M. SAKAI
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CONTINUOUS LINEAR FUNCTIONALS ON THE SPACE OF BOUNDED HARMONIC FUNCTIONS

BY MAKOTO SAKAI

Introduction.

Let $X=(X, \|\ \|)$ be a Banach space, and let T be a continuous linear functional on X. The norm of T is defined by $\|T\|=\sup_{x\in X^1}|T(x)|$, where X^1 is the set of points $x\in X$ such that $\|x\|\leqq 1$. If $y\in X^1$ satisfies $T(y)=\|T\|$, y is called an extremal point (an extremal function if X is a funcion space) of T. The following fundamental assertions are known:

(i) If X is reflexive, then for every continuous linear functional T there exist extremal points of T.

(ii) If X is strictly convex, then for every continuous linear functional T $(\neq 0)$ there exists at most one extremal point of T.

Let $HB(R)$ be the Banach space of all bounded harmonic functions u on a Riemann surface R with the supremum norm:

$$\|u\|_R=\sup_{z\in R}|u(z)|.$$

Then $HB(R)$ is neither reflexive nor strictly convex, in general. Hence there needs the special discussion to obtain the existence and uniqueness theorem of the extremal problems of continuous linear functionals on $HB(R)$.

In this paper we shall deal with the extremal problems of continuous linear functionals on $HB(R)$ and their applications to analytic mappings. In §1 we give the existence and uniqueness theorem of extremal functions of continuous linear functionals of $HB(R)$. To do this, we use the Wiener compactification of Riemann surfaces and the Riesz representation theorem. The definition of absolutely continuous linear functionals is given in §2. Linear functionals which appear in function theory are usually absolutely continuous. In §3 we are concerned with the extensions of continuous linear functionals. As a corollary we see that if $HB(R)$ is of infinite dimension, then $HB(R)$ is not separable. §4 deals with the so-called harmonic lengths. We give two examples of cycles whose extremal functions of harmonic lengths are not determined uniquely. In the last section, §5, we discuss applications of the extremal problems to analytic mappings.

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§ 1. Continuous linear functionals on the space of bounded harmonic functions and their extremal functions.

Let Δ be a compact Hausdorff space, and let $C(\Delta)$ be the Banach space of all continuous real-valued functions f on Δ with the supremum norm:

$$\|f\|_{\Delta}=\sup_{p\in\Delta}|f(p)|.$$

The Riesz representation theorem asserts that to each continuous linear functional T on $C(\Delta)$ there corresponds a unique real regular Borel measure μ on Δ such that

$$T(f)=\int_{\Delta}f\,d\mu \qquad (f\in C(\Delta)).$$

Let μ^{+}, μ^{-} and $|\mu|$ be the positive, negative and total variations of μ, respectively. Then we have $\mu=\mu^{+}-\mu^{-}$ and $\|T\|=|\mu|(\Delta)$. If we denote by $S(\nu)$ the support of a real regular Borel measure ν, we have the following lemmata. The proofs of them are omitted.

LEMMA 1.1. *Let g be an extremal function of T. Then $g=1$ on $S(\mu^{+})$ and $g=-1$ on $S(\mu^{-})$.*

LEMMA 1.2. *There exists an extremal function of T if and only if $S(\mu^{+})\cap S(\mu^{-})=\phi$.*

LEMMA 1.3. *There exists at most one extremal function of T if and only if $S(\mu)\equiv S(\mu^{+})\cup S(\mu^{-})=\Delta$.*

LEMMA 1.4. *There exists a unique extremal function of T if and only if each component of Δ is contained $S(\mu^{+})$ or $S(\mu^{-})$. If Δ is connected and if there exists a unique extremal function of T, then T is positive or negative.*

Let R be an open Riemann surface, and let $HB(R)$ be the Banach space of all bounded harmonic functions on R with the supremum norm. Let R_W be the Wiener compactification of R, and let Δ be the Wiener harmonic boundary of R (cf. Constantinescu-Cornea [3]).

Each function $u\in HB(R)$ can be extended continuously to R_W. We denote it by u_W. We define a mapping π by

$$\pi: u\longmapsto u_W|\Delta,$$

where $u_W|\Delta$ denotes the restriction of u_W to Δ. π is a one-to-one mapping of $HB(R)$ onto $C(\Delta)$. Moreover, π is an isometric isomorphism of $HB(R)$ onto $C(\Delta)$ (cf. Hayashi [5] and Sario-Nakai [10]).

LEMMA 1.5. *Let X and Y be Banach spaces, and let X^* and Y^* be the conjugate spaces of X and Y, respectively. Let π be an isometric isomorphism of X onto Y. Then x is an extremal point of $T\in X^*$ if and only if $\pi(x)$ is an extremal* *point of* $T\circ\pi^{-1}\in Y^*$.

Using these lemmata we have the following theorem:

THEOREM 1.6. *To each continuous linear functional* T *on* $HB(R)$, *there corresponds a unique real regular Borel measure* $\mu=\mu^+-\mu^-$ *on the Wiener harmonic boundary* Δ *of* R *such that*

$$T(u)=\int_{\Delta}\pi(u)d\mu \qquad (u\in HB(R)).$$

Moreover, there exists an extremal function of T *if and only if* $S(\mu^+)\cap S(\mu^-)=\phi$, *and there exists at most one extremal function of* T *if and only if* $S(\mu)=\Delta$.

The measure μ in theorem 1.6 is called the representing measure for T. Let z be a point on R. Then

$$T_z: u\longmapsto u(z)$$

is a continuous linear functional on $HB(R)$. The representing measure for T_z is called the harmonic measure with respect to z, and is denoted by $\omega=\omega_z$.

COROLLARY 1.7. *Let* μ *be the representing measure for a continuous linear functional* T *on* $HB(R)$. *If* ω *is absolutely continuous with respect to* $|\mu|$ *there exists at most one extremal function of* T.

Proof. If ω is absolutely continuous with respect to $|\mu|$, then $S(\omega)\subset S(|\mu|)$. Since $S(\omega)=\Delta$, we have $S(\mu)=\Delta$. Hence the corollary follows from theorem 1.6.

A generalized harmonic measure is a harmonic function u such that $0\leqq u\leqq 1$ on R and the greatest harmonic minorant of u and $(1-u)$ is equal to zero.

COROLLARY 1.8. *If* v *is the unique extremal function of* T, *then the function* $(1+v)/2$ *is a generalized harmonic measure.*

COROLLARY 1.9. *If there exists more than one extremal function of* T, *then for some extremal function* v *the function* $(1+v)/2$ *is not a generalized harmonic measure.*

§ 2. Absolutely continuous linear functionals and their representing measures.

