KÔDAI MATH. SEM. REP.
24 (1972), 331—382

BIEBERBACH CONJECTURE FOR THE EIGHTH COEFFICIENT

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§ 0. Introduction. Let $f(z)$ be a normalized regular function univalent in the unit circle $|z|<1$

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

In 1916 Bieberbach [1] proved $|a_2|\leqq 2$ and stated his famous conjecture $|a_n|\leqq n$ with the equals sigh holding only for the Koebe function $z/(1-z)^2$ and its rotations. Later Löwner [8] established the deeper inequality $|a_3|\leqq 3$ by his parametric method, for which several proofs are now known. Garabedian and Schiffer [3] showed that $|a_4|\leqq 4$ and Charzynski and Schiffer [2] found its elementary proof. In 1968 one of the authors [9] and Pederson [13] proved $|a_6|\leqq 6$ independently and in quite different ways. Several authors proved the local maximality for general a_n or a special a_n at the Koebe function. Among them the method in [7], [10] has real effectivity and then it is supported by our two papers [11], [12]. In [11] we proved that $\Re a_8\leqq 8$ if a_2 is real non-negative. In [12] we proved that $\Re a_8\leqq 8$ if $a_3-3a_2^2/4$ and $a_4-3a_2a_3/2+5a_2^3/8$ are real and $|\arg a_2|\leqq\pi/7$. These works were done as a test for the general case and gave several important informations for the general case.

In this paper we shall prove the following theorem.

THEOREM. $$\Re a_8\leqq 8$$
if $1.9\leqq\Re a_2\leqq 2$ *and* $|\Im a_2/\Re a_2|\leqq 1/20$. *Equality occurs only for the Koebe function* $z/(1-z)^2$.

Our proof of this theorem lies within the elementary level. In the methodological view point there is nothing new and the method is the same as in [9]. However we need here more laborious calculation than in [9]. This paper is the first one for the Bieberbach conjecture for the eighth coefficient. In order to complete this work fully we need more several years. However we believe that the proof of the main part for $\Re a_8\leqq 8$ has been finished up by this paper. We are now continuing the work in order to fill up the remaining part.

Section 1 is devoted to several preparatory lemmas and general inequalities from which we start. Section 2 is concerned with the case $y\geqq 0$, $\xi\leqq 0$ and $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$. Section 3 is concerned with the case $y\geqq 0$, $\xi\geqq 0$ and $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$. Section 4 is devoted to the case $y\leqq 0$ and $1.8\leqq p\leqq 2$, $|x'/p|\leqq 1/10$. Each cases

Received July 12, 1971.

are divided into several subcases.

§ 1. We make use of the same notations as in [11], [12]. Firstly we shall give here several lemmas, which will be used later on. Grunsky's inequality and Golusin's are used here.

LEMMA 1.
$$11(\tau^2+\tau'^2)+9(\varphi^2+\varphi'^2)+7(\xi^2+\xi'^2)+5(\eta^2+\eta'^2)$$
$$+3(y^2+y'^2)+x'^2\leqq 4x-x^2=4-p^2.$$

Proof. This is a simple consequence of the area theorem for $f(1/z^2)^{-1/2}$.

LEMMA 2.
$$\eta+\left(2\beta-\frac{1}{2}p\right)y\leqq(2-p)\beta^2+\frac{1}{12}(8-p^3)-\frac{1}{2}x'y'+\frac{1}{4}px'^2.$$

Proof. By Grunsky's inequality [5], [6]
$$|b_{11}x_1^2+6b_{13}x_1x_3+3b_{33}x_3^2|\leqq|x_1|^2+3|x_3|^2.$$
Here we put $x_1=\beta$ and $x_3=1/3$. Taking the real part we have the desired result.

We need a better inequality than Lemma 2.

