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ON HYPERSURFACES IN EVEN DIMENSIONAL CONTACT RIEMANNIAN MANIFOLDS

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Introduction. An even dimensional differentiable manifold $\tilde{M}$ is called an even dimensional contact manifold, if it admits a 1-form η such that $(d\eta)^n \neq 0$, where $\dim \tilde{M}=2n$. Then there exists naturally an almost Kählerian structure on $\tilde{M}$. Such a manifold was recently studied from the differential geometric point of view by K. Yano and Y. Mutō [8], [9], T. Nagano [1] and S. Sasaki [3].

In the present paper we study hypersurfaces of an even dimensional contact Riemannian manifold. In § 1 we recall first of all the definition of even dimensional contact Riemannian manifolds and some identities which hold in such manifolds and after some preliminaries, § 2 contains some identities which hold for hypersurfaces in an even dimensional contact Riemannian manifold. In § 3 we prove that if the hypersurface satisfies certain condition, it admits a contact structure. In § 4 an integral formula is obtained for closed hypersurfaces with constant mean curvature and applying the integral formula we prove that, under certain conditions, the hypersurface in question is totally umbilical. Finally in § 5 we consider an even dimensional contact Riemannian manifold in which the structure vector field ξ is contravariant almost analytic and study a hypersurface in this manifold.

1. Even dimensional contact Riemannian manifolds.

A $2n$-dimensional differentiable manifold $\tilde{M}$ is said to have an even dimensional contact structure and called an even dimensional contact manifold if there exists a 1-form η, to be called the contact form, on $\tilde{M}$ such that

$$(d\eta)^n \neq 0 \tag{1.1}$$

everywhere on $\tilde{M}$, where $d\eta$ is the exterior derivative of η.

In terms of local coordinate $\{y^\kappa\}$ of $\tilde{M}$ the contact form η and its exterior derivative are expressed as

$$\begin{aligned} \eta &= \eta_\lambda dy^\lambda , \\ d\eta &= F_{\lambda\mu} dy^\lambda \wedge dy^\mu , \end{aligned} \tag{1.2}$$

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where λ, μ run through the range $1, 2, \cdots, 2n$ and we have put

(1.3) $$F_{\lambda\mu}=\partial_\lambda\eta_\mu-\partial_\mu\eta_\lambda, \qquad \partial_\lambda=\partial/\partial y^\lambda.$$

Then $F_{\lambda\mu}$ is a skew-symmetric covariant tensor field on $\tilde{M}$ and has rank $2n$ everywhere over $\tilde{M}$. Therefore, there exists uniquely a skew-symmetric contravariant tensor field $F^{\lambda\kappa}$ over $\tilde{M}$ such that

$$F_{\lambda\mu}F^{\mu\kappa}=-\delta_\lambda^\kappa.$$

We put

$$\xi^\lambda=\eta_\mu F^{\mu\lambda},$$

then ξ^λ is a contravariant vector field over $\tilde{M}$ which vanishes on $\tilde{M}_0$, denoting the set of zero points of η by $\tilde{M}_0$. It is well-known that there exists a Riemannian metric $G_{\lambda\mu}$ and an almost complex structure

$$F_\mu{}^\lambda=F_{\mu\nu}G^{\nu\lambda}$$

such that $(F_\mu{}^\lambda, G_{\lambda\mu})$ is an almost Hermitian structure over $\tilde{M}$ (Sasaki [3]). We call $\tilde{M}$ with such tensor fields an even dimensional contact Riemannian manifold and $(\eta_\mu, F_\mu{}^\lambda, G_{\mu\nu})$ its structure tensors. $(F_\mu{}^\lambda, G_{\mu\nu})$ is an almost Kählerian structure.