Let $\{u_n\}$ be a sequence of bounded harmonic functions on a Riemann surface R. We say that $\{u_n\}$ converges boundedly to u if

(i) $\{u_n\}$ is uniformly bounded on R, and

(ii) $\{u_n\}$ converges uniformly on compact subsets of R to u.

THEOREM 2.1. *Let* T *be a continuous linear functional on* $HB(R)$. *Then the following two conditions are equivalent:*

(a) *If* $\{u_n\}\subset HB(R)$ *converges boundedly to* 0, *then* $\{T(u_n)\}$ *converges to* 0.

(b) *The representing measure* μ *for* T *is absolutely continuous with respect to the harmonic measure* ω.

Let us call a linear functional T *absolutely continuous* if it satisfies the conditions (a) and/or (b). First we show the following lemma:

LEMMA 2.2. *If* $\{u_n\}\subset HB(R)$ *converges boundedly to* 0, *then* $\left\{\int_E \pi(u_n)d\omega\right\}$ *converges to* 0 *for every Borel subset* E *of* $\varDelta$.

Proof. Set $v_n(z)=\pi^{-1}(\chi_E\cdot\pi(u_n))(z)\equiv\int_E \pi(u_n)d\omega_z$, where χ_E denotes the characteristic function of E. Then $\{v_n\}$ is uniformly bounded on R. A uniformly bounded subfamily of $HB(R)$ is a normal family with respect to uniform convergence on compact subsets of R. Assume that a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ converges uniformly on compact subsets of R to v. Then $|v|\leqq\|v\|_R\omega_E$ on R, where $\omega_E=\pi^{-1}(\chi_E)$. The sequence $\{u_{n_k}-v_{n_k}\}$ converges boundedly to $-v$, and hence $|-v|\leqq\|-v\|_R\omega_{\varDelta-E}$ on R, namely $|v|\leqq\|v\|_R(1-\omega_E)$ on R. There fore $v=0$, and $\{v_n\}$ converges boundedly to 0. This proves the lemma.

Proof of Theorem 2.1. (a) implies (b). Let E be a Borel subset of $\varDelta$ such that $\omega(E)=0$. Assume that $\mu(E)\neq0$. By the Hahn decomposition theorem we may assume that $\mu^+(E)>0$ and $\mu^-(E)=0$. Since μ^+ and μ^- are regular there exist a compact set $K\subset E$ such that $\mu^+(K)>0$ and an open set $O\supset E$ such that $\mu^-(O)<\mu^+(K)/2$. Since $\omega(E)=0$ and ω is regular, there exists a sequence $\{O_n\}$ of open subsets of $\varDelta$ such that $E\subset O_n\subset O$ and $\{\omega(O_n)\}$ converges to 0. Let f_n be continuous functions on $\varDelta$ such that $\chi_K\leqq f_n\leqq\chi_{O_n}$ on $\varDelta$. From the inequalities $\int_\varDelta f_n d\omega\leqq\omega(O_n)$ we see that $\{\pi^{-1}(f_n)\}$ converges boundedly to 0. The condition (a) implies that $\left\{T(\pi^{-1}(f_n))=\int_\varDelta f_n d\mu\right\}$ converges to 0. On the other hand, we have the following inequalities:

$$\begin{aligned}\int_\varDelta f_n d\mu&=\int_\varDelta f_n d\mu^+-\int_\varDelta f_n d\mu^-\\&\geqq\mu^+(K)-\mu^-(O_n)\\&\geqq\mu^+(K)-\mu^-(O)\\&>\mu^+(K)/2\,.\end{aligned}$$

This is a contradiction. Hence $\mu(E)=0$, and μ is absolutely continuous with respect to ω.

(b) implies (a). Let $\{u_n\}$ be a sequence of bounded harmonic functions on R such that $\|u_n\|_R\leqq1$ and $\{u_n\}$ converges uniformly on compact subsets of R. If μ is absolutely continuous with respect to ω, then the Radon-Nikodym theorem guarantees the existence of an ω-integrable function h on $\varDelta$ such that $d\mu=hd\omega$. It is sufficient to show that for any given $\varepsilon>0$ there exists a natural number N such that

$$|T(u_n)|=\left|\int_{\Delta}\pi(u_n)hd\omega\right|\leqq 3\varepsilon$$

for every $n\geqq N$. Since h is ω-integrable there exists a positive number M such that

$$\int_{E_M}|h|d\omega\leqq\varepsilon\,,$$

where $E_M=\{\zeta\in\Delta\,|\,|h(\zeta)|\geqq M\}$. Let m be a natural number such that $M<\varepsilon m$, and define E_j $(j=-m, -m+1, \cdots, m-1)$ by

$$E_j=\{\zeta\in\Delta\,|\,\varepsilon j\leqq h\{\zeta\}<\varepsilon(j+1)\}\,.$$

Then

$$\begin{aligned}&\left|\int_{E_j}\pi(u_n)hd\omega-\varepsilon j\int_{E_j}\pi(u_n)d\omega\right|\\ &\leqq\int_{E_j}|\pi(u_n)|\varepsilon d\omega\\ &\leqq\varepsilon\int_{E_j}d\omega\,.\end{aligned}$$

By lemma 2.2, to each E_j $(j\neq 0)$ there exists a natural number N_j such that

$$\left|\int_{E_j}\pi(u_n)d\omega\right|\leqq\frac{1}{(2m-1)|j|}$$

for every $n\geqq N_j$. Hence

$$\left|\int_{E_j}\pi(u_n)hd\omega\right|\leqq\varepsilon\int_{E_j}d\omega+\frac{\varepsilon}{(2m-1)}\,.$$

Put $N=\max_{j\neq 0}\{N_j\}$. Then

$$\begin{aligned}\left|\sum_{j=-m}^{m-1}\int_{E_j}\pi(u_n)hd\omega\right|&\leqq\varepsilon\int_{\Sigma E_j}d\omega+\varepsilon\\ &\leqq 2\varepsilon\end{aligned}$$

for every $n\geqq N$. Therefore

$$\begin{aligned}\left|\int_{\Delta}\pi(u_n)hd\omega\right|&\leqq\left|\sum_{j=-m}^{m-1}\int_{E_j}\pi(u_n)hd\omega\right|+\int_{E_M}|\pi(u_n)|\,|h|\,d\omega\\ &\leqq 2\varepsilon+\varepsilon\\ &=3\varepsilon\end{aligned}$$

for every $n\geqq N$.

