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ON COMPARISON THEOREMS OF NONLINEAR VOLTERRA INTEGRAL EQUATIONS

Dedicated to Professor Yûsaku Komatu on his 60th birthday

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Introduction. In a nonlinear Volterra integral equation

$$x(t)=f(t)+\int_0^t K(t, s, x(s))ds\,, \tag{1}$$

we formally differentiate (1) to obtain an integro-differential equation

$$x'(t)=f'(t)+K(t, t, x(t))+\int_0^t K_t(t, s, x(s))ds\,, \tag{2}$$

where $K_t(t, s, x)\equiv\partial K(t, s, x)/\partial t$. If the function K does not explicity contain a variable t, the equation (2) is reduced to a differential equation. Hence the results concerning (1) or (2) are the generalizations of those corresponding to differential equations. As for the results in differential equations, for example, see [2].

The purpose of this paper is, at first, to obtain some comparison theorems for the integral equation

$$u(t)=f(t)+\int_0^t g(t, s, u(s))ds \tag{3}$$

or

$$u'(t)=f'(t)+g(t, t, u(t))+\int_0^t g_t(t, s, u(s))ds \tag{4}$$

with or without the monotonicity of $g(t, s, u)$ in u, where u is a real variable, and then to show that some of them may be applied to the study of the asymptotic behaviors of solutions and also to the successive approximation method for (1).

§1. Comparison theorems. As to the comparison theorems or existence of the maximal solution for (3), it is usually assumed that $g(t, s, u)$ is nondecreasing in u. For example, see [3]. The following result shows the existence of the maximal solution without the monotonicity of $g(t, s, u)$ in u. It is, however,

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assumed that $f(t)$ is differentiable and $g_t(t, s, u) \equiv \partial g(t, s, u)/\partial t$ is nondecreasing in u.

THEOREM 1. *Suppose that the following conditions are satisfied:*

(i) *$f(t)$ is differentiable for $0 \leqq t \leqq a$ with values in an open subset E of $R = \{x;\ -\infty < x < \infty\}$;*

(ii) *$g(t, s, u)$ and $g_t(t, s, u)$ are continuous for $0 \leqq s \leqq t \leqq a$ and $u \in E$, $|g(t, s, u)| \leqq M$, and $g_t(t, s, u)$ is nondecreasing in u for any fixed t, s.*

Then there exists a constant $\alpha \in (0, a]$ such that the equation (3) *has a continuous solution on $[0, \alpha]$. Furthermore there exists a solution $u^*(t)$ of* (3) *such that the inequality $u(t) \leqq u^*(t)$ is satisfied on $[0, \alpha]$ for any solution $u(t)$ of* (3).

Proof. Let β be an arbitrary number in $(0, a)$. It is easily observed that there exist a constant $\varepsilon_0 > 0$ and a compact subset K of E such that every continuous function $u(t)$ satisfying $|u(t) - f(t)| \leqq \varepsilon_0$ $(0 \leqq t \leqq \beta)$ belongs to K. If we choose $\alpha = \min(\beta, \varepsilon_0/M)$, by the usual method, it follows that the continuity of $f(t)$ and $g(t, s, u)$ implies the existence of continuous solutions of (3) on $[0, \alpha]$.

We next prove the existence of the maximal solution of (3) on $[0, \alpha]$. To this end, for any constant $\varepsilon \in (0, \varepsilon_0)$ we consider an integral equation

$$u(t) = f(t) + \varepsilon + \int_0^t (g(t, s, u(s)) + \varepsilon) ds. \tag{5}$$

Then for any $\gamma \in (0, \alpha)$, if we choose ε sufficiently small, for example, if $\gamma < \min(\beta, (\varepsilon_0 - \varepsilon)/(M + \varepsilon)) \leqq \alpha$, there exists a continuous solution $u(t, \varepsilon)$ of (5) on $[0, \gamma]$.

