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# REMARKS ON ADMISSIBLE DATA FOR CAUCHY PROBLEM

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## § 1. Introduction.

Cauchy problem for hyperbolic operators with constant coefficients for space like initial plane seems to be very much studied (see e.g. [2] Chapter V). On the contrary, few are known for non-characteristic and non-hyperbolic Cauchy problem. Here we have used the terminology "hyperbolic Cauchy problem" as we use the terminology "elliptic boundary value problem". That is, the Cauchy problem is said to be a hyperbolic Cauchy problem if the operator is hyperbolic with respect to $N\in\boldsymbol{R}^n$ which is a normal vector of a given initial plane. In this note we shall study some basic properties of admissible data for such non-hyperbolic Cauchy problems.

Let $P(D)$ be a linear partial differential operator of order $m$ with constant coefficients and let

$$X=X_N=\{x\in\boldsymbol{R}^n;\ \langle x, N\rangle=0\}\ ,$$

where $N\in\boldsymbol{R}^n$, $N\neq 0$. In what follows, unless otherwise stated, we shall assume that $P_m(N)\neq 0$, where $P_m(\xi)$ is the principal part of $P(\xi)$. In other words, $X$ is a non-characteristic hyperplane for $P(D)$. Let $F$ be a closed subset of $X$ which is the closure of some non-empty open connected subset $\omega$ of $X$. By $f\in C^\infty(F)$ we mean that $f$ is the restriction of some $\tilde{f}\in C^\infty(X)$ to $F$; i.e. $\tilde{f}|_F=f$. Now we shall consider several kinds of Cauchy problems and define corresponding spaces of admissible data for $P(D)$.

DEFINITION 1.1 (cf. [3]). $\Phi=(f_0, \cdots, f_{m-1})\in\prod_0^{m-1} C^\infty(F)$ is said to be an admissible data for $(P, F)$ in the half space $H=H_N=\{x\in\boldsymbol{R}^n;\ \langle x, N\rangle>0\}$ if there is some open neighbourhood $\Omega$ of $\omega$ in $\boldsymbol{R}^n$ with $\Omega\cap X=\omega$ and there is a function $u\in C^\infty(\bar{\Omega})$ such that

$$(1.1)\qquad \begin{cases} P(D)u=0 \quad \text{on} \quad \Omega^+=\Omega\cap H, \\ u|_F=f_0, \cdots, \partial^{m-1}u|_F=f_{m-1}, \end{cases}$$

where $\partial^k u$ is the $k$-th derivative of $u$ to the inner normal direction of $F$, i.e.

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$\partial=\langle N, \partial/\partial x\rangle/|N|$. We shall denote $D^{+}(P, F)$ the vector space of admissible data for $(P, F)$ in $H$.

The element $\Phi\in D^{+}(P, F)$ is, in a sense, *one side* admissible data. Next we define *two side* admissible data as follows.

DEFINITION 1.2. $\Phi=(f_0, \cdots, f_{m-1})\in \prod_0^{m-1} C^{\infty}(F)$ is said to be an admissible data for $(P, F)$ if there is some open neighbourhood $\Omega$ of $\omega$ in $\boldsymbol{R}^n$ with $\Omega\cap X=\omega$ and there is a function $u\in C^{\infty}(\bar{\Omega})$ such that

$$\left\{\begin{array}{ll} P(D)u=0 & \text{on } \Omega, \\ u|_F=f_0, \cdots, \partial^{m-1}u|_F=f_{m-1}. \end{array}\right. \tag{1.2}$$

We shall denote $D(P, F)$ the vector space of all admissible data for $(P, F)$.

Thus the condition $\Phi\in D^{+}(P, F)$ means that the Cauchy problem with data $\Phi$ can be solved for one side of $X$ (i.e. have a solution in one side of $X$) near $F$, and $\Phi\in D(P, F)$ means that it can be solved for two sides of $X$ near $F$. Essentially, these are local properties of data as we see in § 3. We also define global admissible data as follows.