As mentioned in the proof of lemma 2.2, a uniformly bounded subfamily of $HB(R)$ is a normal family with respect to uniform convergence on compact subsets of R. Therefore we have:

PROPOSITION 2.3. *Let T be an absolutely continuous linear functional on*

$HB(R)$. Then there exists an extremal function of T.

Remark. In § 4, we shall construct an absolutely continuous linear functional which has more than one extremal function.

A topological space Δ is called extremely disconnected if the closure of every open subset of Δ is again open. It is known that the Wiener harmonic boundary Δ is extremely disconnected (cf. Dixmier [4], Hayashi [5] and Mori [9]). We shall give an alternative proof of this fact.

COROLLARY 2.4. *The Wiener harmonic boundary Δ is extremely disconnected.*

Proof. Let O be an open subset of Δ, and let S be the support of $\chi_0 d\omega$. By definition, we have $\Delta - S \supset (\Delta - O)^\circ$, where $(\Delta - O)^\circ$ denotes the open kernel of $(\Delta - O)$. Since $S(\omega) = \Delta$, we have $\Delta - S \subset (\Delta - O)$. Hence $\Delta - S \subset (\Delta - O)^\circ$. Therefore $S = \Delta - (\Delta - O)^\circ$, namely S is equal to the closure $\bar{O}$ of O. We consider next the measure $d\mu = \chi_O d\omega - \chi_{\Delta - O} d\omega$. It is clear that positive and negative variations of $d\mu$ are equal to $\chi_O d\omega$ and $\chi_{\Delta - O} d\omega$, respectively. Hence $S(\mu) = \Delta$ and $S(\chi_O d\omega) = \bar{O}$. By theorem 1.6 and proposition 2.3 we have $S(\chi_O d\omega) \cap S(\chi_{\Delta - O} d\omega) = \phi$. Therefore $\bar{O} = \Delta - S(\chi_{\Delta - O} d\omega)$ is open.

§ 3. Extensions of continuous linear functionals.

Let A be a linear subspace of $HB(R)$, let T be a continuous linear functional on A and set

$$\|T\|_A = \sup_{u \in A^1} |T(u)|,$$

where $A^1 = \{u \in A \mid \|u\|_R \leqq 1\}$. A continuous linear functional $\hat{T}$ on $HB(R)$ is called an extension of T if

(i) $\hat{T}(u) = T(u)$ for every $u \in A$, and

(ii) $\|\hat{T}\| = \|T\|_A$.

The Hahn-Banach theorem asserts that for every continuous linear functional T there exists an extension of T.

In this section we consider the following problem: *Under what conditions does there exist an absolutely continuous extension of T?*

Let us call a linear functional T on A *absolutely continuous* if it satisfies the following condition:

(a) If $\{u_n\} \subset A$ converges boundedly to $u \in HB(R)$, then $\{T(u_n)\}$ converges.

THEOREM 3.1. *For every absolutely continuous linear functional T on every linear subspace of $HB(R)$ there exists an absolutely continuous extension of T if and only if $HB(R)$ is of finite dimension.*

Proof. Sufficiency. Assume that $HB(R)$ is of finite dimension. Then $HB(R)$ has a base consisting of HB-minimal functions (cf. Constantinescu-Cornea [3]).

Hence every boundedly convergent sequence converges uniformly on R. Therefore every extension of a continuous linear functional T is absolutely continuous.

Necessity. Assume that $HB(R)$ is of infinite dimension. It is easy to construct a continuous function g on Δ such that $0\leqq g\leqq 1$ on Δ, $\|g\|=1$ and $\omega(g^{-1}(1))=0$. Set $A=\{\alpha\cdot\pi^{-1}(g)\,|\,\alpha \text{ is real}\}$ and define T by $T(\alpha\pi^{-1}(g))=\alpha$. Then T is an absolutely continuous linear functional on A of the norm 1. Assume that there exists an absolutely continuous extension $\hat{T}$ of T. Then $\pi^{-1}(g)$ is an extremal function of $\hat{T}$, and by theorem 1.6 we see that $S(\hat{\mu}^+)\subset g^{-1}(1)$ and $S(\hat{\mu}^-)\subset g^{-1}(-1)=\phi$, where $\hat{\mu}^+$ and $\hat{\mu}^-$ are the positive and negative variations of the representing measure $\hat{\mu}$ for $\hat{T}$, respectively. Hence $\omega(S(\hat{\mu}^+))=0$ and $S(\hat{\mu}^-)=\phi$. Since $\hat{\mu}$ is absolutely continuous with respect to ω, we have $\hat{\mu}(S(\hat{\mu}^+))=0$. Therefore $\hat{T}\equiv 0$. This is a contradiction.

We give now two extension theorems.

THEOREM 3.2. *Let A be the image of a linear mapping ρ of $HB(R)$ into itself such that*

(i) $\rho\circ\rho=\rho$,

(ii) $\|\rho(u)\|_R\leqq\|u\|_R$ *for every $u\in HB(R)$, and*

(iii) *if $\{u_n\}\subset HB(R)$ converges boundedly to 0, then $\{\rho(u_n)\}$ converges boundedly to 0.*

Then for every absolutely continuous linear functional T on A there exists an absolutely continuous extension $\hat{T}$ of T.

Proof. We define $\hat{T}$ by $\hat{T}=T\circ\rho$. Then the assertion is evident.

Example 3.3. Let E be a Borel subset of Δ. We define a linear mapping ρ of $HB(R)$ into itself by $\rho(u)=\pi^{-1}(\chi_E\cdot\pi(u))$. It is easy to see that ρ satisfies the conditions (i) and (ii). lemma 2.2 implies that ρ satisfies the condition (iii).

