} \operatorname{Re} \left(\frac{\bar{a}_n - \bar{c}}{r} - \frac{r}{a_n - c} \right)$$

for every positive r. Hence from (1), we have

$$\operatorname{Re} \frac{f_c'(0)}{f_c(0)} = \operatorname{Re} \frac{f'(c)}{f(c)} \geqq -4 \frac{T(r, f_c)}{r}.$$

Therefore, since the assumption (*) also gives

$$\liminf_{r \to \infty} \frac{T(r, f_c)}{r} = 0,$$

we conclude that

$$\operatorname{Re} \frac{f'(z)}{f(z)} \geqq 0$$

for each point z in R.

Here, notice that $\operatorname{Re}(f'(z)/f(z))$ is harmonic in R. Then we obtain (2) excepting when

$$f(z) = \exp(az+b).$$

This completes the proof of Lemma 1.

Let $f(z)$ be a nonconstant entire function satisfying the hypotheses of Theorem. Then by Lemma 1,

$$\operatorname{Re} \frac{f'(z)}{f(z) - w_n} > 0 \qquad (n=1, 2, \cdots), \tag{3}$$

in the right half plane R. In particular, the first derivative $f'(z)$ has no zeros there.

Now we consider the argument of $f'(z)$ which is denoted by $u(z)$. Let us set

$$\gamma_n = \arg w_n \qquad (n=1, 2, \cdots).$$

Then the inequalities (3) will be

$$\mathrm{Re}\frac{f(z)}{f'(z)} > \left|\frac{w_n}{f'(z)}\right| \cos(u(z)-\gamma_n),$$

so that

$$|f(z)| > |w_n| \cos(u(z)-\gamma_n) \qquad (n=1, 2, \cdots), \tag{4}$$

for every point z in R. These inequalities (4) are essential to our proof.

Here we assume that there exist four points a, b, c and d in the right half plane R such that

$$u(a) = u(c) - \pi \tag{5}$$

and

$$u(a) < u(b) < u(c) < u(d) < u(a) + 2\pi. \tag{6}$$

Then it is possible to find a positive number ε such that

$$\begin{aligned} &u(a)+\varepsilon < u(b) < u(c)-\varepsilon, \\ &u(c)+\varepsilon < u(d) < u(a)+2\pi-\varepsilon. \end{aligned} \tag{7}$$

According to the inequalities (4), for each n $(n=1, 2, \cdots)$,

$$|f(a)| > |w_n| \cos(u(a)-\gamma_n),$$

$$|f(c)| > |w_n| \cos(u(c)-\gamma_n).$$

Therefore, since the sequence $\{w_n\}$ is unbounded, infinitely many terms of $\{w_n\}$ must satisfy

$$\pi - \varepsilon \leqq 2|\gamma_n - u(a)| \leqq \pi + \varepsilon.$$

But this clearly contradicts (4) and (7). Hence we cannot take four points in R satisfying (5) and (6). By this fact, we easily have the following lemma.

LEMMA 2. *Let $f(z)$ be a nonconstant entire function satisfying the hypotheses of Theorem. Then it is possible to find a real number γ such that*

$$|\arg f'(z) - \gamma| \leqq \frac{\pi}{2}$$

for every point z in the right half plane R.

3. Proof of Theorem. We may assume that $f(z)$ is not linear. Then by Lemma 2, there exists a real number γ such that

$$|\arg f'(z)-\gamma| \leqq \frac{\pi}{2} \tag{8}$$

for every point z in R. Set

$$\begin{aligned} v_{2n-1}&=n \exp\left(i\gamma+i\frac{2}{3}\pi\right), \\ v_{2n}&=n \exp\left(i\gamma-i\frac{2}{3}\pi\right) \qquad (n=1, 2, \cdots). \end{aligned} \tag{9}$$

Then it follows from (8) that all the roots of equations

$$f'(z)=v_n \qquad (n=1, 2, \cdots),$$

belong to the half plane

$$\{z \mid \operatorname{Re} z \leqq 0\}.$$

Further by an elementary estimation, we also have

$$\liminf_{r\to\infty} \frac{T(r, f'(z))}{r}=0.$$

Hence by the same argument which is developed in the section 2, the second derivative $f''(z)$ has no zeros in the right half plane R and

$$\operatorname{Re} \frac{f'(z)}{f''(z)} > \operatorname{Re} \frac{v_n}{f''(z)} \qquad (n=1, 2, \cdots), \tag{10}$$

there. Thus from the definition (9) of the sequence $\{v_n\}$ and the inequalities (10), we obtain

$$|\arg f''(z)-\gamma| \leqq \frac{\pi}{6} \tag{11}$$

for each point z in R. Therefore by (11), using the same argument once more, we easily conclude that

$$f''(z)=C,$$

which yields the desired result.

4. Remarks. Finally, it might be of interest to mention that our Lemma 1 is sufficient to yield the following facts which are analogues of Lucas' theorem [1].

(I) Let $f(z)$ be a nonconstant entire function satisfying

$$\liminf_{r\to\infty} \frac{T(r, f)}{r}=0.$$

Then the smallest convex set which contains the zeros of $f(z)$ also contains the zeros of $f'(z)$.

(II) Let $f(z)$ be a nonconstant entire function which satisfies

$$\liminf_{r\to\infty} \frac{T(r, f)}{r} = 0 .$$

Then the smallest convex set which contains the zeros and ones of $f(z)$ also contains all the roots of equations

$$f(z)=t \qquad (0 \leqq t \leqq 1) .$$

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