LEMMA 3.
$$72p^3\left\{\eta+\frac{1}{6}(\beta-3p)y\right\}$$
$$\leqq 192+4\beta^2-\beta^2p^2-3p^6-6\beta p(4-p^2)y$$
$$+\{-45p^4-3(\beta-6p)^2-144+90p^2x'^2-108x'^2\}y^2$$
$$+\{-30(\beta-6p)p^2-72(\beta-6p)+30(\beta-6p)x'^2\}y\eta$$
$$+\{-5(\beta-6p)^2-432\}\eta^2+(-360p^2+360x'^2)y\xi-120(\beta-6p)\eta\xi$$
$$-720\xi^2+(-9p^4-\beta^2-9p^2x'^2-3x'^4)x'^2$$
$$+\{-36p^3-18(\beta-6p)p^2-24\beta+108px'^2+6(\beta-6p)x'^2\}x'y'$$
$$+\{-45p^4-3(\beta-6p)^2-144+90p^2x'^2-108x'^2\}y'^2+(-216p^2+72x'^2)x'\eta'$$
$$+\{-30(\beta-6p)p^2-72(\beta-6p)+30(\beta-6p)x'^2\}y'\eta'+\{-5(\beta-6p)^2-432\}\eta'^2$$
$$+(-360p^2+360x'^2)y'\xi'-120(\beta-6p)\eta'\xi'-720\xi'^2$$
$$+[\{18(\beta-6p)p+108p^2-36x'^2\}x'^2+180p^2y'^2$$
$$+\{-60(\beta-6p)p-180p^2+432\}x'\eta'+120(\beta-6p)y'\eta'$$
$$-720px'\xi'+1440y'\xi']y$$
$$+[216px'^2+\{60(\beta-6p)p+180p^2-432\}x'y'-60(\beta-6p)y'^2+720x'\xi']\eta$$

$$+[720px'y'-720x'\eta'-720y'^2]\xi+180p^2y^3+60(\beta-6p)y^2\eta+720y^2\xi.$$

Proof. By Golusin's inequality [4], [6] we have

$$|b_{11}x_1+b_{31}x_3|^2+3|b_{13}x_1+b_{33}x_3|^2+5|b_{15}x_1+b_{35}x_3|^2\leqq|x_1|^2+3|x_3|^2.$$

Here we put $x_1=2\beta$, $x_3=8$. Then we have

$$\begin{aligned}&(\beta p+12y)^2+(\beta x'+12y')^2+3(\beta y-6py+12\eta+6x'y'+p^3-3px'^2)^2\\&+3(\beta y'-6x'y-6py'+12\eta'+3p^2x'-x'^3)^2+5(3p^2y+\beta\eta-6p\eta+12\xi)^2\\&+10(3p^2y+\beta\eta-6p\eta+12\xi)(-3x'^2y+6x'\eta'-6y^2+6y'^2-6px'y')\\&+5(3p^2y'+\beta\eta'-6p\eta'+12\xi')^2+10(3p^2y'+\beta\eta'-6p\eta'+12\xi')\\&\quad\times(-3x'^2y'+6px'y-6x'\eta-12yy')\leqq4\beta^2+192.\end{aligned}$$

By a simple calculation we have the desired result.

Next we shall give an inequality, from which we start. Grunsky's inequality with $m=7$, $x_1=\gamma$, $x_3=\delta/3$, $x_5=p/5$, $x_7=1/7$, $x_2=x_4=x_6=0$ leads us to the following inequality