Example 3.4. Consider an open Riemann surface R', and let R be a subregion of R' such that $R'-R$ is compact in R'. In the following the symbol "$'$" stands for "with respect to R'". We denote by $\bar{R}$ the closure of R in R'_W. Then there exists a unique continuous mapping j of R_W onto $\bar{R}$ which fixes R elementwise (cf. Sario-Nakai [10], Theorem IV. 5C). The mapping j is a homeomorphism of $R\cup j^{-1}(\Delta')$ onto $R\cup\Delta'$ such that if E is a Borel subset of Δ', then $\omega'(E)>0$ if and only if $\omega(j^{-1}(E))>0$. In other words, j^{-1} is a measure preserving transformation from a measure space (Δ',ω') onto a measure space $(j^{-1}(\Delta'), \omega'\circ j)$, and $\omega'\circ j$ is absolutely continuous with respect to ω. We define a linear mapping ρ of $HB(R)$ into itself by

$$\rho(u)=\pi'^{-1}\{(\pi(u)\,|\,j^{-1}(\Delta'))\circ j^{-1}\}\,|\,R\,.$$

Then ρ satisfies evidently the conditions (i) and (ii). Let z be a point of R, set $\omega=\omega_z$ and $\omega'=\omega'_z$, and define a linear functional T on $HB(R)$ by $T(u)=\rho(u)(z)$. Then

$$T(u)=\int_{\Delta'}\left\{(\pi(u)|j^{-1}(\Delta'))\circ j^{-1}\right\}d\omega'$$

$$=\int_{j^{-1}(\Delta')}\pi(u)d(\omega'\circ j)\,.$$

Since $\omega'\circ j$ is absolutely continuous with respect to ω, T is absolutely continuous. By theorem 2.1 ρ satisfies the condition (iii).

Every linear subspace A of $HB(R)$ is not always the image of a linear mapping ρ of $HB(R)$ into itself which satisfies the conditions (i)$\sim$(iii). Let $\hat{A}$ be the class of all bounded harmonic functions on R to which some sequence $\{u_n\}\subset A$ converges boundedly. If A is the image of such a mapping ρ, then we have $\hat{A}=A$.

Example 3.5. Assume that $R\notin O_{AB}$ and define $Re\,AB$ by

$$Re\,AB=\{Re\,f|f\in AB\}\,,$$

where AB denotes the class of all bounded analytic functions on R. Then $Re\,AB\subsetneqq\widehat{Re\,AB}$, and hence $Re\,AB$ is not the image of any linear mapping which satisfies the conditions (i)$\sim$(iii). In fact, there exists a nonconstant unbounded analytic function g on R such that $-1\leqq Re\,g\leqq 1$ on R. $Re\,g$ does not belong to $Re\,AB$. Let ϕ_n be conformal mappings of $\{w|-1<Re\,w<1\}$ into $\{w|-1<Re\,w<1, -n<Im\,w<n\}$ such that ϕ_n converges uniformly on compact subsets of R to the identity mapping. Then $\phi_n\circ g\in AB$, and $Re\,(\phi_n\circ g)$ converges boundedly to $Re\,g$. Hence $Re\,AB\subsetneqq\widehat{Re\,AB}$.

Let $\hat{\hat{A}}$ be the class of all bounded harmonic functions on R to which some sequence $\{u_n\}\subset A$ converges uniformly on compact subset of R.

THEOREM 3.6. *Let A be a linear subspace of $HB=HB(R)$ such that*

(i) $\hat{\hat{A}}=A$, *and*

(ii) *there exists a countable subset S of HB such that $A\cup S$ is dense in HB.*

Let T be a linear functional on A satisfying the following condition:

(a′) *If $\{u_n\}\subset A$ converges uniformly on compact subsets of R to 0, then $\{T(u_n)\}$ converges to 0.*

Then there exists an absolutely continuous extension $\hat{T}$ of T.

Proof. Let $\{R_n\}$ be an exhaustion of R. Set $\|u\|_{R_n}=\sup_{z\in R_n}|u(z)|$ and $\|u\|=\|u\|_R$, and define norms $\|u\|_k$ ($k=1, 2, \cdots$) by

$$\|u\|_k=\sum_{n=1}^{\infty}(\|u\|_{R_{k+n-1}}/2^n)\,.$$

Then

(1) $\quad\|u\|_k\leqq\|u\|\qquad$ for every k,

(2) $\|u\| \leqq 1$ if $\|u\|_k \leqq 1$ for every k,

(3) $k \leqq k'$ implies $\|u\|_k \leqq \|u\|_{k'} \leqq 2^{k'-k}\|u\|_k$,

(4) $\{u_n\}$ converges uniformly on compact subsets of R to 0 if $\{\|u_n\|_1\}$ converges to 0,

(5) $\|u\|_k \leqq 1$ if $\{u_n\}$ converges uniformly on compact subsets of R to u and $\|u_n\|_k \leqq 1$ for every n,

(6) $\{\|u_n\|_1\}$ converges to 0 if $\{u_n\}$ converges boundedly to 0.

Let H be the class of all harmonic functions on R, and set $H_k = \{u \in H \mid \|u\|_k < \infty\}$. Then (3) implies $H_k = H_1$ for every k, and (1) implies $HB \subset H_1$. Let $\{u_n\}$ be a sequence of functions in A, and let $u \in A$. If $\{\|u_n - u\|_k\}$ converges to 0, then (3) and (4) imply that $\{u_n\}$ converges uniformly on compact subsets of R to u. Hence $\{T(u_n)\}$ converges to $T(u)$. Therefore T is continuous with respect to the norms $\| \ \|_k$, and there exist the norms $\|T\|_k$ of T such that

$$|T(u)| \leqq \|T\|_k \cdot \|u\|_k \qquad (u \in A).$$

The Hahn-Banach theorem asserts that T can be extended to linear functionals $\hat{T}_k$ on H_1 of the same norms $\|T\|_k$. Hence

$$|\hat{T}_k(u)| \leqq \|T\|_k \cdot \|u\|_k \qquad (u \in H_1).$$

We restrict $\hat{T}_k$ to HB, and denote them again by $\hat{T}_k$. Then $\hat{T}_k$ are absolutely continuous. We denote the norms of $\hat{T}_k$ on HB by $\|\hat{T}_k\|_{HB}$. Then $\|T\|_A \leqq \|\hat{T}_k\|_{HB} \leqq \|T\|_k$ for every k. We shall see that $\{\|\hat{T}_k\|_{HB}\}$ converges to $\|T\|_A$. It is sufficient to show that $\|T\|_k \downarrow \|T\|_A$ as $k \to \infty$. By (3), $k \leqq k'$ implies $\|T\|_k \geqq \|T\|_{k'}$. Hence $\{\|T\|_k\}$ converges and satisfies $\lim_k \|T\|_k \geqq \|T\|_A$. Let v_k be functions of A such that $\|v_k\|_k \leqq 1$ and $T(v_k) \geqq \|T\|_k - 1/k$. Then $\|v_k\|_{R_n} \leqq 2^n$, and hence $\{v_k\}$ is uniformly bounded on R_n. The diagonal process guarantees the existence of a subsequence $\{v_{k_j}\}$ of $\{v_k\}$ such that $\{v_{k_j}\}$ converges uniformly on compact subsets of R to $v \in H$. By (3) and (5) we have $\|v\|_k \leqq 1$ for every k. Hence by (2) we have $\|v\| \leqq 1$. Since $\hat{\hat{A}} = A$, v belongs to A. From the inequality

