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NOTE ON SPACES WITH $H^*(\ ;Z)=E[x_1, x_2]$

Dedicated to Prof. Y. Komatu on his 60-th birthday

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§1. Let K be a 1-connected CW complex such that $H^*(K;Z)$ is the exterior algebra over Z generated by two elements x_i, $i=1,2$ with $\dim x_i=n_i$ $(n_1<n_2)$. In this note we shall consider the problem:

When does there exist a map $K\to S^{n_2}$ so that $H_*(K;Z)$ maps onto $H_*(S^{n_2};Z)$

(this is the special case of $r=2$ of the problem 50 in [2]).

It is clear that there exists such a map if and only if K has the same homotopy type as the total spaee of a n_1-spherical fibre space over S^{n_2}, so this problem has already solved in some sence. Our purpose is to obtain one of sufficient conditions for the existence of such a map.

Since we may regard K as a CW complex of the form $S^{n_1}\cup e^{n_2}\cup e^{n_1+n_2}$, we denote by L the subcomplex of $K\times K=S^{n_1}\cup e^{n_2}\times S^{n_1}$. Now we consider the following condition:

(A) there exists a map $h:L\to K$ such that $h|S^{n_1}\cup e^{n_2}\times(*)$ and $h|(*)\times S^{n_1}$ are both the identity.

We say that K is of type (A) if K satisfies the condition (A). We shall prove

THEOREM. *If K is of type* (A), *then there exists a map $K\to S^{n_2}$ so that $H_*(K;Z)$ maps onto $H_*(S^{n_2})$.*

For example, if K is an H-space K is of type (A). Therefore we have

COROLLARY. *If K is an H-space K has the same homotopy type as the total space of a S^{n_1}-spherical fibre space over S^{n_2}.*

Remark. Mimura, Nishida and Toda has classified H-spaces with rank 2 up to homotopy.

Now we denote by α and β the attaching maps for the cells e^{n_2}, $e^{n_1+n_2}$ respectively.

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§ 2. The case of $n_1=n_2$.

Since $n_1=n_2$ K has the form $S^{n_1}\vee S^{n_2}\cup e^{n_1+n_2}$ and

$$\beta=\iota_1\circ\beta_1+\iota_2\circ\beta_2+[\iota_1, \iota_2], \qquad \beta_i\in\pi_{2n-1}(S^n)$$

where ι_j denotes the inclusion $S^n=S^{n_1}\to S^{n_1}\vee S^{n_2}$ and $n=n_1=n_2$.

LEMMA 1. *If K is of type (A), then K is $S^n\times S^n$ $(n=3, 7)$ up to homotopy. Hence K is also an H-space.*

Proof. Let ι be the generator of $\pi_n(S^n)$ and let i be the inclusion map $S^{n_1}\vee S^{n_2}\to K$. Since $L=S^{n_1}\vee S^{n_2}\times S^{n_1}$ the condition (A) means that $i_*([\iota_1, \iota_1])=0$ $=i_*([\iota_1, \iota_2])$ where $[\ ,\ ]$ denotes Whitehead product. Therefore there exist two integers a, b such that

$$\iota_{1*}([\iota, \iota])=a(\iota_1\circ\beta_1+\iota_2\circ\beta_2+[\iota_1, \iota_2]) \tag{2.1}$$

$$[\iota_1, \iota_2]=b(\iota_1\circ\beta_1+\iota_2\circ\beta_2+[\iota_1, \iota_2]). \tag{2.2}$$

From (2.1) we have $a=0$, i.e. $\iota_{1*}([\iota, \iota])=0$. Since ι_{1*} is monic we have $[\iota, \iota]=0$, i.e. $n=3$ or 7. Moreover, from (2.2), we have that $b=1$ and $\beta_1=\beta_2=0$. Thus the proof is completed.

Now Theorem follows from lemma 1.

§ 3. The case $n_1<n_2$ (since n_j must be odd this means $n_2>n_1+1$).

Let $\bar{\alpha}:(D^{n_2}, S^{n_2-1})\to(K, S^{n_1})$ and $\bar{\beta}:(D^{n_1+n_2}, S^{n_1+n_2-1})\to(K, S^{n_1}\cup e^{n_2})$ be the characteristic map for the cells e^{n_2} and $e^{n_1+n_2}$ respectively.

Consider the part of the homotopy exact sequence of the pair $(S^{n_1}\cup e^{n_2}, S^{n_1})$:

$$\pi_*(S^{n_1}) \xrightarrow[\iota_{1*}]{} \pi_*(S^{n_1}\cup e^{n_2}) \xrightarrow[j_*]{} \pi_*(S^{n_1}\cup e^{n_2}, S^{n_1})$$

where $*$ means n_1+n_2-1.