Let $u(t)$ and $u(t, \varepsilon)$ $(0 \leqq t \leqq \gamma)$ be the continuous solutions of (3) and (5) respectively. Then it is clear that $u(t)$ and $u(t, \varepsilon)$ are the solutions of the following integro-differential equations respectively:

$$u'(t) = f'(t) + g(t, t, u(t)) + \int_0^t g_t(t, s, u(s)) ds, \qquad u(0) = f(0), \tag{6}$$

$$u'(t) = f'(t) + g(t, t, u(t)) + \varepsilon + \int_0^t g_t(t, s, u(s)) ds, \qquad u(0) = f(0) + \varepsilon. \tag{7}$$

We first prove an inequality $u(t) \leqq u(t, \varepsilon)$ $(0 \leqq t \leqq \gamma)$. Let $t^* = \inf\{t \in [0, \gamma];\ u(t) \geqq u(t, \varepsilon)\}$. It is clear that $0 < t^* \leqq \gamma$, since $u(0) < u(0, \varepsilon)$. We claim that $t^* = \gamma$.

On the contrary, suppose that $0 < t^* < \gamma$. From the continuity of solutions, we have $u(t^*) = u(t^*, \varepsilon)$ and $u(t) \leqq u(t, \varepsilon)$ $(0 \leqq t \leqq t^*)$. Then

$$\begin{aligned} & f'(t^*) + g(t^*, t^*, u(t^*)) + \int_0^t g_t(t^*, s, u(s)) ds \\ & \quad = u'(t^*) = \lim_{t \to t^* - 0} \frac{u(t) - u(t^*)}{t - t^*} \geqq \lim_{t \to t^* - 0} \frac{u(t, \varepsilon) - u(t^*, \varepsilon)}{t - t^*} \\ & \quad = u'(t^*, \varepsilon) = f'(t^*) + g(t^*, t^*, u(t^*, \varepsilon)) + \varepsilon + \int_0^{t^*} g_t(t^*, s, u(s, \varepsilon)) ds. \end{aligned}$$

which is a contradiction, since $\varepsilon>0$, $u(t^*)=u(t^*, \varepsilon)$, and $g_t(t^*, s, u)$ is nondecreasing in u. Hence $t^*=\gamma$.

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a decreasing sequence such that $\varepsilon_n\to 0$ $(n\to\infty)$. Then by means of the same reason as above, it follows that the corresponding sequence of solutions, $\{u(t, \varepsilon_n)\}_{n=1}^{\infty}$, is nonincreasing on $[0, \gamma]$. Since the family of functions, $\{u(t, \varepsilon_n)\}$, is equicontinuous and uniformly bounded on $[0, \gamma]$, the sequence itself converges uniformly on $[0, \gamma]$. It is clear that the limiting function $u^*(t)$ is a solution of (3) on $[0, \gamma]$, and satisfies an inequality $u(t)\leqq u^*(t)$ $(0\leqq t\leqq\gamma)$ for any solution $u(t)$ of (3). Since $\gamma\in(0, \alpha)$ is arbitrary, we obtain the required result.

If $f(t)$ is a constant, we obtain the following result, for which the proof of the existence of solutions may be done as an easy application of the Schauder-Tychonov's fixed point theorem.

COROLLARY. *Suppose that*

(i) $g(t, s, u)$ *is continuous for* $0\leqq s\leqq t\leqq a$ *and* $|u-u_0|\leqq b$, *and* $|g(t, s, u)|\leqq M$;

(ii) $g_t(t, s, u)$ *is continuous for* $0\leqq s\leqq t\leqq a$ *and* $|u-u_0|\leqq b$, *and nondecreasing in* u *for any fixed* t, s.

Then there exists the maximal solution of

$$u(t)=u_0+\int_0^t g(t, s, u(s))ds$$

on $[0, \alpha]$, *where* $\alpha=\min(a, b/M)$.

The following result is a comparison theorem without the assumption of the monotonicity of $g(t, s, u)$ in u.