$$T(v) = \lim_j T(v_{k_j}) \geqq \lim_k \|T\|_k$$

we have $\|T\|_A \geqq \lim_k \|T\|_k$, and hence $\|T\|_k \downarrow \|T\|_A$ as $k \to \infty$. Therefore we have shown that $\hat{T}_k$ are absolutely continuous linear functionals on HB and $\{\|\hat{T}_k\|_{HB}\}$ converges $\|T\|_A$. From (ii) we see that there exists a subsequence $\{\hat{T}_{k_j}\}$ of $\{\hat{T}_k\}$ such that $\{\hat{T}_{k_j}\}$ converges to a continuous linear functional $\hat{T}$ on HB in the weak-star topology on HB^*, namely $\lim_j \hat{T}_{k_j}(u) = \hat{T}(u)$ for each $u \in HB$. Since $\hat{T}$ is an extension of T, the theorem will be proved if we show the following lemma:

LEMMA 3.7. *Let $\{\mu_n\}$ be a sequence of regular Borel measures on Δ such that*

(i) *each of which is absolutely continuous with respect to the harmonic measure* ω,

(ii) $\{|\mu_n|(\Delta)\}$ *is uniformly bounded.*

Let μ *be a regular Borel measure, and suppose that* $\left\{\int_\Delta u d\mu_n\right\}$ *converges to* $\int_\Delta u d\mu$ *for every* $u\in C(\Delta)$. *Then* μ *is absolutely continuous with respect to* ω.

Proof. Let E be a Borel subset of Δ. Then there exist a continuous function h_E on Δ such that $h_E=\chi_E$ ω-a.e. on Δ (cf. Sario-Nakai [10], Theorem IV. 4D). Hence

$$\mu_n(E)=\int_\Delta \chi_E d\mu_n=\int_\Delta h_E d\mu_n\,,$$

and $\{\mu_n(E)\}$ converges for every Borel subset E of Δ. We denote its limit by $\nu(E)$. The Vitali-Hahn-Saks theorem asserts that ν is a Borel measure with finite total variation and is absolutely continuous with respect to ω. It is easy to see that $\nu=\mu$, and hence we have the lemma.

COROLLARY 3.8. *If* $HB(R)$ *is of infinite dimension, then* $HB(R)$ *is not separable.*

Proof. Assume that $HB(R)$ is of infinite dimension, and consider a linear subspace A and linear functional T on A which have been defined in the proof of theorem 3.1. We have seen that there does not exist any absolutely continuous extension of T. If $HB(R)$ is separable theorem 3.6 asserts that there exists an absolutely continuous extension of T. This is a contradiction.

Prof. K. Hayashi suggested to the author that corollary 3.8 should be extended to $C(\Delta)$ of an arbitrary extremely disconnected compact Hausdorff space (an arbitrary Stonian space). We show finally the following proposition:

PROPOSITION 3.9. *Let* Δ *be an extremely disconnected locally compact Hausdorff space. If* Δ *is infinite, then* $CB(\Delta)$ *is not separable, where* $CB(\Delta)$ *denotes the Banach space of all bounded continuous functions on* Δ *with the supremum norm.*

Proof. Assume that Δ is infinite, and let $\mathcal{O}$ be the class of all simultaneously open and closed subsets of Δ. Let p and q be two distinct points of Δ. Then the Hausdorff separation axiom implies the existence of disjoint open sets U and V such that $p\in U$ and $q\in V$. Since $\bar{U}$ is again open we have that $\bar{U}\in\mathcal{O}$ and $\Delta-\bar{U}$ $(\neq\phi)\in\mathcal{O}$. Hence Δ is decomposed by two elements of $\mathcal{O}$. Since Δ is infinite, at least one element of them is infinite and is decomposed again by two elements of $\mathcal{O}$. Repeating this process we have a sequence $\{O_n\}$ of mutually disjoint sets $O_n\in\mathcal{O}$. Let $\alpha=\{a_n\}$ be a sequence of numbers 0 and 1. We define $f_\alpha^N\in CB(\Delta)$ by

$$f_\alpha^N(p)=\begin{cases} a_n & (p\in O_n,\ n\leqq N) \\ 0 & (p\in\Delta-\bigcup_{n=1}^N O_n)\,. \end{cases}$$

Since $\varDelta$ is extremely disconnected, every bounded subset of $CB(\varDelta)$ has a supremum in $CB(\varDelta)$ relative to the natural ordering for real-valued functions. Set $f_\alpha=\sup_N f_\alpha^N\in CB(\varDelta)$. Then we have $f_\alpha(p)=a_n$ ($p\in O_n$) for every n. In fact, it is sufficient to prove when $a_n=0$. Let p be a point of O_n. Since $\varDelta$ is a locally compact Hausdorff space, there exists a function $g\in CB(\varDelta)$ such that $0\leqq g\leqq 1$ on $\varDelta$, $g=1$ on $\varDelta-O_n$ and $g(p)=0$. Since $f_\alpha^N\leqq g$ for every N we have $f_\alpha\leqq g$, and hence $f_\alpha(p)=0$. Therefore we see that $\|f_\alpha-f_{\alpha'}\|_\varDelta=1$ if and only if $\alpha\neq\alpha'$. Set

$$D_\alpha=\left\{f\in CB(\varDelta)\mid \|f-f_\alpha\|_\varDelta<\frac{1}{2}\right\}.$$

Then $\alpha\neq\alpha'$ implies $D_\alpha\cap D_{\alpha'}=\phi$. Let S be a dense subset of $CB(\varDelta)$. Then for every sequence α of numbers 0 and 1 there exists at least one function $f\in S$ such that $f\in D_\alpha$. Since the class of all such sequences is uncountable, S is uncountable.

§ 4. **Harmonic length.**

Let R be a Riemann surface, and let c be a cycle on R. We define the harmonic length $h(c)$ of c by

$$h(c)=\sup_{u\in U}\int_c du^*,$$

where U denotes the class of all harmonic functions on R such that $0\leqq u\leqq 1$ on R, and du^* denotes the conjugate differential of du. A function $v\in U$ satisfying

$$h(c)=\int_c dv^*$$

is called an extremal function of the harmonic length of c (or briefly an extremal function of c).

The notion of harmonic length was introduced by Landau-Osserman [8]. They showed the uniqueness of extremal functions of cycles on Dirichlet domains. Suita [11] improved this result and Kato [7] obtained the uniqueness theorem for cycles on finite Riemann surfaces (the definition is found in lemma 5.2).