If we put

$$\xi_\mu=G_{\mu\lambda}\xi^\lambda, \qquad \eta^\lambda=\eta_\mu G^{\mu\lambda},$$

we can easily verify the following relations:

$$G_{\nu\mu}\xi^\nu\eta^\mu=0,$$
$$F_\mu{}^\lambda\xi^\mu=-\eta^\lambda, \qquad F_\mu{}^\lambda\eta^\mu=\xi^\lambda,$$
$$F_\mu{}^\lambda\xi_\lambda=\eta_\mu, \qquad F_\mu{}^\lambda\eta_\lambda=-\xi_\mu.$$

2. Hypersurfaces. Let M be a $(2n-1)$-dimensional oriented differentiable manifold and i be an immersion of M into a $2n$-dimensional contact Riemannian manifold $\tilde{M}$ with the structure $(\eta_\mu, F_\mu{}^\lambda, G_{\mu\nu})$. We assume throughout the paper that the sets M and $\tilde{M}_0$ has no intersection. In terms of local coordinates $(x^1, \cdots, x^{2n-1})$ of M and $(y^1, \cdots, y^{2n})$ of $\tilde{M}$ the immersion $\imath$ is locally expressed by

$$y^\kappa=y^\kappa(x^1, \cdots, x^{2n-1}).$$

If we put $B_i{}^\lambda=\partial_i y^\lambda$, $\partial_i=\partial/\partial x^\imath$, then $B_i{}^\lambda$ are $(2n-1)$ local vector fields in $\tilde{M}$ spanning the tengent space at each point of M. A Riemannian metric g on M ishnaturally induced from the Riemannian metric G on $\tilde{M}$ by the immersion in such a way that

(2.1) $$g_{ji}=G_{\mu\lambda}B_j{}^\mu B_i{}^\lambda.$$

We take the unit normal C^λ to M in such a way that $B_1{}^\lambda, B_2{}^\lambda, \cdots, B_{2n-1}{}^\lambda, C^\lambda$ give the positive orientation of $\tilde{M}$.

Let H_{ij} be the second fundamental tensor of the immersion ι. Then the equations of Gauss and those of Weingarten are written as

$$\nabla_j B_i^\lambda = H_{ji} C^\lambda, \tag{2.2}$$

$$\nabla_i C^\lambda = -H_i^j B_j^\lambda, \tag{2.3}$$

where ∇_j is the so-called van der Waerden-Bortolotti covariant differentiation, where $\nabla_j B_i$ and $\nabla_j C$ are defined respectively by

$$\nabla_j B_i^\lambda = \partial_j B_i^\lambda - \left\{ {h \atop j\ i} \right\} B_h^\lambda + \widetilde{\left\{ {\lambda \atop \mu\ \nu} \right\}} B_j^\mu B_i^\nu ,$$

$$\nabla_j C^\lambda = \partial_j C^\lambda + \widetilde{\left\{ {\lambda \atop \mu\ \nu} \right\}} B_j^\mu C^\nu ,$$

$\left\{ {h \atop j\ i} \right\}$ and $\widetilde{\left\{ {\lambda \atop \mu\ \nu} \right\}}$ being the Christoffel's symbols of M and $\tilde{M}$ respectively.

The transform $F_\mu^\lambda B_i^\mu$ of the tangent vector field B_i^μ by F_μ^λ can be represented as a sum of its tangential part and its normal part, that is,

$$F_\mu^\lambda B_i^\mu = f_i^h B_h^\lambda + f_i C^\lambda, \tag{2.4}$$

By the similar way, we can easily see that

$$F_\mu^\lambda C^\mu = -f^i B_i^\lambda, \qquad (f^i = f_j g^{ji}). \tag{2.5}$$

On the other hand, η^λ being tangent to $\tilde{M}$ is expressed as a linear combination of B_i^λ and C^λ. Hence we can put

$$\eta^\lambda = u^h B_h^\lambda + r C^\lambda, \tag{2.6}$$

which implies

$$u_i = \eta_\lambda B_i^\lambda, \qquad r = \eta_\lambda C^\lambda. \tag{2.7}$$

Moreover we have

$$\xi^\lambda = F_\mu^\lambda \eta^\mu = (u^i f_i^j - r f^j) B_j^\lambda + u^i f_i C^\lambda. \tag{2.8}$$

It is well-known that an orientable hypersurface in an almost Hermitian manifold have an almost contact metric structure (Tashiro [5]). In our case, we find that (f_i^h, f_i, g_{ji}) is an almost contact Riemannian structure and so the following relations:

$$\begin{aligned} & f_i^j f_j^h = -\delta_i^h + f_i f^h, \qquad f_i^j f_j = 0, \\ & f^i f_i^j = 0, \qquad f^i f_i = 1 \end{aligned} \tag{2.9}$$

hold good.