$$\begin{aligned}\Re a_8\leqq&\frac{2}{7}+\frac{2}{5}p^2+\frac{2}{3}\delta^2-\frac{1}{12}\delta^2p^3-\frac{1}{80}p^7+\frac{27}{64\cdot7}p^7+x\gamma^2\\&+\left(\frac{5}{4}p^2-2\delta\right)\varphi+\left(\frac{11}{8}p^3-\delta p-2\gamma\right)\xi+\left(\frac{7}{8}p^4+\frac{1}{2}\delta p^2-\delta^2-2\gamma p\right)\eta\\&+\left(\frac{11}{16}p^5-\frac{1}{2}\delta p^3+\frac{1}{2}\delta^2p-2\gamma\delta\right)y\\&+\left(\frac{3}{2}p^3+\frac{1}{2}\delta p\right)y^2+\frac{9}{4}p\eta^2+\left(\frac{27}{8}p^2+2\delta\right)y\eta+\frac{9}{2}py\xi+4\eta\xi+3y\varphi+\frac{9}{8}py^3\\&-\left(\frac{73}{64}p^5-\frac{1}{4}\delta^2p\right)x'^2-\left(\frac{3}{2}p^3+\frac{1}{2}\delta p\right)y'^2-\frac{9}{4}p\eta'^2-\left(\frac{53}{16}p^4-\delta p^2-\frac{1}{2}\delta^2\right)x'y'\\&-\frac{15}{4}p^3x'\eta'-\left(\frac{29}{8}p^2+\delta\right)x'\xi'-\frac{7}{2}px'\varphi'-2x'\tau'-\left(\frac{27}{8}p^2+2\delta\right)y'\eta'\\&-\frac{9}{2}py'\xi'-4\eta'\xi'-3y'\varphi'\\&-\left(\frac{21}{2}p^2-\delta\right)x'y'y-\frac{1}{2}x'\eta'\eta-\frac{27}{4}px'y'\eta-\frac{27}{4}px'\eta'y-\frac{3}{2}x'y'\xi-\frac{3}{2}x'\xi'y\\&-\frac{51}{8}x'y'y^2+\frac{17}{8}x'y'^3-\left(\frac{13}{2}p^3-\frac{\delta}{2}p\right)x'^2y-\left(\frac{23}{4}p^2-\frac{1}{2}\delta\right)x'^2\eta\end{aligned}\tag{A}$$

$$-\frac{29}{8}px'^2\xi-\frac{5}{4}x'^2\varphi-\frac{21}{4}px'^2y^2+\frac{21}{4}px'^2y'^2-\frac{39}{8}x'^2y\eta+\frac{39}{8}x'^2y'\eta'$$

$$+\frac{13}{2}p^2x'^3y'+\frac{15}{4}px'^3\eta'+\frac{11}{8}x'^3\xi'+3x'^3y'y+\frac{53}{16}px'^4y+\frac{7}{8}x'^4\eta$$

$$+\frac{131}{64}p^3x'^4-\frac{11}{16}x'^5y'-\frac{27}{64}px'^6-\frac{27}{8}pyy'^2.$$

§ **2.** In this section we shall be concerned with the case $y\geqq 0$, $\xi\leqq 0$ and $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$. We put, in (A),

$$\delta=\frac{5}{8}p^2+Ay,\qquad \gamma=\frac{3-\alpha}{8}p^3+Bpy+C\eta,$$

where A, B, C and α are constants to be fixed in later parts. Then we have

$$\begin{aligned}
\Re a_8 \leqq\, & U+\frac{\alpha^2-6\alpha}{64}p^6x+\frac{5}{48}Axp^2(p^2+2p+4)y+\frac{3-\alpha}{4}Bp^4xy+\frac{3-\alpha}{4}Cp^3x\eta\\
&+\frac{1}{128}(13+20\alpha)p^5y+\frac{1}{64}(3+16\alpha)p^4\eta+\frac{1}{4}\alpha p^3\xi\\
&+\left\{\left(\frac{29}{16}-\frac{5-2\alpha}{8}A-\frac{5}{4}B\right)p^3+\frac{1}{12}A^2x(p^2+2p+4)+B^2xp^2\right.\\
&\quad\left.+\left(-\frac{21}{4}+\frac{1}{2}A+\frac{1}{4}A^2\right)px'^2\right\}y^2\\
&+\left\{\left(\frac{37}{8}-\frac{3}{4}A-2B-\frac{5}{4}C\right)p^2+2BCxp+\left(-\frac{39}{8}+\frac{A}{2}\right)x'^2\right\}y\eta\\
&+\left\{\left(\frac{9}{4}-2C\right)p+C^2x\right\}\eta^2+\left(\frac{9}{2}-2B-A\right)py\xi+(4-2C)\eta\xi+(3-2A)y\varphi\\
&+\left(-\frac{267}{256}p^5+\frac{131}{64}p^3x'^2-\frac{27}{64}px'^4\right)x'^2+\left(-\frac{319}{128}p^4+\frac{13}{2}p^2x'^2-\frac{11}{16}x'^4\right)x'y' \qquad \text{(B)}\\
&+\left(-\frac{29}{16}p^3+\frac{21}{4}px'^2\right)y'^2+\left(-\frac{15}{4}p^3+\frac{15}{4}px'^2\right)x'\eta'+\left(-\frac{37}{8}p^2+\frac{39}{8}x'^2\right)y'\eta'\\
&-\frac{9}{4}p\eta'^2+\left(-\frac{17}{4}p^2+\frac{11}{8}x'^2\right)x'\xi'-\frac{9}{2}py'\xi'-4\eta'\xi'-\frac{7}{2}px'\varphi'-3y'\varphi'-2x'\tau'\\
&+Sy+T\eta+V\xi-\frac{5}{4}x'^2\varphi+\left(\frac{9}{8}+\frac{1}{2}A+\frac{1}{2}A^2-2AB\right)py^3\\
&+(2A-A^2-2AC)y^2\eta+\left(-\frac{51}{8}+A+\frac{1}{2}A^2\right)x'y'y^2+\frac{17}{8}x'y'^3,
\end{aligned}$$