THEOREM 2. *In addition to the hypothese in Theorem* 1, *suppose that* $m(t)$ *is a continuous function on* $[0, \alpha]$, *and satisfies an inequality*

$$D^+m(t)\leqq f'(t)+g(t, t, m(t))+\int_0^t g_t(t, s, m(s))ds$$

on $[0, \alpha]$, *where* α *is the same number as in Theorem 1, and* $D^+m(t)$ *represents a Dini's derivative such that*

$$D^+m(t)=\overline{\lim_{h\to+0}}\frac{m(t+h)-m(t)}{h}.$$

Then the inequality $m(t)\leqq u^*(t)$ *is satisfied on* $[0, \alpha]$, *provided that* $m(0)\leqq f(0)$.

Proof. It is sufficient to prove an inequality $m(t)\leqq u(t, \varepsilon)$ on $[0, \gamma]$, where $\gamma\in(0, \alpha)$ is an arbitrary number and $u(t, \varepsilon)$ is a solution of (5) existing on $[0, \gamma]$ for suffiently small ε. Let $t^*=\inf\{t\in[0, \gamma]; m(t)\geqq u(t, \varepsilon)\}$. It is clear that $0<t^*\leqq\gamma$ since $m(0)<f(0)+\varepsilon=u(0, \varepsilon)$. We claim that $t^*=\gamma$.

On the contrary, suppose that $0<t^*<\gamma$. Then from the definition of t^* there exists a decreasing sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n\to t^*$ $(n\to\infty)$ and $m(t_n)\geqq u(t_n, \varepsilon)$. By the continuity of $m(t)$ and $u(t, \varepsilon)$, we have $m(t^*)=u(t^*, \varepsilon)$. Hence

$$f'(t^*)+g(t^*, t^*, m(t^*))+\int_0^{t^*} g_t(t^*, s, m(s))ds$$

$$\geqq D^+m(t^*)=\overline{\lim_{t_n\to t^*+0}}\frac{m(t_n)-m(t^*)}{t_n-t^*}\geqq\overline{\lim_{t_n\to t^*+0}}\frac{u(t_n, \varepsilon)-u(t^*, \varepsilon)}{t_n-t^*}$$

$$=u'(t^*, \varepsilon)=f'(t^*)+g(t^*, t^*, u(t^*, \varepsilon))+\varepsilon+\int_0^{t^*} g_t(t^*, s, u(s, \varepsilon))ds,$$

which is a contradition, since $\varepsilon>0$, $m(t^*)=u(t^*, \varepsilon)$, and $g_t(t^*, s, u)$ is nondecreasing in u. Hence $t^*=\gamma$. By the monotonicity of $u(t, \varepsilon)$ in ε, we obtain $m(t)\leqq u^*(t)$ on $[0, \gamma]$ as $\varepsilon\to 0$. Since $\gamma\in(0, \alpha)$ is arbitrary, we obtain the required result.

§2. **Applications of comparison theorem.** In this section, we will show some applications of Theorem 2. The following result will be useful for the study of asymptotic behaviors of solutions of (3) as in the theory of differential equations.

THEOREM 3. *Suppose that the following conditions are satisfied:*

(i) *$g(t, s, u)$ and $g_t(t, s, u)$ are continuous for $0\leqq s\leqq t<\infty$ and $0\leqq u<\infty$, and $g_t(t, s, u)$ is nondecreasing in u for any fixed t, s;*

(ii) *$K(t, s, x)$ and $K_t(t, s, x)$ are continuous for $0\leqq s\leqq t<\infty$ and $x\in R^n$;*

(iii) *$V(t, x)$ is a nonnegative function continuous for $[0, \infty)\times R^n$ and locally Lipschitzian in x, $a(r)$ is nonnegative, continuous, and strictly increasing for $0\leqq r<\infty$, and an inequality*

$$a(|x|)\leqq V(t, x) \tag{8}$$

is satisfied for $[0, \infty)\times R^n$;

(iv) *for any solution $x(t)$ of*

$$x(t)=x_0+\int_0^t K(t, s, x(s))ds, \tag{9}$$

an inequality

$$DV(t, x(t))\leqq g(t, t, V(t, x(t)))+\int_0^t g_t(t, s, V(s, x(s)))ds$$

is satisfied, where

$$DV(t, x(t))=\overline{\lim_{h\to+0}}\frac{1}{h}\Big(V\Big(t+h, x(t)+h\Big(K(t, t, x(t)) +\int_0^t K_t(t, s, x(s))ds\Big)\Big)-V(t, x(t))\Big);$$

(v) *$u^*(t)$ is the maximal solution of*

$$u(t)=u_0+\int_0^t g(t, s, u(s))ds \tag{10}$$

existing for $[0, \infty)$.