From (2.4) we have

$$f_i^h = F_{\nu\mu} B_i^\nu B_j^\mu g^{jh}, \qquad f_i = C_\lambda F_\mu^\lambda B_i^\mu. \tag{2.10}$$

Differentiating $(2.7)_1$ covariantly and making use of (2.2), $(2.7)_2$, we have

$$\nabla_j u_i = (\tilde{\nabla}_\mu \eta_\lambda) B_j{}^\mu B_i{}^\lambda + r H_{ji}$$

or

$$(\tilde{\nabla}_\mu \eta_\lambda) B_j{}^\mu B_i{}^\lambda = \nabla_j u_i - r H_{ji}\,, \tag{2.11}$$

where we denote by $\tilde{\nabla}_\mu$ the operator of covariant differentiation with respect to $\widetilde{\left\{ {\lambda \atop \mu\ \nu} \right\}}$. Substituting (1.3) into (2.10) we have

$$\begin{aligned} f_i{}^h &= (\tilde{\nabla}_\nu \eta_\mu - \tilde{\nabla}_\mu \eta_\nu) B_i{}^\nu B_j{}^\mu g^{jh} \\ &= (\nabla_i u_j - \nabla_j u_i) g^{jh} \qquad \text{(using (2.11))}\,, \end{aligned}$$

or

$$f_{ji} = \nabla_j u_i - \nabla_i u_j\,. \tag{2.12}$$

from which we see that the 2-form $f = f_{ji} dx^j \wedge dx^i$ is closed.

3. Contact hypersurfuces. We consider an orientable hypersurface M in $\tilde{M}$. We assume in this section that the unit normal C^λ of M is represented as a linear combination of the vector fields η^λ and ξ^λ and that η^λ is not perpendicular to M. Then C^λ can be expressed as

$$C^\lambda = \alpha \eta^\lambda + \beta \xi^\lambda\,, \tag{3.1}$$

where α and β are some scalar functions. By means of (2.6) and (2.8), we have

$$C^\lambda = \{\alpha u^j + \beta(u^i f_i{}^j - r f^j)\} B_j{}^\lambda + (\alpha r + \beta u^i f_i) C^\lambda\,,$$

from which, by assumption, we have

$$\alpha u^j + \beta(u^i f_i{}^j - r f^j) = 0\,, \tag{3.2}$$

$$\alpha r + \beta u^i f_i = 1\,. \tag{3.3}$$

Suppose that $\beta = 0$ at some point of M. In that case, from (3.2) we have $u^j = 0$, because of $\alpha \neq 0$. Hence by virtue of (2.7) η^λ is perpendicular to M, which is a contradiction for our assumption. Consequently we see that $\beta \neq 0$ at each point of M.

Transvecting (3.2) with f_j and using (2.9), we have

$$\alpha u^j f_j - \beta r = 0\,. \tag{3.4}$$

From (3.3) and (3.4), we have

$$r = \alpha/(\alpha^2 + \beta^2)\,, \qquad u^i f_i = \beta/(\alpha^2 + \beta^2)\,. \tag{3.5}$$

Transvecting (3.2) with $f_j{}^k$ and making use of (2.9) and $(3.5)_2$, we have

$$\alpha u^j f_j{}^k - \beta u^k + (\beta^2/(\alpha^2 + \beta^2)) f^k = 0\,. \tag{3.6}$$

Again transvecting (3.6) with $f_k{}^i$ we have

$$\alpha u^i - \alpha u^j f_j f^i + \beta u^j f_j{}^i = 0 ,$$

or making use of $(3.5)_2$

$$\alpha u^k - (\alpha\beta/(\alpha^2+\beta^2)) f^k + \beta u^j f_j{}^k = 0 . \tag{3.7}$$