DEFINITION 1.3. $\Phi=(f_0, \cdots, f_{m-1})\in \prod_0^{m-1} C^{\infty}(X)$ is said to be a global admissible data (resp. global admissible data in $H_N$) if there is a function $u\in C^{\infty}(\boldsymbol{R})$ (resp. $u\in C^{\infty}(\bar{H}_N)$) such that

$$\left\{\begin{array}{ll} P(D)u=0 & \text{on } \boldsymbol{R}^n \text{ (resp. } H_N), \\ u|_X=f_0, \cdots, \partial^{m-1}u|_X=f_{m-1}. \end{array}\right. \tag{1.3}$$

We shall denote $D(P)$ (resp. $D^{+}(P)$) the vector space of global admissible data (resp. global admissible data in $H_N$).

Note that, in general, $D(P)\subsetneq D(P, X)$ as we shall see later.

Next we recall some facts about hyperbolic Cauchy problem. $P(D)$ is called hyperbolic with respect to $N$ when $P_m(N)\neq 0$ and $P(\xi+tN)\neq 0$ when $\xi\in\boldsymbol{R}^n$, $\operatorname{Im} t<C_0$ where $C_0$ is a constant which is independent to $\xi$. For fundamental properties of hyperbolic operators and hyperbolic Cauchy problems we refer to [2], Chapter V. We shall review some of them which we use in § 2, Theorem 2.4. Cauchy problem for operator $P(D)$ with constant coefficients is $C^{\infty}$-well posed when $P(D)$ is hyperbolic with respect to time like direction which is normal to the initial hyperplane. By the word "well posed" we usually mean the following two equivalent properties:

(I) For every $C^{\infty}$-data $\Phi$, there is one and only one $C^{\infty}$-solution $u$ of the Cauchy problem. That is, $D(P, F)=\prod_0^{m-1} C^{\infty}(F)$.

(II) The map $\Phi\rightarrow u$ is continuous in usual topology of the $C^{\infty}$-space.

If we consider non-hyperbolic Cauchy problem, above two conditions do not hold. In connection with (II), John [4] studied some continuity properties for solutions and data. Concerning to (I), John [3] also studied various properties of admissible data for the operator

$$P(D)=\frac{\partial^2}{\partial t^2}-c^2\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\Big)+k\,, \tag{1.4}$$

$c, k\in \boldsymbol{R}$, $c\neq 0$, with the initial plane $X=\{(t, x, y, z)\,;\, x=0\}$. He studied especially dependence of the spaces $D(P, F)$, $D^+(P, F)$ on $c$ and $k$. Further, in [3], he constructed an example of one side admissible data $\in D^+(P, X)$ which is not two side admissible data in any neighbourhood of any point in $X$ when $P(D)$ is defined by (1.4). We shall generalize this for any homogeneous operator which is not hyperbolic with respect to $N$ and show some facts which follow from this example. Though some other facts are known by Volterra (1894), Hadamard (1923), Titt (1939) and others for such problems for wave operator (see [1] p. 247-, [3]), it seems that there are no systematic study on structure of admissible data for Cauchy problems in general situations.

After reviewing some fundamental results for Cauchy problems in § 2, we shall prove rather negative results for non-characteristic and non-hyperbolic Cauchy problems in § 3. More precisely, we shall show that the following conditions are equivalent for homogeneous operators:

(i) $P(D)$ is hyperbolic with respect to $N$.

(ii) $D(P, F)=D^+(P, F)$ for some (and every) $F$.

(iii) $D(P, F)=D(P)|_F$ for some (and every) $F$.

(iv) $D(P, F_1)|_{F_2}=D(P, F_2)$ for some (and every) convex $F_1$ and $F_2$ with $F_2\subsetneqq F_1$.