In this section we shall give first an alternative proof of Suita's theorem and construct next two examples of cycles whose extremal functions are not determined uniquely.

We consider the following functional:

$$T_c:\ u\longmapsto\frac{1}{2}\int_c du^*.$$

Then T_c is absolutely continuous linear functional on $HB(R)$. Set $H^1=\{u\in H\mid \|u\|_R\leqq 1\}$. Then $u\in U$ if and only if $2u-1\in H^1$. Hence $h(c)=\|T_c\|$, and v is an extremal function of the harmonic length of c if and only if $2v-1$ is an extremal function of T_c.

THEOREM 4.1 (Suita [11]). *Let R_S be the Stoilow compactification of a Riemann surface R. Let c be a dividing cycle which divides the boundary R_S-R into two closed sets A and B in such a way that the intersection number of c with any curve starting from A to B is equal to* 1. *If $h(c)>0$, the function $v\in U$ satisfying*

$$h(c)=\int_c dv^*$$

is unique and coincides with the harmonic measure of B.

Proof. Since the Wiener compactification R_W is of Stoilow type, the cycle c divides the Wiener harmonic boundary Δ into two closed disjoint sets S_A and S_B. Let $O(\neq\phi)$ be an open subset of S_B. Then there exists a non-negative continuous function $f(\neq 0)$ on Δ whose support is contained in O. Since $u=\pi^{-1}(f)$ is positive on c and since $f=0$ on S_A, we have

$$T_c(u)=\frac{1}{2}\int_c du^*>0$$

(cf. Ahlfors-Sario [1], III. 4C). Therefore $S_B\subset S(\mu_c^+)$, where μ_c^+ is the positive variation of the representing measure μ_c for T_c. Similarly we have $S_A\subset S(\mu_c^-)$, where μ_c^- is the negative variation of μ_c. Hence $S_B=S(\mu_c^+)$ and $S_A=S(\mu_c^-)$, and from theorem 1.6 and corollary 1.8 we have the theorem.

If $R\in O_{HB}$, then $h(c)=0$ for every cycle c on R. If $R\in O_{HB}^2-O_{HB}$ and if $h(c)>0$ the extremal function of cycle c is unique. We construct now an example of a cycle c on $R\in O_{HB}^3-O_{HB}^2$.

Example 4.2. Let $R_0\in O_{HB}-O_G$ be a Riemann surface which has one ideal boundary component. Let l_0 be an analytic Jordan arc on R_0, and let R_j-l_j ($j=1, 2, 3$) be copies of R_0-l_0. We joint these copies along their analytic arcs identifying the upper edges of l_j of R_j with the lower edges of l_{j+1} of R_{j+1} (mod 3). Thereby a Riemann surface R is constructed as a covering surface (R, ϕ) of R_0 (cf. Ahlfors-Sario [1]). R is of class $O_{HB}^3-O_{HB}^2$ (cf. Constantinescu-Cornea [2]). Let $c_0=c_0(t)$ $(0\leqq t\leqq 1)$ be an analytic Jordan curve on R_0 which lies in a small neighborhood of l_0 and encircles l_0 once. Let $c_1=(\phi^{-1}\circ c_0)(t)$ $(0\leqq t\leqq 1)$ be an analytic Jordan curve on R which is a component of $\phi^{-1}(c_0)$ lying on R_1, and let $c_2=(\phi^{-1}\circ c_0)(1-t)$ $(0\leqq t\leqq 1)$ be an analytic Jordan curve on R which is a component of $\phi^{-1}(c_0)$ lying on R_2. We set $c=c_1+c_2$. Let p_j be points of the Wiener harmonic boundary Δ of R corresponding to the ideal boundary of R_j, and set $\omega_j=\pi^{-1}(\chi_{\{p_j\}})$. Then $\int_c d\omega_3^*=0$, and $\int_c d\omega_1^*=-\int_c d\omega_2^*\neq 0$. Hence $S(\mu_c)=\{p_1, p_2\}\neq\Delta$, where μ_c denotes the representing measure for T_c. From theorem 1.6 we find that the extremal function of c are not determined uniquely.

Let W be a plane region, and set $\widetilde{W}=\{z|\bar{z}\in W\}$, where $\bar{z}$ is the complex conjugate of z. Let u be a function on W, let $\omega(z)=a(z)dx+b(z)dy$ be a first order differential on W and let $c=\sum_{i=1}^{n} c_i(t)$ $(0\leqq t\leqq 1)$ be a differentiable 1-chain on W. We define a function $\tilde{u}$ on $\widetilde{W}$, a differential $\tilde{\omega}$ on $\widetilde{W}$ and a 1-chain $\tilde{c}$ on $\widetilde{W}$ as follows:

$$\tilde{u}(z)=u(\bar{z}),$$

$$\tilde{\omega}(z)=a(\bar{z})dx-b(\bar{z})dy,$$

$$\tilde{c}=\sum_{i=1}^{n}\overline{c_i(t)}.$$

It is easy to show that

(1) $d\tilde{u}=\widetilde{du}$ for $u\in C^1(W)$,

(2) $\tilde{\omega}^*=-\tilde{\omega}^*$ for every differential ω on W,

(3) $\int_{\tilde{c}}\tilde{\omega}=\int_c \omega$ for every differential ω on W and for every differentiable 1-chain c on W.

Using (1)$\sim$(3) we have the following lemma:

LEMMA 4.3. *Let W be a plane region such that $\widetilde{W}=W$, let u be a harmonic function on W such that $\tilde{u}=u$ (or $\tilde{u}=-u$) and let c be a differentiable 1-chain such that $\tilde{c}$ is homologous to c (or $\tilde{c}$ is homologous to $-c=c(1-t)$, resp.). Then we have*

$$\int_c du^*=0.$$

Example 4.4. Let E be a compact subset of the closed interval $[-1, 1]$ such that $E\in N_B$ and E is of positive capacity. Set

$$D_\pm=\{z|\,|z\mp 3i|\leqq 1\}$$

$$W=\{z|\,|z|\leqq\infty\}-(D_+\cup D_-\cup E),$$

and define cycles $c_\pm=c_\pm(t)$, $c=c(t)$ $(0\leqq t\leqq 1)$ by

$$c_\pm(t)=\pm 3i+2e^{\pm 2\pi i t},$$

$$c=c_-+c_+.$$

We shall show that the extremal functions of the harmonic length of c are not determined uniquely. From theorem 1.6 it is sufficient to show that $S(\mu_c)\neq\Delta$, where μ_c is the representing measure for T_c.

