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HARMONIC AND BIHARMONIC DEGENERACY

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1. In the biharmonic classification theory of Riemannian manifolds, an interesting problem is to relate the biharmonic and harmonic null classes to one another. In the present paper we consider bounded and Dirichlet finite functions and ask: Does a biharmonic degeneracy imply, or is it implied by, the corresponding harmonic degeneracy? We shall show that the answer is in the negative. Explicitly, let O_{HB}^N, O_{HD}^N be the classes of Riemannian manifolds of dimension $N\geqq 2$ which do not carry (nonconstant) bounded or Dirichlet finite harmonic functions, and $O_{H^2B}^N$, $O_{H^2D}^N$ the corresponding (nonharmonic) biharmonic null classes. For any null class O^N let $\tilde{O}^N$ be its complement. Then for $X=B, D$ and $Y=B, D$, the classes $O_{HX}^N \cap O_{H^2Y}^N$, $O_{HX}^N \cap \tilde{O}_{H^2Y}^N$, $\tilde{O}_{HX}^N \cap O_{H^2Y}^N$, $\tilde{O}_{HX}^N \cap \tilde{O}_{H^2Y}^N$ are all nonvoid for every N.

The crucial classes are $\tilde{O}_{HX}^N \cap O_{H^2Y}^N$. We shall show their non-emptyness by proving that $\tilde{O}_{HBD}^N \cap O_{H^2B}^N \cap O_{H^2D}^N \neq \emptyset$.

The essence of the problem lies in finding a manifold and a metric which satisfy the following simultaneous conditions: (1) the Laplace-Beltrami equation is explicitly integrable, (2) the metric grows in such a manner that both H^2B- and and H^2D-functions are excluded, yet HBD-functions are not, (3) the argument holds for an arbitrary N.

2. We introduce the slit N-torus

$$T=\{|x|<1,\ |y_i|\leqq\pi,\ i=1,2,\cdots,N-1\}$$

endowed with the metric

$$ds^2=\lambda^2dx^2+\sum_{i=1}^{N-1}\lambda^{2/(N-1)}dy_i^2.$$

Here $y_i=-\pi$ and $y_i=\pi$ are identified for each i, and $\lambda=\lambda(x)\in C^2((-1,1))$.

For a trial solution of $\Delta h=0$ we set $h=f(x)\Pi_{i=1}^{N-1}g_i(y_i)$ and obtain

$$\Delta h=-\lambda^{-2}\left\{f''\prod_{i=1}^{N-1}g_i+\lambda^{2(N-2)/(N-1)}f\sum_{i=1}^{N-1}g_i''\prod_{j\neq i}g_j\right\}$$

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$$=-\lambda^{-2/(N-1)}h\left\{\lambda^{-2(N-2)/(N-1)}f''f^{-1}+\sum_{i=1}^{N-1}g_i''g_i^{-1}\right\}=0.$$

In { }, each term depends on one variable only and is therefore constant. For g_i, the eigenvalues give $g_i''=-n_i^2g_i$, where the n_i are integers $\geqq 0$. The solutions are

$$g_{i1}=\cos n_iy_i, \qquad g_{i2}=\sin n_iy_i.$$

For $n=(n_1,\cdots,n_{N-1})$ with $0=(0,\cdots,0)$ and for a function j: $n\to\{1,2\}$, set

$$G_{nj}=\prod_{i=1}^{N-1}g_{ij}(n_i).$$

Then $f_nG_{nj}\in H(T)$ if $f_n(x)$ satisfies $f_n''=\eta^2\lambda^{2(N-2)/(N-1)}f_n$, where $\eta^2=\sum_{i=1}^{N-1}n_i^2$. For the present we assume $N>2$, choose

$$\lambda(x)=(1-x^2)^{-(N-1)/(N-2)}$$

and designate by T_0 this particular T. To solve the resulting equation

$$(1-x^2)^2f_n''=\eta^2f_n$$

we make the substitutions $f_n=(1-x^2)^{1/2}t(z)$, $dx/dz=1-x^2$, which transform our equation into $t''(z)=(1+\eta^2)t(z)$. The functions $t=e^{\pm\sqrt{1+\eta^2}z}=[(1+x)/(1-x)]^{\pm\sqrt{1+\eta^2}/2}$ give the solutions

$$\begin{cases} f_{n1}=(1+x)^{(1+\sqrt{1+\eta^2})/2}(1-x)^{(1-\sqrt{1+\eta^2})/2}, \\ f_{n2}=(1+x)^{(1-\sqrt{1+\eta^2})/2}(1-x)^{(1+\sqrt{1+\eta^2})/2}. \end{cases}$$