That (ii), (iii), (iv) follow from (i) is trivial from well known fact that $D(P, F)=\prod C^\infty(F)$ when $P(D)$ is hyperbolic with respect to $N$. So our task is to prove that (i) follows from one of the conditions (ii), (iii) and (iv). That (ii)$\Rightarrow$(i) means that if all one side admissible data are two side admissible data, then $P(D)$ must be a hyperbolic operator (Theorem 3.2). That (iii)$\Rightarrow$(i) means that if all local admissible data on $F$ can be continued to some data on $X$ which is global admissible data, then $P(D)$ must be a hyperbolic operator (Corollary 3.4). That (iv)$\Rightarrow$(i) means that if all local admissible data on some convex set can be continued to some local admissible data on some wider set, then $P(D)$ must be a hyperbolic operator (Theorem 3.5). The proof will be done by constructing some special type of solutions and corresponding data.

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## § 2. Review of some results on Cauchy problem.

In this section we shall see, at first, some uniqueness and non-uniqueness theorems for Cauchy problem which will be needed in § 3. Next we shall see some results concerning well posed Cauchy problem (i. e. hyperbolic Cauchy problem) which will be compared with the results in § 3. All the statements and proofs will follow from results written in Chapter V of [2] and we shall omit the proofs.

Though classical Holmgren's theorem is only a local uniqueness theorem, we can get rather global results for operators with constant coefficients.

THEOREM 2.1. *For any given data* $\Phi=(f_0, \cdots, f_{m-1})\in\prod C^\infty(X)$, *the solution of the Cauchy problem* (1.3) *is, if exists, unique. The same is true for solution of* (1.1) *with* $F=X$.

This follows from Corollary 5.3.1 of [2].

THEOREM 2.2. *Let* $X=X_N$ *be a non-characteristic hyperplane with respect to* $P(D)$. *Then there is an open convex cone* $\Gamma$ *with vertex* 0 *in* $\boldsymbol{R}^n$ *such that*

(i) $\Gamma\subset H_N^-=\{x\in\boldsymbol{R}^n;\ \langle x, N\rangle<0\}$,

(ii) *Let* $y\in\boldsymbol{R}^n$ *be any point with* $\langle y, N\rangle>0$ *and let* $F=(\{y\}+\Gamma)\cap X$. *Then it follows that*

(2.1) *for every solutions* $u_1, u_2\in C^\infty$ *of* $P(D)u=0$ *defined in an open neighbourhoods* $\Omega_1, \Omega_2$ *of* $F$ *with same data* $\Phi=(f_0, \cdots, f_{m-1})$ *on* $F$, $u_1\equiv u_2$ *on* $\Omega_1\cap\Omega_2\cap(\{y\}+\Gamma)$.

*Proof.* Since $X$ is a non-characteristic hyperplane we can take an open convex cone $\Gamma'$ $(\neq\phi)$ with vertex 0 in $\xi$-space $\boldsymbol{R}^n$ such that

$$\Gamma'\subset H_N^-=\{\xi\in\boldsymbol{R}^n;\ \langle\xi, N\rangle<0\}\,,$$

$$\Gamma'\cap\{\xi\in\boldsymbol{R}^n;\ P_m(\xi)=0\}=\phi\,,$$

$$\Gamma'\ni-N\,.$$

Then we know that

$$\Gamma=\{x\in\boldsymbol{R}^n;\ \langle x, \xi\rangle>0 \text{ for all } \xi\in\Gamma'\}$$

is a desired open convex cone from Corollary 5.3.3 of [2].

REMARK. From the above theorem, it follows that for any given $F=\bar{\omega}\subset X$ we can get an open neighbourhood $\Omega_0$ of $\omega$ in $H_N$ which has the property (3.1) stated above $((\{y\}+\Gamma)$ is replaced by $\Omega_0)$. In fact we have only to take

$$\Omega_0=\bigcup_y(\{y\}+\Gamma)\,,$$

where the union is taken for all $y\in H_N$ with $(\{y\}+\Gamma)\cap X\subset F$.

On the other hand we know the following non-uniqueness theorem for characteristic Cauchy problem.