Let U be a small simply connected neighborhood of E such that $\widetilde{U}=U$, and let $\overline{U-E}$ be the closure of $U-E$ in the Wiener compactification of W. Let f be an arbitrary continuous function on Δ whose support is contained in $\overline{U-E}\cap\Delta$, and set $u=\pi^{-1}(f)$. Then $\tilde{u}=u$. In fact, the function $v=u-\tilde{u}$ satisfies $\tilde{v}=-v$, and every cycle $\gamma=\gamma(t)$ $(0\leqq t\leqq 1)$ on $U-E$ is homologous to $-\tilde{\gamma}=\tilde{\gamma}(1-t)$ $(0\leqq t\leqq 1)$. Hence by lemma 4.3 we have

$$\int_{\gamma} dv^*=0$$

for every cycle γ on $U-E$. Therefore we have a single-valued analytic function $g=v+iv^*$ on $U-E$ such that $Re\, g=v$ is bounded, where v^* denotes a conjugate harmonic function of v. Since $E\in N_B$, g can be extended analytically onto U, and hence v is a bounded harmonic function on $W\cup E$. Since the boundary values of v is equal to zero we have $v=0$ on $W\cup E$, and therefore $\tilde{u}=u$ on W. Since $\tilde{c}$ is homologous to c, by lemma 4.3 we have

$$\int_{c} du^*=0\,.$$

Therefore $S(\mu_c)\subset\Delta-(\overline{U-E}\cap\Delta)\neq\Delta$.

§5. Applications of the extremal problems to analytic mappings.

Applications of the extremal problems of harmonic lengths to analytic mappings is studied by Landau-Osserman [8], Suita [11] and Kato [7]. In this section we study applications of the extremal problems of continuous linear functionals on HB to analytic mappings.

Let ϕ be an analytic mapping of a Riemann surface R into a Riemann surface S, and let T be a continuous linear functional on $HB(R)$. We define a continuous linear functional $\phi_*(T)$ on $HB(S)$ by

$$\phi_*(T)(u)=T(u\circ\phi) \qquad (u\in HB(S))\,.$$

Then we have the following theorem:

THEOREM 5.1. *Let $\phi_*(T)$ be a continuous linear functional on $HB(S)$ defined above. Then*

(1) $\|\phi_*(T)\|\leqq\|T\|$,

(2) *if T is absolutely continuous, then $\phi_*(T)$ is absolutely continuous,*

(3) *if $\|\phi_*(T)\|=\|T\|$ and if the extremal function u of T is unique, then there exists at most one extremal function v of $\phi_*(T)$,*

(4) *in addition to the assumption of* (3) *if u is nonconstant and if there exists an extremal function of $\phi_*(T)$, then v is nonconstant and ϕ is of type Bl in the sence of Heins* [6].

Proof. (1) If $v\in HB(S)$ satisfies $\|v\|_S\leqq 1$, then $v\circ\phi\in HB(R)$ satisfies $\|v\circ\phi\|_R\leqq 1$. Hence $\|\phi_*(T)\|\leqq\|T\|$.

(2) If $\{v_n\}\subset HB(S)$ converges boundedly to v, then $\{v_n\circ\phi\}\subset HB(R)$ converges boundedly to $v\circ\phi$. Hence we have (2).

(3) The assumption of (3) implies $u=v\circ\phi$. If ϕ is a constant mapping,

then u is constant. By the uniqueness of the extremal function of T, u is equal to ± 1. From the maximum principle for harmonic functions v is equal to ± 1, and hence v is determined uniquely. If ϕ is nonconstant the image $\phi(R)$ of R under ϕ is open, and hence v is determined uniquely.

(4) We set $u_0=(1+u)/2$ and $v_0=(1+v)/2$. Then, by corollary 1.8, u_0 and v_0 are generalized harmonic measures and satisfy $u_0=v_0\circ\phi$. If u_0 is nonconstant, then v_0 is nonconstant and ϕ is of type Bl (cf. Heins [6]).

We consider now an application of theorem 5.1. Let z and z' be two points of a Riemann surface R. We define an absolutely continuous linear functional $T_{zz'}$ on $HB(R)$ by

$$T_{zz'}(u)=u(z)-u(z'),$$

and set $\rho_R(z, z')=\|T_{zz'}\|$. Then $\rho_R=\rho_R(z, z')$ is a pseudo-metric on R. Let E be a compact set defined in example 4.4, and set $G=\{z \mid |z|\leqq\infty\}-E$. Then for every $z\in G$ we have $\rho_G(z, \bar{z})=0$. Hence ρ_R is not a metric on R, in general. Let W be a plane region defined in example 4.4. Then ρ_W is a metric on W and for every point z the extremal functions of $T_{z\bar{z}}$ are not determined uniquely. We show first the following lemma:

LEMMA 5.2. *Let R be a finite Riemann surface, namely a proper subregion of a compact Riemann surface whose boundary ∂R consists of a finite number of analytic Jordan curves. Then*

(i) *ρ_R is a metric on R,*

(ii) *for every pair of distinct points z, z' the extremal function of $T_{zz'}$ is determined uniquely.*

Proof. Let $\hat{u}$ denote the boundary values (the non-tangential limits) of $u\in HB(R)$ on ∂R. Then

$$u(z)=\frac{1}{2\pi}\int_{\partial R}\hat{u}(\zeta)\frac{\partial g(\zeta, z)}{\partial n_\zeta}\,ds_\zeta \qquad (u\in HB(R)),$$

where $g(\zeta, z)=g_R(\zeta, z)$ denotes the Green's function of R, and the derivative $\partial/\partial n_\zeta$ is in the direction of the left normal. From an argument similar to § 1, it is sufficient to show that if $z\neq z'$, then $\partial g(\zeta, z)/\partial n_\zeta-\partial g(\zeta, z')/\partial n_\zeta$ does not vanish on a set of positive measure. Since the Green's function of R vanishes on ∂R, we have

$$d(g(\zeta, z)+ig^*(\zeta, z))-d(g(\zeta, z')+ig^*(\zeta, z'))$$

$$=-i\Big(\frac{\partial g(\zeta, z)}{\partial n_\zeta}-\frac{\partial g(\zeta, z')}{\partial n_\zeta}\Big)(ds_\zeta+idn_\zeta) \qquad \text{on } \partial R,$$

where g^* denotes a conjugate function of g. From the fact that the above differential ($\neq 0$) is analytic on a neighborhood of ∂R we obtain that $\partial g(\zeta, z)/\partial n_\zeta-\partial g(\zeta, z')/\partial n_\zeta$ has at most a finite number of zeros on ∂R.