Then for any solution $x(t)$ *of* (9), *the inequality*

$$V(t, x(t)) \leqq u^*(t), \qquad 0 \leqq t < \infty$$

is satisfied, provided that $V(0, x_0) \leqq u_0$.

Proof. By the standard method, it follows that for any given $\varepsilon > 0$ sufficiently small there exists a solution $u(t, \varepsilon)$ of

$$u(t) = u_0 + \varepsilon + \int_0^t (g(t, s, u(s)) + \varepsilon) ds$$

for $[0, \infty)$. It is easily observed that there exists an interval $[0, \alpha)$, in which the equation (9) has a continuous solution $x(t)$. Hence we first prove an inequality $V(t, x(t)) \leqq u(t, \varepsilon)$ for $0 \leqq t < \alpha$.

Let $m(t) = V(t, x(t))$. From the hypotheses (iii) and (iv),

$$\begin{aligned} D^+ m(t) &= \overline{\lim_{h \to +0}} \frac{1}{h} (V(t+h, x(t+h)) - V(t, x(t))) \\ &= \overline{\lim_{h \to +0}} \frac{1}{h} \Big(V\Big(t+h, x(t) + h\Big(K(t, t, x(t)) \\ &\qquad + \int_0^t K_t(t, s, x(s)) ds\Big) + o(h)\Big) - V(t, x(t))\Big) \\ &\leqq DV(t, x(t)) \leqq g(t, t, m(t)) + \int_0^t g_t(t, s, m(s)) ds . \end{aligned}$$

Hence by Theorem 2, it follows that $m(t) \leqq u(t, \varepsilon)$ $(0 \leqq t < \alpha)$, provided that $m(0) = V(0, x_0) < u_0 + \varepsilon = u(0, \varepsilon)$. Thus the inequality $V(t, x(t)) \leqq u^*(t)$ $(0 \leqq t < \alpha)$ is obtained as $\varepsilon \to 0$.

We next prove that under the condition (8) the solution $x(t)$ is continuable to the whole interval $0 \leqq t < \infty$. On the contrary, suppose that there exists a finite interval $[0, t^*)$ $(0 < t^* < \infty)$, in which the solution $x(t)$ of (9) exists, but it is not continuable to the right beyond $t = t^*$.

For any fixed $T > t^*$, the solution $u^*(t)$ is bounded on $[0, T]$. Hence, if $u^*(t) \leqq M$ on $[0, T]$, the function $K(t, s, x)$ is bounded for $0 \leqq s \leqq t \leqq T$ and $|x| \leqq L$, where $L > a^{-1}(K)$ is a fixed constant. Then by the usual method, it follows that the limit of $x(t)$ as $t \to t^* - 0$ exists and $x(t)$ is continuable to the right beyond $t = t^*$, which is a contradiction. Hence we have $t^* = \infty$ and $V(t, x(t)) \leqq u(t, \varepsilon)$ for $0 \leqq t < \infty$. The required result is immediately obtained as $\varepsilon \to 0$.

COROLLARY. $g \equiv 0$ *is admissible in Theorem 3, and* $V(t, x(t)) \leqq V(0, x_0)$ $(0 \leqq t < \infty)$.

By means of the same method as above, we easily obtain the following

THEOREM 4. *Suppose that the following conditions are satisfied:*

(i) $g(t, s, u)$, $V(t, x)$, *and* $K(t, s, x)$ *satisfy the conditions* (i), (ii), (iii) *in Theorem 3;*

(ii) *$\varphi(r)$ is nonnegative and continuous for $0 \leqq r < \infty$ and an inequality*

$$DV(t, x(t)) + \varphi(|x(t)|) \leqq g(t, t, |x(t)|) + \int_0^t g_s(t, s, |x(s)|)ds, \qquad 0 \leqq t < \infty$$

is fulfilled for any continuous solution $x(t)$ of (9).