$$U=8-\frac{31}{4}x-\frac{81}{8}x^2+\frac{1111}{48}x^3-\frac{863}{48}x^4+\frac{2291}{320}x^5$$

$$-\left(\frac{133}{128}+\frac{25}{96}+\frac{7}{40}\right)x^6+\left(\frac{9}{112}+\frac{25}{64\cdot 12}+\frac{1}{80}\right)x^7,$$

$$S=\left\{\left(-\frac{99}{16}+\frac{5}{16}A\right)p^3+\frac{53}{16}px'^2\right\}x'^2+\left\{\left(-\frac{79}{8}+\frac{13}{8}A\right)p^2+3x'^2\right\}x'y'$$

$$+\left(-\frac{27}{8}-\frac{1}{2}A\right)py'^2-\frac{27}{4}px'\eta'-2Ay'\eta'-\left(\frac{3}{2}+A\right)x'\xi',$$

$$T=\left(-\frac{87}{16}p^2+\frac{7}{8}x'^2\right)x'^2-\frac{27}{4}px'y'-\frac{1}{2}x'\eta',$$

$$V=-\frac{29}{8}px'^2-\frac{3}{2}x'y'.$$

This is our starting inequality in the case $y\geqq 0$.

In the case $\xi\leqq 0$ $V\xi$ is an obstructive term, then we choose α to be a suitable positive number. Really we put $\alpha=7/160$ in (B). We divide this case into several subcases.

Case 1. $\eta\geqq 0$.

We start from (B) with $\alpha=7/160$. Applying Lemma 3 to the term $(37p^4/640)(\eta+15py/8)$ we have

$$\Re a_8\leqq U+\frac{37}{240}p+\frac{13357}{640\cdot 32}p^3-\frac{13357}{640\cdot 128}p^5-\frac{37}{240\cdot 64}p^7-\frac{6671}{64\cdot 25600}p^6x$$

$$+\frac{5}{48}A(p^4+2p^3+4p^2)xy-\frac{703}{640\cdot 16}(p^4+2p^3)xy+\frac{473}{640}Bp^4xy$$

$$+\frac{473}{640}Cp^3x\eta+\frac{7}{640}p^3\xi$$

$$+\left\{-\frac{37}{128\cdot 8}p^5+\left(\frac{135049}{128\cdot 640}-\frac{393}{640}A-\frac{5}{4}B\right)p^3+\frac{1}{12}A^2x(p^2+2p+4)+B^2xp^2\right.$$

$$\left.-\frac{37}{320}p+\frac{37}{64\cdot 8}p^3x'^2+\left(-\frac{6831}{320\cdot 4}+\frac{1}{2}A+\frac{1}{4}A^2\right)px'^2\right\}y^2$$