THEOREM 2.3. *Let* $X=\{x\in \boldsymbol{R}^n;\ \langle x, N\rangle=0\}$ *be a characteristic hyperplane for* $P(D)$ (*i.e.* $P_m(N)=0$). *Then there is a solution* $u\in C^\infty(\boldsymbol{R}^n)$ *of* (1.3) *with data* $\Phi=(0, \cdots, 0)$ *such that* $0\in \operatorname{supp} u$.

This is Theorem 5.2.2 of [2].

Finally we shall recall a fundamental theorem for hyperbolic operators.

THEOREM 2.4. *Let* $P(D)$ *be an operator with constant coefficients and* $X=\{x\in \boldsymbol{R}^n;\ \langle x, N\rangle=0\}$. *Then the following conditions are equivalent.*

(i) $P(D)$ *is hyperbolic with respect to* $N$ *in the following sense:* $P_m(N)\neq 0$ *and there is a constant* $C_0$ *such that*

$$P(\xi+tN)\neq 0 \qquad \text{if} \quad \xi\in \boldsymbol{R}^n,\ \operatorname{Im} t<C_0. \tag{2.2}$$

(ii) $D(P)=\prod C^\infty(X)$.

(iii) $D^+(P)=\prod C^\infty(X)$.

This follows from Theorem 5.4.1, Corollary 5.6.2 and Theorem 5.5.1 of [2]. (Note that the equivalence of (ii) and (iii) follows from the fact that if $P(D)$ is hyperbolic with respect to $N$, then it is also hyperbolic with respect to $-N$).

From Theorem 2.4 we have

COROLLARY 2.5. *For hyperbolic Cauchy problem,*

$$D(P, \bar{\omega})=D^+(P, \bar{\omega})=\prod C^\infty(\bar{\omega}), \tag{2.3}$$

*for any open set* $\omega$ *in* $X$.

REMARK 1. It is known that the converse of the above corollary is true. That is, for given $P(D)$, if (2.3) holds for some open set, then $P(D)$ is hyperbolic. It was proved by P. D. Lax.

REMARK 2. If $P(D)$ is a homogeneous operator, condition (3.2) can be replaced by the following condition:

$$P(\xi+tN)\neq 0 \qquad \text{if} \quad \xi\in \boldsymbol{R}^n,\ \operatorname{Im} t\neq 0. \tag{2.4}$$

## § 3. Admissible data.

At first we note the following fact which is a trivial consequence of Holmglen's theorem.

PROPOSITION 3.1. *Let* $\Phi=(f_0, \cdots, f_{m-1})\in \prod C^\infty(F)$. *If for each point* $x\in F$ *there is a neighbourhood* $\omega_x$ *of* $x$ *in* $X$ *such that* $\Phi\in D(P, \bar{\omega}_x)$, *then* $\Phi\in D(P, F)$. *The same is true when we replace* $D(\cdot)$ *to* $D^+(\cdot)$.

This proposition shows that $\Phi\in D(P, F)$ is a local property of $\Phi$.

When $P(D)$ is hyperbolic with respect to $N$, we have seen in Corollary 2.5 that any data $\Phi\in D(P, F)$ can be continued across the plane $X=\{x\in \boldsymbol{R}^n;\ \langle x, N\rangle =0\}$ at any point of $F$. We shall see an example that this is not true in general.

EXAMPLE. Let $P(D)$ be an elliptic operator of order $m=2h$ and let $\omega$ be a bounded non-empty open set in $X$. We know from general theory of elliptic boundary value problems (see Chapter X of [2]) that for any given data $f_0, \cdots, f_{h-1}\in C^\infty(\bar{\omega})$, there is a solution of

$$(3.1)\qquad \begin{cases} P(D)u=0 & \text{on some half neighbourhood } \Omega^+ \text{ of } \omega . \\ u|_{\bar{\omega}}=g_0, \cdots, \partial^{h-1}u|_{\bar{\omega}}=g_{h-1}, & \end{cases}$$

such that $u\in C^\infty(\bar{\Omega})$. On the other hand, for elliptic operator, it is known that the solution $u\in C^\infty(\Omega)$ of $P(D)u=0$ on some open set $\Omega$ is real analytic on $\Omega$. Thus if we take (at least) one $g_\iota$ such that it is $C^\infty$ but is never real analytic at each point in $\omega$, we cannot continue the solution $u$ of (3.1) to any full neighbourhood of $\omega$ across $\omega$.