We show now a corollary of theorem 5.1.

COROLLARY 5.3. *Let ϕ be an analytic mapping of a Riemann surface R into a Riemann surface S. Then*

$$\rho_S(\phi(z), \phi(z')) \leqq \rho_R(z, z')$$

for every pair of points z, z'. Moreover, if R satisfies (i) *and* (ii) *in lemma* 5.2 *and if equality holds for some pair of distinct points z, z', then ϕ is of type Bl.*

Proof. It is easy to see that $\phi_*(T_{zz'}) = T_{\phi(z)\phi(z')}$. By theorem 5.1 we have the corollary.

We show finally the following theorem:

THEOREM 5.4. *Let R and S be finite Riemann surfaces, and let ϕ be an analytic mapping of R into S. If $\rho_S(\phi(z), \phi(z')) = \rho_R(z, z')$ for some pair of distinct points z, z', then ϕ is an n-to-one mapping of R onto S for some natural number n. Moreover, if $R=S$, then $n=1$ namely ϕ is a conformal mapping of R onto itself.*

Proof. Let u and v be the extremal functions of $T_{zz'}$ and $\phi_*(T_{zz'})$, respectively. We define $\partial R^{\pm}$ by

$$\partial R^{\pm} = \left\{ \zeta \in \partial R \,\middle|\, \frac{\partial g_R(\zeta, z)}{\partial n_\zeta} - \frac{\partial g_R(\zeta, z')}{\partial n_\zeta} \gtrless 0 \right\},$$

and define similarly $\partial S^{\pm}$ with respect to the points $\phi(z)$, $\phi(z')$. Then $u(\zeta) = \pm 1$ if $\zeta \in \partial R^{\pm}$, and $v(\eta) = \pm 1$ if $\eta \in \partial S^{\pm}$. If $\zeta \in \partial R^{\pm}$, by the reflection principle, u can be extended harmonically onto some simply connected neighborhood U of ζ. Let $\{z_n\} \subset R$ be a sequence which converges to ζ. Since $u = v \circ \phi$, $\{\phi(z_n)\}$ tends to ∂S. Let $\eta \in \partial S$ be an accumulating point of $\{\phi(z_n)\}$. Then ϕ can be extended analytically onto some neighborhood $U' \subset U$ of ζ and $\eta = \phi(\zeta) \in \partial S^{\pm}$. In fact, assume that $\eta \in \partial S - (\partial S^+ \cup \partial S^-)$. Then there is a neighborhood V of η such that $V - \partial S^{\mp}$ is simply connected and v can be extended harmonically onto $V - \partial S^{\mp}$. Hence there exist single-valued analytic functions $f = u + iu^*$ on U and $g = v + iv^*$ on $V - \partial S^{\mp}$ such that $\phi = g^{-1} \circ f$. This contradicts the assumption of η. Therefore $\eta \in \partial S^+ \cup \partial S^-$. Hence v can be extended harmonically onto some simply connected neighborhood of η. We have again $\phi = (v + iv^*)^{-1} \circ (u + iu^*)$ on some neighborhood U' of ζ. This implies the assertion.

Let $\hat{R}$ and $\hat{S}$ be the double of R and S, respectively. Then, by the above formula, ϕ can be extended analytically onto $\hat{R}$. The extension $\hat{\phi}$ is an analytic mapping $\hat{R}$ onto $\hat{S}$ such that $\hat{\phi}(\partial R) = \partial S$. Hence ϕ is an n-to-one analytic mapping of R onto S for some natural number n. If $R = S$ and if $\hat{R}$ is of positive genus, $\hat{\phi}$ is univalent and a fortiori ϕ is univalent. If $\hat{R}$ is of genus zero R is conformally equivalent to the unit disc D. Without loss of generality we may assume that $R = S = D$, $z = \phi(z) = 0$ and z', $\phi(z')$ are real positive numbers. Let α and β be real positive numbers less than 1. Then $\alpha = \beta$ if and only if $\rho_D(0, \alpha) = \rho_D(0, \beta)$. In fact, assume that $\alpha < \beta$. Let u be the extremal function of $T_{0\alpha}$ and define a harmonic function v by $v(z) = u(\alpha/\beta \cdot z)$. Then $\|v\|_D < 1$, and hence

$$\rho_D(0, \alpha)=u(0)-u(\alpha)$$
$$=v(0)-v(\beta)$$
$$<\rho_D(0, \beta).$$

This implies that $\alpha=\beta$ if and only if $\rho_D(0, \alpha)=\rho_D(0, \beta)$. Therefore we have $\phi(z')=z'$. From the Schwarz lemma ϕ is the identity mapping of D. This completes the proof.

REFERENCES

[1] AHLFORS, L. V. AND L. SARIO, Riemann surfaces. Princeton Univ. Press, Princeton (1960).
[2] CONSTANTINESCU, C. AND A. CORNEA, Über den idealen Rand und einige seiner Anwendungen bei der Klassifikation der Riemannschen Flächen. Nagoya Math. J. 13 (1958), 169-233.
[3] CONSTANTINESCU, C. AND A. CORNEA, Ideale Ränder Riemannschen Flächen. Springer-Verlag, Berlin-Göttingen-Heidelberg (1963).
[4] DIXMIER, J., Sur certains espaces considérés par M. H. Stone. Sum. Bras. Math. 2 (1951), 151-182.
[5] HAYASHI, K., Une frontière des surfaces de Riemann ouvertes et applications conformes. Kōdai Math. Sem. Rep. 14 (1962), 169-188.
[6] HEINS, M., On the Lindelöf principle. Ann. Math. (2) 61 (1955), 440-473.
[7] KATO, T., On the uniqueness of the extremal function of harmonic length problem and its application. Kōdai Math. Sem. Rep. 23 (1971), 480-485.
[8] LANDAU, H. J. AND R. OSSERMANN, On analytic mappings of Riemann surfaces. J. Analyse Math. 7 (1959/60), 249-279.
[9] MORI, S., On a compactification of an open Riemann surface and its application. J. Math. Kyoto Univ. 1 (1961), 21-42.
[10] SARIO, L. AND M. NAKAI, Classification theory of Riemann surfaces. Springer-Verlag, Berlin-Heidelberg-New York (1970).
[11] SUITA, N., Analytic mapping and harmonic length. Kōdai Math. Sem. Rep. 23 (1971), 351-356.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.