From (3.6) and (3.7) we have

$$u^k = (\beta/(\alpha^2+\beta^2)) f^k .$$

Now if we define 1-form f and 2-form ϕ on M by

$$f = f_i dx^i , \qquad \phi = f_{ij} dx^j dx^i ,$$

then (f, ϕ) is an almost cosymplectic structure (Okumura [2]), that is,

$$f \wedge \phi^{n-1} \neq 0 .$$

Moreover if we define a 1-form u by $u = u_i dx^i$, we have

$$u \wedge \phi^{n-1} = (\beta/(\alpha^2+\beta^2)) f \wedge \phi^{n-1} \neq 0 ,$$

which together with (2.12) shows that 1-form u is a contact 1-form on M and then M is a contact hypersurface. Summing up the result above, we have the following:

THEOREM 3.1. *Let M be an orientable hypersurface in an even dimensional contact Riemannian manifold. If the unit normal of M is represented as a linear conbination of the two vectors η and ξ and η is not prependicular to M, then the hypersurface admits a contact structure.*

As a special case of Theorem 3.1, we have

THEOREM 3.2. *A $(2n-1)$-dimensional sphere in a $2n$-dimensional Euclidean space is a contact hypersurface.*

Proof. We put in a $2n$-dimensional Euclidean space E^{2n}

$$\eta = \sum_{\alpha=1}^{n} (x^{n+\alpha} dx^\alpha - x^\alpha dx^{n+\alpha})/2$$

where $x^1, \cdots, x^{2n}$ are Cartesian coordinates. Then we see that

$$\eta_\mu = \left(\frac{1}{2} x^{n+\alpha}, -\frac{1}{2} x^\alpha\right), \qquad \xi^\lambda = \left(\frac{1}{2} x^\alpha, \frac{1}{2} x^{n+\alpha}\right),$$

$$F_\mu{}^\lambda = \begin{pmatrix} 0 & -\delta_\beta{}^\alpha \\ \delta_\beta{}^\alpha & 0 \end{pmatrix}, \qquad G_\mu{}^\lambda = \begin{pmatrix} \delta_{\beta\alpha} & 0 \\ 0 & \delta_{\beta\alpha} \end{pmatrix}.$$

By virtue of these equations we can easily see that E^{2n} admits an even dimensional contact Riemannian structure (Sasaki [3]).

Now, we consider a $(2n-1)$-dimensional sphere S in E^{2n}:

$$\sum_{\lambda=1}^{2n}(x^\lambda)^2=4.$$

Then ξ is a unit normal to S and η is a tangent to S. Consequently our assertion follows immediately from Theorem 3.1.

4. Closed hypersurfaces in which covariant differential of ξ is proportional to the displacement along the hypersurfaces. We consider a closed orientable hypersurface M immersed in $\tilde{M}$. We always assumed in this section that the covariant differential of the vector field ξ is proportional to the displacement along M and that the mean curvature is a constant. By assumption, we have

$$\nabla_i \xi^\lambda = \rho B_i{}^\lambda, \tag{4.1}$$

ρ being a scalar function on M.

In (2.8) putting

$$v^j = u^i f_i{}^j - r f^j, \qquad s = u^i f_i,$$

we have

$$\begin{aligned}\nabla_i \xi^\lambda &= \nabla_i(v^j B_j{}^\lambda + sC^\lambda)\\ &= (\nabla_i v^j - sH_i{}^j)B_j{}^\lambda + (v^j H_{ij} + \nabla_i s)C^\lambda.\end{aligned}$$

By assumption we find

$$\nabla_i v_j = \rho g_{ij} + sH_{ij}, \tag{4.2}$$

$$H_{ij}v^j + \nabla_i s = 0. \tag{4.3}$$

From (4.2) we have

$$\nabla_i v^i = (2n-1)\rho + sH_i{}^i.$$

Integrating over M, we find

$$\int_M [(2n-1)\rho + sH_i{}^i]\,dV = 0, \tag{4.4}$$

where dV denotes the volume element of M.