Since $x_1\cup x_2$ is a generator of $H^{n_1+n_2}(K; Z)$ there exists an element γ of $\pi_*(D^{n_2}, S^{n_2-1})$ such that

$$j_*(\beta)=[\bar{\alpha}, \iota_1]_r+\bar{\alpha}\circ\gamma$$

where $[\ ,\ ]_r$ denotes relative Whitehead product and $i_*(\iota_1)=\iota_1$.

Let $p: S^{n_1}\cup e^{n_2}\to S^{n_2}$ and $q: D^{n_2}\to S^{n_2}$ be the natural pinching maps. Then we have

$$p_*(\beta)=q_*(\gamma). \tag{3.1}$$

Now let $h: L\to K$ be the map satisfying the condition (A).

Since

$$
\begin{aligned}
h^*(x_1\cup x_2)&=h^*(x_1)\cup h^*(x_2)\\
&=(1\otimes x_1+x_1\otimes 1)\cup(x_2\otimes 1)\\
&=x_2\otimes x_1
\end{aligned}
$$

h is a map of degree 1 on $e^{n_1+n_2}=e^{n_1}\times e^{n_2}$. Let $\tilde{L}$ be the subcomplex $L-e^{n_1+n_2}$ of L and let θ be the attaching class for the cell $e^{n_1+n_2}$ of L ($\theta\in\pi_*(\tilde{L})$).

Then, from the above argument, we have

$$\tilde{h}_*(\theta)=\beta \qquad (\tilde{h}=h\,|\,L)\,. \tag{3.2}$$

Since we can regard the restriction $h\,|\,S^{n_1}\times S^{n_1}$ as a map $S^{n_1}\times S^{n_1}\to S^{n_1}\cup e^{n_2}$ we have a map $\lambda: S^{2n_1}\to S^{n_2}$ defined by the following diagram

$$
\begin{tikzcd}
S^{n_1}\times S^{n_1} \arrow[r] \arrow[d, "h"'] & S^{n_1}\times S^{n_1}/S^{n_1}\vee S^{n_1}=S^{2n_1} \arrow[d] \arrow[d, shift left=8ex, "\lambda"] \\
S^{n_1}\cup e^{n_2} \arrow[r] & S^{n_1}\cup e^{n_2}/S^{n_1} \quad =S^{n_2}\,.
\end{tikzcd}
$$

LEMMA 2. *$\alpha\circ(E^{-1}\lambda)=[\iota_{n_1}, \iota_{n_1}]$ in $\pi_{2n_1-1}(S^{n_1})$, where E denotes the suspension isomorphism: $\pi_{2n_1-1}(S^{n_2-1})\to\pi_{2n_1}(S^{n_2})$.*

Proof. Consider the following diagram

$$
\begin{tikzcd}
 & \pi_{2n_1}(S^{n_1}\times S^{n_1}, S^{n_1}\vee S^{n_1}) \arrow[r, "\partial"] \arrow[d, "h|_*"] \arrow[dl] & \pi_{2n_1-1}(S^{n_1}\vee S^{n_1}) \arrow[d, "(\iota_{n_1}\vee\iota_{n_1})_*"] \\
\pi_{2n_1}(S^{2n_1}) \arrow[d, "\lambda^*"] & \pi_{2n_1}(S^{n_1}\cup e^{n_2}, S^{n_1}) \arrow[r, "\partial"] \arrow[dl, "p_*"'] & \pi_{2n_1-1}(S^{n_1}) \\
\pi_{2n_1}(S^{n_2}) & \pi_{2n_1}(D^{n_2}, S^{n_2-1}) \arrow[l, "q_*"] \arrow[r, "\partial"] \arrow[u, "\bar{\alpha}_*"'] & \pi_{2n_1-1}(S^{n_2-1}) \arrow[u, "\alpha_*"']
\end{tikzcd}
$$

Then, the proof follows from the commutativity in each block and $\partial^{-1}\circ q_*=E$.

LEMMA 3.

$$q_*(\gamma)=\lambda\circ E^{n_1}\alpha\,.$$

Proof. From pinching $S^{n_1}\vee S^{n_1}$ to a point we obtain a map $\tilde{p}:\tilde{L}\to S^{n_2}\vee S^{2n_1}$ such that the following diagram is commutative

$$
\begin{tikzcd}
\tilde{L} \arrow[r] \arrow[d, "\tilde{h}"'] \arrow[dr, phantom, "\tilde{p}"] & S^{n_2}\vee S^{2n_1} \arrow[d, "\iota_{n_2}\vee\lambda"] \\
S^{n_1}\cup e^{n_2} \arrow[r, "p"'] & S^{n_2}
\end{tikzcd}
$$