REMARK. When $P(D)$ is elliptic, it is thus necessary and sufficient for $\Phi=(f_0, \cdots, f_{m-1})\in D(P, F)$ that all $f_\iota$ $(i=0, 1, \cdots, m-1)$ are real analytic on some neighbourhood of $F$. But we have seen that for any given $g_0, \cdots, g_{h-1}\in C^\infty(F)$ we can add some function $g_h, \cdots, g_{m-1}\in C^\infty(F)$ such that $(g_0, \cdots, g_{h-1}, \cdots, g_{m-1})\in D^+(P, F)$. More will be known, for degree of freedom of $\Phi\in D^+(P, F)$, when we note "Lopatinski-Schapiro condition". Further we can get, by using results on hyperbolic mixed problems, some facts about degree of freedom of $\Phi\in D^+(P, F)$ for hyperbolic operator $P(D)$ which is not hyperbolic with respect to $N$ (=a normal vector of $F$). (see [5] for hyperbolic mixed problems).

We shall now prove that for non-hyperbolic Cauchy problem for any homogeneous operator $P(D)$, there are data $\Phi\in D^+(P, F)$ which cannot be continued to any neighbourhood of $x\in\omega$ across $F$.

THEOREM 3.2. *Let $P(D)$ be a homogeneous operator and let $X=X_N=\{x\in \boldsymbol{R}^n;\ \langle x, N\rangle=0\}$ be a non-characteristic initial plane. If $P(D)$ is not hyperbolic with respect to $N$, $D(P, F)\subset D^+(P, F)$, $D(P)\subset D^+(P)$ are proper inclusions.*

To prove the theorem, it is sufficient to prove that there is a solution $u\in C^\infty(\bar{H}_N)$ of $P(D)u=0$ in $\bar{H}_N=\{x\in \boldsymbol{R}^n;\ \langle x, N\geqq 0\}$ which cannot be continued to any full neighbourhood of $x\in X$ in $\boldsymbol{R}^n$ across $X$. Thus the following lemma will prove the theorem.

LEMMA 3.3. *Let $P(D)$ be a homogeneous operator which is not hyperbolic with respect to $N$. Then there is a function $u\in C^\infty(\bar{H}_N)$ such that*

(i) $P(D)u=0$ *in* $\bar{H}_N$,

(ii) *For every* $x\in X$, *there are no neighbourhood* $\Omega$ *of* $x$ *in* $\boldsymbol{R}^n$ *such that there is a function* $\tilde{u}\in C^\infty(\bar{H}_N\cup\Omega)$ *with* $P(D)\tilde{u}=0$ *in* $\bar{H}_N\cup\Omega$ *and* $\tilde{u}|_H=u$.

*Proof.* Without loss of generality we can assume that $N=(1, 0, \cdots, 0)$ and $X=\{(0, x')=(0, x_2, \cdots, x_n); x_i\in\boldsymbol{R}\}$. As $X$ is a non-characteristic hyperplane, we can factor $P(\xi)$ as follows;

$$P(\xi)=P(\xi_1, \cdots, \xi_n)=P(1, 0)\prod_{j=1}^{m}(\xi_1-\lambda_j(\xi')),$$

where $\xi'=(\xi_2, \cdots, \xi_n)$. When $P(D)$ is not hyperbolic with respect to $N$, we can take some $(\xi_2^0, \cdots, \xi_n^0)=\xi^{0\prime}\in\boldsymbol{R}^{n-1}$ such that $\operatorname{Im}\lambda_j(\xi^{0\prime})\neq0$ for some $j=1, \cdots, m$ (see (3.4)). Note that $\xi^{0\prime}\neq0$. Without loss of generality, we can assume that $\xi_2^0\neq0$. Putting $\xi^{0\prime}\to-\xi^{0\prime}$ if necessary, we can assume that $\operatorname{Im}\lambda_j(\xi^{0\prime})>0$ since $P(\xi)$ is homogeneous. Now take a function $F(\zeta)$ of one complex variable $\zeta$ which is