We now compute $\nabla_i(H_j{}^i v^j)$:

$$\nabla_i(H_j{}^i v^j) = \rho H_j{}^j + sH^{ji}H_{ji} + (\nabla_i H_j{}^i)v^j.$$

But, from the equation of Codazzi:

$$R_{\nu\mu\lambda\kappa}B_l{}^\nu B_i{}^\mu B_h{}^\lambda C^\kappa = \nabla_j H_{ih} - \nabla_i H_{jh},$$

$R_{\nu\mu\lambda\kappa}$ being the curvature tensor of $\tilde{M}$, we have, making use of $B_i{}^\mu B_h{}^\lambda g^{ih} = G^{\mu\lambda} - C^\mu C^\lambda$,

$$R_{\nu\kappa}B_j{}^\nu C^\kappa = -\nabla_i H_j{}^i$$

and consequently we have

$$\nabla_i(H_j{}^i v^j) = \rho H_j{}^j + sH_{ji}H^{ji} - R_{\nu\kappa}B_j{}^\nu v^j C^\kappa.$$

Thus, integrating over M, we have

$$\int_M R_{\nu\kappa}B_j{}^\nu v^j C^\kappa dV=\int_M [\rho H_i{}^i+sH_{ji}H^{ji}]dV\,. \tag{4.5}$$

If $H_i{}^i=0$, M is totally geodesic and consequently totally umbilical. In the case where $H_i{}^i\neq 0$, forming (4.5)—(4.4)$\times(1/(2n-1))H_i{}^i$, we have

$$\int_M R_{\nu\kappa}B_j{}^\nu v^j C^\kappa dV=\int_M s(H^{ji}H_{ji}-(1/(2n-1))H_i{}^iH_j{}^j)dV\,,$$

or

$$\int_M R_{\nu\kappa}B_j{}^\nu v^j C^\kappa dV=\int_M s\Big(H^{ji}-\frac{1}{2n-1}H_a{}^a g^{ji}\Big)\Big(H_{ji}-\frac{1}{2n-1}H_b{}^b g_{ji}\Big)dV\,. \tag{4.6}$$

Suppose that

$$\int_M R_{\nu\kappa}B_j{}^\nu v^j C^\kappa dV\leqq 0 \text{ (resp. } \geqq 0)\,,\qquad s>0 \text{ (resp. } <0)\,,$$

then we have

$$H_{ji}-\frac{1}{2n-1}H_b{}^b g_{ji}=0\,,$$

that is, the hypersurface under consideration is totally umbilical. From $(2.7)_1$ and $(2.10)_2$ we have $s=u^i f_i=C_\lambda \xi^\lambda$. Thus we have

THEOREM 4.1. *Let M be a closed orientable hypersurface of an even dimensional contact manifold and assume that the covariant differential of the structure vector field ξ is proposional to the displacement along M. If*

(i) *the mean curvature is constant,*

(ii) $\int_M R_{\nu\kappa}B_j{}^\nu v^j C^\kappa dV\leqq 0$ $(\geqq 0)$,

(iii) *the scalar product $C_\lambda\xi^\lambda$ does not change the sign (and is not $\equiv 0$), then the hypersurface is totally umbilical.*

N. B. (i) If the ambient manifold $\tilde{M}$ is a Kähler-Einstein manifold, then $R_{\nu\kappa}B_j{}^\nu v^j C^\kappa=0$ holds and consequently the second condition of Theorem 4.1 is automatically satisfied.

(ii) In a sphere S appeared at the last section, the vector field ξ satisfies the equation (4.1) together with $\rho=1/2$.

5. The case where ξ is almost analytic.

In this section we consider a $2n$-dimensional contact Riemannian manifold $\tilde{M}$ in which the vector field ξ is a contravariant almost analytic one with respect to the almost complex structure F (Yano [7]). In an even dimensional Euclidean space appeared in §3 the vector ξ is just contravariant almost analytic.

By assumption

$$\mathcal{L}(\xi)F_\mu{}^\lambda=0 \tag{5.1}$$

holds good, denoting Lie derivation with respect to ξ by $\mathcal{L}(\xi)$. A vector field ξ is called to be homothetic Killing if $\mathcal{L}(\xi)G_{\mu\lambda}=2cG_{\mu\lambda}$, c=constant and c to be homothetic constant. The following theorems are known:

THEOREM 5.1 (Sasaki [4]). *In a 2n-dimensional contact Riemannian manifold, the vector field ξ is contravariant almost analytic if and only if ξ is homothetic Killing with the homothetic constant* 1/2.