LEMMA 1. *$h_{njk}=f_{nk}G_{nj}\in H(T_0)$ for $k=1,2$.*

3. In the case $n=0$ we have $h_{011}=1+x$, $h_{012}=1-x$, and

$$D(h_{01k})=\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\int_{-1}^{1}\lambda^{-2}h_{01k}'^2\lambda^2dxdy_1\cdots dy_{N-1}=2^N\pi^{N-1}.$$

LEMMA 2. *$T_0\in\tilde{O}_{HBD}^N$ for $N>2$.*

We could also have obtained this result by $\Delta h(x)=-\lambda^{-2}h''(x)=0$, which gives $h(x)=ax+b\in HBD(T_0)$ for every λ.

4. An arbitrary harmonic function h on T_0 restricted to a fixed x has the eigenfunction expansion $h=\sum_n\sum_jc_{nj}(x)G_{nj}$. This readily seen to imply:

LEMMA 3. *Every $h\in H(T_0)$ has an expansion on T_0*

$$h=\sum_n\sum_j\sum_k a_{njk}f_{nk}G_{nj}.$$

5. We claim:

LEMMA 4. *$T_0 \in O^N_{H^2D}$ for $N>2$.*

Suppose $u \in H^2D(T_0)$, with $\Delta u = h \in H(T_0)$. For all $\varphi \in C^1(T_0)$ with $\operatorname{supp} \varphi \subset \Omega \subset \bar{\Omega} \subset T_0$, we have at once (cf. Nakai-Sario [1])

$$0=\int_{\partial\Omega} \varphi * du = (d\varphi, du)_\Omega + (\varphi, h)_\Omega,$$

$$|(h, \varphi)| / \sqrt{D(\varphi)} \leqq \sqrt{D(u)} < \infty.$$

Let h have the expansion of Lemma 3. For constants $0<\beta<\gamma<1$ let $\rho_0 \in C_0^2((-1,1))$, $\operatorname{supp} \rho_0 \subset (\beta, \gamma)$. If $a_{nj1} \neq 0$ for some (n, j), set $\rho_t(x)=\rho_0((1-x)/t), t>0$, and $\varphi_t = \rho_t G_{nj}$ for $n \neq 0$, $\varphi_t = \rho_t$ for $n=0$. Then

$$|(h, \varphi_t)| = C\int_{1-\gamma t}^{1-\beta t} (a_{nj1} f_{n1} + a_{nj2} f_{n2}) \rho_t \lambda^2 dx.$$

Here and later C is a constant, not always the same. As $t \to 0$, $f_{n1} \to \infty$ if $n \neq 0$, and $f_{n1} \to 2$ if $n=0$, whereas $f_{n2} \to 0$ for each n. Therefore $\lim_{x\to 1} |a_{nj1} f_{n1} + a_{nj2} f_{n2}| > |a_{nj1}| > 0$.

We have for all sufficiently small t

$$|(h, \varphi_t)| > C\int_{1-\gamma t}^{1-\beta t} \rho_t (1-x)^{-2(N-1)/(N-2)} dx$$

$$= Ct^{-2(N-1)/(N-2)+1} = Ct^{-N/(N-2)},$$

$$D(\varphi_t) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \int_{-1}^{1} \left(\lambda^{-2} \rho_t'^2 G_{nj}^2 + \lambda^{-2/(N-1)} \rho_t^2 \sum_{i=1}^{N-1} \left(\frac{\partial G_{nj}}{\partial y_i} \right)^2 \right) \lambda^2 dx dy \cdots dy_{N-1}$$

$$< \int_{1-\gamma t}^{1-\beta t} (c_1 \rho_t'^2 + c_2 (1-x)^{-2} \rho_t^2) dx = O(t^{-1}) + O(t^{-1}),$$

and $\sqrt{D(\varphi_t)} = O(t^{-1/2})$. Since $-N/(N-2) < -1/2$, $|(h, \varphi_t)| / \sqrt{D(\varphi_t)} \to \infty$ as $t \to 0$, a contradiction. Therefore $a_{nj1}=0$ for every (n, j).