(i) analytic for $|\zeta|<1$,

(ii) of class $C^\infty$ for $|\zeta|\geqq1$ and

(iii) has the circle $|\zeta|=1$ as the natural boundary.

For example we may take

$$F(\zeta)=\sum_{n=0}^{\infty}e^{-2n!}\zeta^{(2n!)2}.$$

Put

$$u(x)=F(e^{i\langle x,\xi^0\rangle}),$$

where $\xi_1^0=\lambda_j(\xi_2^0, \cdots, \xi_n^0)$. This is what we call a plane wave solution. From the properties of $F(\zeta)$ and $\lambda_j(\xi^{0\prime})$, $u(x)$ is defined and $C^\infty$ for $x_1\geqq0$. Further, $u(x)$ also satisfies

$$Q(D)u=0 \qquad \text{when} \quad x_1\geqq0,$$

where

$$Q(D)=(\xi_1^0)^2D_{x_2}^2-(\xi_2^0)^2D_{x_1}^2.$$

Note that $Q(D)$ is an elliptic operator acting on $(x_1, x_2)$-space because $\operatorname{Im}\xi_1^0\neq0$ and $\xi_2^0\neq0$. Suppose $u(x)$ is continued across the plane $x_1=0$ as a solution of $P(D)u=0$ to some neighbourhood $\Omega$ of some point $y\in X$. For such solution $\tilde{u}(x)$, it is clear that

$$\begin{cases} P(D)Q(D)\tilde{u}=0 \qquad \text{in} \quad H, \\ Q(D)\tilde{u}|_X=0,\ D_{x_1}Q(D)\tilde{u}|_X=0, \cdots, D_{x_1}^{m-1}Q(D)\tilde{u}|_X=0. \end{cases}$$

Thus applying Holmgren's theorem to the solution $Q(D)u$, we have $Q(D)u=0$ in a full neighbourhood of $y\in X$ in $\boldsymbol{R}^n$. On the other hand, since $Q(D)$ is elliptic, $u(x)$ must be real analytic in some neighbourhood of $y$ as a function of $x_1$ and $x_2$ and thus $F(\zeta)$ must be holomorphic in some neighbourhood of $\zeta=e^{i\langle x,\xi^0\rangle}$. This is a contradiction since $|\zeta|=|e^{ix_1\xi_1^0}|>1$ for $x_1<0$. This proves the lemma.

As an easy consequence of Lemma 3.3, we can get the following corollary which will be compared to the hyperbolic case.

COROLLARY 3.4. *Let $P(D)$ be a homogeneous operator which is not hyperbolic with respect to $N$. Then $D(P)\subset D(P, X)$ and $D^+(P)\subset D^+(P, X)$ are proper inclusions.*

In fact, to prove this, it is sufficient to take data associated to the solution $u\in C^\infty(\bar{H})$ of Lemma 3.3 with

$$H=\{x\in \boldsymbol{R}^n;\ \langle x, N\rangle > c\},$$

for some $c>0$ and use Theorem 2.1.

Finally we shall consider on continuability of admissible data.

EXAMPLE 1. Let $P(D)$ be hyperbolic with respect to $N$. Then every data $\Phi\in D(P, F)$ $(=D(P, F)=\prod C^\infty(F))$ can be continued to some data $\tilde{\Phi}\in D(P, X)$ $=D(P)$.