THEOREM 5.2 (Yano [6], [11]). *Let $\tilde{M}$ be an n-dimensional orientable Riemannian manifold and M a closed hypersurface in $\tilde{M}$ whose first mean curvature is constant. We suppose that $\tilde{M}$ admits a 1-parameter group of homothetic transformations such that the inner product of the generating vector ξ and the normal C to the hypersurface has constant sign on M, and the Ricci curvature R(C, C) with respect to the normal C is non-negative on M. Then every point of M is umbilical and R(C, C)=0 on M.*

From the last two theorems we have the following:

THEOREM 5.3. *Let $\tilde{M}$ be a 2n-dimensional contact Riemannian manifold in which the vector field ξ is contravariant almost analytic and M a closed hypersurface in $\tilde{M}$ whose first mean curvature is constant. We suppose that the inner product of the vector ξ and the normal C to the hypersurface has constant sign on M, and the Ricci curvature R(C, C) with respect to the normal C is non-negative on M. Then every point of M is umbilical and R(C, C)=0 on M.*

The vector fields ξ and η determine a 2-dimensional plane at each point of $\tilde{M}-\tilde{M}_0$. We denote it by σ-plane. Evidently the σ-plane is invariant by the almost complex structvre F. We cenote the distribution of σ-planes by D.

PROPOSITION 5.4. *In a 2n-dimensional contact Riemannian manifold with contravariant almost analytic vector field ξ, the distribution D is involutive.*

Proof. By assumption we have

$$\mathcal{L}(\xi)F_{\mu}{}^{\lambda}=0\,.$$

Then

$$[\xi,\eta]^{\lambda}=\mathcal{L}(\xi)\eta^{\lambda}=\mathcal{L}(\xi)(-F_{\mu}{}^{\lambda}\xi^{\mu})$$

$$=-(\mathcal{L}(\xi)F_{\mu}{}^{\lambda})\xi^{\mu}-F^{\nu\lambda}\mathcal{L}(\xi)\xi^{\mu}=0\,,$$

from which our assertion is proved.

By Proposition 5.4, we see that through each point of the hypersurface there passes a unique integral submanifold of the distribution D. Therefore, we can speak of this integral submanifold L. As the tangent space at each point of L is holomorphic, L is an invariant submanifold. Moreover it is well-known that in an almost Kählerian manifold, every invariant submanifold is always minimal (Schouten and Yano [4]).

Let L be locally expressed as $y^{\kappa}=y^{\kappa}(x^1, x^2)$. As L is invariant, we have

$$F_{\mu}{}^{\lambda}B_{i}{}^{\mu}=f_{i}{}^{j}B_{j}{}^{\lambda},$$

from which we can easily see that $(f_{i}{}^{j})$ is an almost complex structure of L. But, in general, an almost complex manifold of dimension 2 is necessarily complex (Yano and Mutō [10]). So L admits a complex structure.

Now, η^{λ} being tangent to L, we can put as $\eta^{\lambda}=u^{i}B_{i}{}^{\lambda}$, and then we have $u_{i}=\eta_{\lambda}B_{i}{}^{\lambda}$. Differentiating the equation covariantly along L, we have

$$\nabla_{j}u_{i}=(\tilde{\nabla}_{\mu}\eta_{\lambda})B_{j}{}^{\mu}B_{i}{}^{\lambda}+\sum_{A=3}^{2n}\eta_{\lambda}H_{Aji}C_{A}{}^{\lambda}=(\tilde{\nabla}_{\mu}\eta_{\lambda})B_{j}{}^{\mu}B_{i}{}^{\lambda},$$

where $C_{A}{}^{\lambda}$ are $2n-2$ orthonormals of L and H_{Aji} the second fundamental tensors with respect to L. Hence we have

$$f_{ji}=\nabla_{j}u_{i}-\nabla_{i}u_{j}. \tag{5.2}$$

If we put

$$\phi=f_{ji}dx^{j}dx^{i}, \qquad u=u_{i}dx^{i},$$

then from (1.1) and $\eta=\eta_{\lambda}dy^{\lambda}=u_{i}dx^{i}=u$ it follows that $\phi=du\neq 0$. Consequently L is a 2-dimensional contact Riemannian manifold. Thus we have the following:

PROPOSITION 5.5. *In a 2n-dimensional contact Riemannian manifold with contravariant almost analytic vector field ξ, the integral submanifold L of the distribution D is a Riemannian surface and admits a 2-dimensional contact Riemannian structure.*

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