Then the inequality

$$V(t, x(t)) + \int_0^t \varphi(|x(s)|)ds \leqq u^*(t), \qquad 0 \leqq t < \infty$$

holds, provided that $V(0, x_0) \leqq u_0$, where $u^*(t)$ is the maximal solution of (10).

The following result is a direct consequence of Theorem 2 which assures the uniqueness of solutions.

THEOREM 5. *Suppose that the following conditions are satisfied:*

(i) *$f(t)$ is differentiable for $0 \leqq t \leqq a$;*

(ii) *$K(t, s, x)$, $K_t(t, s, x)$ are continuous for $0 \leqq s \leqq t \leqq a$ and $x \in R^n$;*

(iii) *$g(t, s, u)$, $g_t(t, s, u)$ are continuous for $0 \leqq s \leqq t \leqq a$ and $0 \leqq u < \infty$, $g_t(t, s, u)$ is nondecreasing in u for any fixed t and s, $g(t, s, 0) \equiv 0$, and an inequality*

$$\left| K(t, t, x(t)) - K(t, t, y(t)) + \int_0^t (K_t(t, s, x(s)) - K_t(t, s, y(s)))ds \right|$$
$$\leqq g(t, t, |x(t) - y(t)|) + \int_0^t g_t(t, s, |x(s) - y(s)|)ds$$

is satisfied for any continuous solutions $x(t)$ and $y(t)$ of

$$x(t) = f(t) + \int_0^t K(t, s, x(s))ds \tag{11}$$

existing on $0 \leqq t \leqq \alpha$ $(\leqq a)$;

(iv) *$u(t) \equiv 0$ is the maximal solution of*

$$u(t) = \int_0^t g(t, s, u(s))ds. \tag{12}$$

Then the integral equation (11) *has at most one solution.*

Proof. Let $x(t)$ and $y(t)$ be two solutions of (11) and $m(t) = |x(t) - y(t)|$. Then

$$\begin{aligned} D^+m(t) &\leqq |x'(t) - y'(t)| \\ &= \left| K(t, t, x(t)) - K(t, t, y(t)) + \int_0^t (K_t(t, s, x(s)) - K_t(t, s, y(s)))ds \right| \\ &\leqq g(t, t, m(t)) + \int_0^t g_t(t, s, m(s))ds. \end{aligned}$$

Hence by Theorem 2, it follows that $m(t) \leqq u^*(t)$, where $u^*(t)$ is the maximal solution of (12) which is identicaly equal to zero. This implies $x(t) \equiv y(t)$.

§ 3. **Successive approximation method.** As in the theory of differential equations, in order to apply comparison theorems to the successive approximation method originally due to Ważewsky, it is necessary to establish a comparison theorem, in which the monotonicity of $g(t, s, u)$ in u is assumed. By means of the same reason as before, the following Lemma will easily be proved.

LEMMA. *Suppose that*

(i) *$g(t, s, u)$ is continuous for $0 \leqq s \leqq t \leqq a$ and $0 \leqq u \leqq b$, $0 \leqq g(t, s, u) \leqq M$, and nondecreasing in u for any fixed t, s;*

(ii) *$m(t)$ is a continuous function satisfying an inequality*

$$m(t) \leqq \int_0^t g(t, s, m(s))ds \tag{13}$$

on $[0, \alpha]$, where $\alpha = \min(a, b/M)$.

Then an inequality $m(t) \leqq u^*(t)$ $(0 \leqq t \leqq \alpha)$ is satisfied, where $u^*(t)$ $(0 \leqq t \leqq \alpha)$ is the maximal solution of (12).