If $a_{nj2} \neq 0$ for some (n, j), take ρ_0 as above but $\rho_t(x) = \rho_0((x+1)/t)$. Then $\lim_{x\to -1} |a_{nj1} f_{n1} + a_{nj2} f_{n2}| > |a_{nj2}| > 0$ and

$$|(h, \varphi_t)| > Ct^{-2(N-1)/(N-2)} \int_{-1+\beta t}^{-1+\gamma t} \rho_t dx = Ct^{-N/(N-2)}.$$

Since $\sqrt{D(\varphi_t)} = O(t^{-1/2})$ as before, we again have $a_{nj2}=0$ for all (n, j), and Lemma 4 follows.

6. We proceed to show:

LEMMA 5. *$T_0 \in O^N_{H^2B}$ for $N>2$.*

Suppose $u \in H^2B(T_0)$ and set $\Delta u = h \in H(T_0)$. Trivially $(h, \varphi) = (\Delta u, \varphi) = (u, \Delta\varphi)$ and therefore (cf. Nakai-Sario [2])

$$\frac{|(h, \varphi)|}{(1, |\Delta\varphi|)} \leqq \sup_{T_0} |u| < \infty,$$

for every $\varphi \in C_0^2(T_0)$. If some $a_{nj1} \neq 0$, choose φ_t as in the first case above so as to obtain $|(h, \varphi_t)| \sim Ct^{-N/(N-2)}$. On the other hand,

$$\Delta\varphi_t = -\lambda^{-2}\left(\rho_t'' G_{nj} + \lambda^{2(N-2)/(N-1)} \sum_{i=1}^{N-1} \rho_t \frac{\partial^2 G_{nj}}{\partial y_i^2}\right),$$

$$(1, |\Delta\varphi_t|) \sim \int_{1-\gamma t}^{1-\beta t} (c_1|\rho_t|'' + c_2 t^{-2}\rho_t)dx = c_1 t^{-1} + c_2 t^{-1}.$$

Since $-N/(N-2) < -1$, $|(h, \varphi_t)|/(1, |\Delta\varphi_t|) \to \infty$ as $t \to 0$, a contradiction. Therefore $a_{nj1} = 0$ for all (n, j). If some $a_{nj2} \neq 0$, we again take $\rho_t(x) = \rho_0((x+1)/t)$ and arrive at a contradiction. Lemma 5 follows.

7. We are ready to state:

THEOREM. *For $X = B, D$ and $Y = B, D$, the spaces*

$$O_{HX}^N \cap O_{H^2Y}^N,\ O_{HX}^N \cap \tilde{O}_{H^2Y}^N,\ \tilde{O}_{HX}^N \cap O_{H^2Y}^N,\ \tilde{O}_{HX}^N \cap \tilde{O}_{H^2Y}^N$$

are all nonvoid for every $N \geqq 2$.

In fact, for $N > 2$, Lemmas 2, 4, 5 give $\tilde{O}_{HX}^N \cap O_{H^2Y}^N \neq \emptyset$. For $N = 2$, the unit disk with any conformal metric is trivially in $\tilde{O}_{HBD}^N$, since such a metric affects neither harmonicity nor the Dirichlet integral. Thus we are free to choose this metric such that the H^2B-and H^2D-functions are excluded (Nakai-Sario [1, 2]).

It is known that $O_G^N \cap O_{H^2Y}^N \neq \emptyset$ and $O_G^N \cap \tilde{O}_{H^2Y}^N \neq \emptyset$ for $Y = B, D$, with O_G^N the class of parabolic N-manifolds (Sario-Wang [4, 5]). In view of $O_G^N \subset O_{HB}^N \subset O_{HD}^N$ (e.g., Sario-Nakai [3]), we have $O_{HX}^N \cap O_{H^2Y}^N \neq \emptyset$ and $O_{HX}^N \cap \tilde{O}_{H^2Y}^N \neq \emptyset$. The remaining relation $\tilde{O}_{HX}^N \cap \tilde{O}_{H^2Y}^N \neq \emptyset$ is trivial by virtue of the Euclidean N-ball.

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