EXAMPLE 2. Let $P(D)$ be elliptic of order $m=2h$ and let $F_1=\bar{\omega}_1$, $F_2=\bar{\omega}_2$ be any compact sets in $X$ with $\omega_1\subsetneqq\omega_2$. Then $D(P, F_2)|_{F_1}\subset D(P, F_1)$ is a proper inclusion. In fact take $\Omega$ as an open neighbourhood of $F_1$ such that $\omega_2\backslash(\Omega\cap X)$ $\neq\phi$ and $\partial\Omega\in C^\infty$. Take $g_0\in C^\infty(\partial\Omega)$ such that it is not analytic in any neighbourhood of $y\in\partial\Omega$. Then the solution $u\in C^\infty$ of Dirichlet problem

$$\begin{cases} P(D)u=0 \quad \text{on } \Omega, \\ u|_{\partial\Omega}=g_0,\ (\partial/\partial\nu)u|_{\partial\Omega}=0, \cdots, (\partial/\partial\nu)^{h-1}u|_{\partial\Omega}=0, \end{cases}$$

cannot be continued (as a solution of $P(D)u=0$) across $\Omega$. Thus for such $u$, $\Phi$ $=(u|_{F_1}, \cdots, \partial^{m-1}u|_{F_1})\in D(P, F_1)$ but $\Phi\notin D(P, F_2)$. If we take $\Omega^+=\Omega\cap H$ instead of $\Omega$, we have that $D^+(P, F_2)|_{F_1}\subset D^+(P, F_2)$ is also proper.

Let us now show that when we restrict $P(D)$ homogeneous and $F_1$, $F_2$ convex, $D(P, F_2)|_{F_1}\subsetneqq D(P, F_1)$ is true for general operator $P(D)$.

THEOREM 3.5. *Let $P(D)$ be a homogeneous operator which is not hyperbolic with respect to $N$. Then for every open convex sets $\omega_1$ and $\omega_2$ in $X=X_N$ with $\omega_1\subset\omega_2$, we have proper inclusions;*

$$D(P, F_2)|_{F_1}\subsetneqq D(P, F_1), \qquad D^+(P, F_2)|_{F_1}\subsetneqq D^+(P, F_1),$$

*where $F_1=\bar{\omega}_1$, $F_2=\bar{\omega}_2$.*

*Proof.* By convexity of $F_1$ and Theorem 2.2, we can take a hyperplane $Y=Y_\vartheta=\{x\in\boldsymbol{R}^n;\ \langle x, \vartheta\rangle=0\}$ with the following properties:

(i) $Y\cap F_1=\{y\}$, $F_1\subset G=\{x\in\boldsymbol{R}^n;\ \langle x, \vartheta\rangle\leqq 0\}$,

(ii) $\Omega_0\cap\Omega_x\cap Y=\phi$ for any neighbourhood $\Omega_x$ of $x$,

where $\Omega_0$ is the domain constructed as in Remark after Theorem 2.2. Take $u$

as in Lemma 3.3 with $X=X_N$ replaced by $Y=Y_\vartheta$. Then $\Phi=(u|_{F_1}, \cdots, \partial^{m-1}u|_{F_1}) \in D(P, F_1)$ cannot be continued to any data $\in D(P, F_1\cup\bar{\omega}_x)$ where $\omega_x$ is any neighbourhood of $x$ in $X$. In fact suppose $\Phi$ can be continued to some $\tilde{\Phi}\in D(P, F_1\cup\bar{\omega}_x)$ for some $\omega_x$, there is a neighbourhood $\Omega_x$ of $\omega_x$ and there is $u\in C^\infty(\bar{\Omega}_x\cup G)$ with $P(D)u=0$ on $\Omega_x\cup G$. Then $\Psi=(u|_{Y\cap\bar{\Omega}_x}, \cdots, \partial^{m-1}u|_{Y\cap\bar{\Omega}_x})\in D(P, Y\cap\bar{\Omega}_x)$ which is a contradiction.

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