THEOREM 6. *Suppose that the following conditions are satisfied:*

(i) *$f(t)$ is continuous on $0 \leqq t \leqq a$;*

(ii) *$K(t, s, x)$ is defined and continuous for $0 \leqq s \leqq t \leqq a$, $x \in R^n$, and $|K(t, s, x)| \leqq M$ for any t, s, x;*

(iii) *in addition to the assumptions in Lemma, $g(t, s, 0) \equiv 0$ for any t, s, and the integral equation* (12) *has the unique solution $u(t) \equiv 0$;*

(iv) $$|K(t, s, x) - K(t, s, y)| \leqq g(t, s, |x-y|)$$

for any t, s, x, y.

Then the sequence $\{x_k(t)\}_{k=0}^{\infty}$ defined by

$$x_0(t) = f(t),$$
$$x_{k+1}(t) = f(t) + \int_0^t K(t, s, x_k(s))ds \qquad (k = 0, 1, \cdots)$$

is well defined on $[0, \alpha]$ as continuous functions on $[0, \alpha]$, where $\alpha = \min(a, b/M)$. Furthermore, the sequence $\{x_k(t)\}_{k=0}^{\infty}$ converges uniformly to the unique solution of the integral equation

$$x(t) = f(t) + \int_0^t K(t, s, x(s))ds \tag{14}$$

on $[0, \alpha]$.

Proof. In order to prove the above result, the analogous method to that in [1] will be used.

By the easy induction, it is clear that every function $x_k(t)$ is well defined on $[0, \alpha]$ as a continuous function.

We now define a sequence $\{u_k(t)\}_{k=0}^{\infty}$ on $[0, \alpha]$ as follows:

$$u_0(t)=Mt,$$

$$u_{k+1}(t)=\int_0^t g(t, s, u_k(s))ds \qquad (k=0, 1, \cdots).$$

Then every function $u_k(t)$ is well defined as a continuous function, and by the induction, it is easily proved that $0\leqq u_{k+1}(t)\leqq u_k(t)$ $(k=0, 1, \cdots)$ are satisfied on $[0, \alpha]$.

From the properties of g, it follows that the sequence $\{u_k(t)\}_{k=0}^{\infty}$ is uniformly bounded and equicontinuous. Since the sequence is nonincreasing, the sequence itself converges uniformly on $[0, \alpha]$. It is clear that the limiting function is a solution of (12), which is supposed to be unique and identically equal to zero.

Now from the definitions of $\{x_k(t)\}_{k=0}^{\infty}$, $\{u_k(t)\}_{k=0}^{\infty}$, and the monotonicity of g, it follows by the induction that $|x_{k+1}(t)-x_k(t)|\leqq u_k(t)$ $(k=0, 1, \cdots)$ on $[0, \alpha]$. Hence, if we put $\varphi(t)=\overline{\lim_{k\to\infty}}|x_{k+1}(t)-x_k(t)|$, by the uniform convergence of $u_k(t)$ to zero, the above inequality yields $\varphi(t)\equiv 0$ $(0\leqq t\leqq\alpha)$, which implies that the sequence $\{x_k(t)\}_{k=0}^{\infty}$ converges uniformly on $[0, \alpha]$. If we denote by $x^*(t)$ the limiting function, it is clear that $x^*(t)$ is a continuous solution of (14).

We next prove the uniqueness of solutions. Suppose that there are two solutions $x(t)$ and $y(t)$ of (14). Then we have

$$|x(t)-y(t)|\leqq\int_0^t |K(t, s, x(s))-K(t, s, y(s))|ds$$

$$\leqq\int_0^t g(t, s, |x(s)-y(s)|)ds.$$

If we put $m(t)=|x(t)-y(t)|$, we have an inequality

$$m(t)\leqq\int_0^t g(t, s, m(s))ds.$$

Then by the above lemma, it follows that the inequality $m(t)\leqq u^*(t)$ $(0\leqq t\leqq\alpha)$ is satisfied for the maximal solution $u^*(t)$ of (12), which is identically equal to zero. Hence $m(t)\equiv 0$, which implies $x(t)\equiv y(t)$ $(0\leqq t\leqq\alpha)$.

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