Then we have

$$\tilde{P}_*(\theta)=\iota_{2n_1}\circ E^{n_1}\alpha\,, \tag{3.3}$$

where ι_{2n_1} denotes the inclusion $S^{2n_1}\to S^{n_2}\vee S^{2n_1}$. For, let $q_1: S^{n_2}\vee S^{2n_1}\to S^{n_2}=S^{n_2}\vee S^{2n_1}/S^{2n_1}$ be the pinching map. Since it is clear that the composition

$$L=S^{n_1}\cup e^{n_2}\times S^{n_1}\xrightarrow[\text{pro.}]{} S^{n_1}\cup e^{n_2}\xrightarrow[p]{} S^{n_2}$$

is an extension of the map $q_1\circ\tilde{p}$ over L, we have

$$q_{1*}\tilde{p}_*(\theta)=0\,. \tag{3.4}$$

Next, let $q_2: S^{n_2}\vee S^{2n_1}\to S^{2n_1}=S^{n_2}\vee S^{2n_1}/S^{n_2}$ be the another pinching map. Then $q_2\circ\tilde{p}$ is extendable over L to the reduced join $S^{n_1}\cup e^{n_2}\times S^{n_1}=E^{n_1}(S^{n_1}\cup e^{n_2})=S^{2n_1}\cup e^{n_1+n_2}$. Hence we have

$$q_{2*}\tilde{p}_*(\theta)=E^{n_1}\alpha\,. \tag{3.5}$$

Now (3.3) follows from (3.4) and (3.5).

Since, from (3.1), (3.2) and (3.3), we have

$$q_*(\gamma)=p_*(\beta)=p_*\tilde{h}_*(\theta)=(\iota_{n_2}\vee\lambda)_*\tilde{p}_*(\theta)=\lambda\circ E^{n_1}\alpha\,.$$

Thus the proof is completed.

Now, by lemma 2, we have

$$E\alpha\circ\lambda=E[\iota_{n_1},\iota_{n_1}]=0\,.$$

Therefore, by the equality

$$E^{n_2+1}\alpha\circ E^{n_2}\lambda=\pm E^{n_1+1}\lambda\circ E^{2n_1+1}\alpha=\pm E^{n_1+1}(\lambda\circ E^{n_1}\alpha)\,,$$

we obtain

$$E^{n_1+1}(\lambda\circ E^{n_1}\alpha)=0\,.$$

Since E^{n_1+1} is isomorphic $(n_2>n_1+1)$ this means

$$\lambda\circ E^{n_1}\alpha=0\,. \tag{3.6}$$

On the other hand, it is clear that if $p_*(\beta)=0$ there exists a map $K\to S^{n_2}$ so that $H_*(K;Z)$ maps onto $H_*(S^{n_2})$. Hence the proof of Theorem is completed by lemma 3 and (3.6).

§ 4. Addendum.

Lemma 1 shows that if K is of type (A) $(n_1=n_2)$, K is an H-space. But this is not true in general $(n_2>n_1)$. In fact, lemma 2 and (3.2) show how to construct a complex of type (A). Let $[\iota_{n_1},\iota_{n_1}]$ be decomposable as follows $(s-1>n_1)$

$$[\iota_{n_1},\iota_{n_1}]=\alpha\circ\alpha'\,,\qquad \alpha\in\pi_{s-1}(S^{n_1})\quad\text{and}\quad\alpha'\in\pi_{2n_1-1}(S^{s-1})\,.$$

We consider the complex $S^{n_1}\cup e^s$ which is obtained from attaching e^s to S^{n_1} by α. Since $[\iota_{n_1}, \iota_{n_1}]=0$ in $\pi_{2n_1-1}(S^{n_1}\cup e^s)$ there exists a map $k: S^{n_1}\times S^{n_1}\to S^{n_1}\cup e^s$ such that $k|S^{n_1}\vee S^{n_1}$ is the identity map. We define the map $\tilde{k}: \tilde{L}\to S_u{}^1\cup e^s$ by

$$\tilde{k}|S^{n_1}\times S^{n_1}=k\,,$$

$$\tilde{k}|S^{n_1}\cup e^s\times s_0=\text{projection to the first factor}\,.$$

Let β be the element $\tilde{k}_*(\theta)$ of $\pi_{n_1+s-1}(S^{n_1}\cup e^s)$ and let K be the complex which is obtained from attaching e^{n_1+s} to $S^{n_1}\cup e^s$ by β. If n and s are both odd, $H^*(K; Z)$ is $E[x_1, x_2]$ and K is type of (A). For example, let σ be the generator of $\pi_8(S^7)$. Then we have

$$[\iota_7, \iota_7]=0=\sigma\circ 0\,.$$

Hence there exists a complex $K=S^7\cup e^9\cup e^{16}$ of type (A) with $H_*(K; Z)=E[x_1, x_2]$. Then K is not an H-space by the Theorem of Adams in [1].

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