$$+\left\{-\frac{407}{128\cdot 16}p^4+\left(\frac{10619}{128\cdot 20}-\frac{3}{4}A-2B-\frac{5}{4}C\right)p^2\right.$$

$$\left.+2BCxp+\frac{407}{128\cdot 16}p^2x'^2+\left(\frac{A}{2}-\frac{39}{8}\right)x'^2\right\}y\eta$$

$$+\left\{-\frac{4477}{128\cdot128}p^3+\left(\frac{609}{320}-2C\right)p+C^2x\right\}\eta^2$$

$$+\left\{-\frac{37}{128}p^3+\left(\frac{9}{2}-A-2B\right)p+\frac{37}{128}px'^2\right\}y\xi$$

$$+\left\{-\frac{407}{32\cdot16}p^2+(4-2C)\right\}\eta\xi+(3-2A)y\varphi-\frac{37}{64}p\xi^2$$

$$+\left(-\frac{5377}{640\cdot8}p^5-\frac{13357}{640\cdot128}p^3+\frac{10443}{640\cdot8}p^3x'^2-\frac{6517}{640\cdot24}px'^4\right)x'^2$$

$$+\left(-\frac{27037}{640\cdot16}p^4-\frac{703}{640\cdot4}p^2+\frac{13571}{128\cdot16}p^2x'^2-\frac{11}{16}x'^4\right)x'y'$$

$$+\left(-\frac{37}{128\cdot8}p^5-\frac{161911}{128\cdot640}p^3-\frac{37}{320}p+\frac{37}{64\cdot8}p^3x'^2+\frac{6609}{640\cdot2}px'^2\right)y'^2$$

$$(\mathrm{B}_1)\qquad +\left(-\frac{2511}{640}p^3+\frac{2437}{640}px'^2\right)x'\eta'+\left(-\frac{407}{128\cdot16}p^4-\frac{13061}{128\cdot20}p^2\right.$$

$$\left.+\frac{407}{128\cdot16}p^2x'^2+\frac{39}{8}x'^2\right)y'\eta'$$

$$+\left(-\frac{4477}{128\cdot128}p^3-\frac{831}{320}p\right)\eta'^2+\left(-\frac{17}{4}p^2+\frac{11}{8}x'^2\right)x'\xi'$$

$$+\left(-\frac{37}{128}p^3-\frac{9}{2}p+\frac{37}{128}px'^2\right)y'\xi'$$

$$+\left(-\frac{407}{32\cdot16}p^2-4\right)\eta'\xi'-\frac{37}{64}p\xi'^2-\frac{7}{2}px'\varphi'-3y'\varphi'-2x'\tau'$$

$$+S_1y+T_1\eta+V_1\xi-\frac{5}{4}x'^2\varphi+\left\{\frac{37}{32\cdot8}p^3+\left(\frac{9}{8}+\frac{1}{2}A+\frac{1}{2}A^2-2AB\right)p\right\}y^3$$

$$+\left(\frac{407}{32\cdot32}p^2+2A-A^2-2AC\right)y^2\eta+\frac{37}{64}py^2\xi+\left(-\frac{51}{8}+A+\frac{1}{2}A^2\right)y^2x'y'$$

$$+\frac{17}{8}x'y'^3,$$

$$S_1=\left\{\left(-\frac{61251}{640\cdot16}+\frac{5}{16}A\right)p^3+\frac{4203}{640\cdot2}px'^2\right\}x'^2+\left\{\left(\frac{13}{8}A-\frac{79}{8}\right)p^2+3x'^2\right\}x'y'$$

$$+\left\{\frac{37}{128\cdot2}p^3-\left(\frac{27}{8}+\frac{1}{2}A\right)p\right\}y'^2+\left(-\frac{555}{128\cdot8}p^3-\frac{2049}{320}p\right)x'\eta'$$

$$+\left(\frac{407}{32\cdot16}p^2-2A\right)y'\eta'$$

$$+\left(-\frac{37}{64}p^2-\frac{3}{2}-A\right)x'\xi'+\frac{37}{32}py'\xi',$$

$$T_1=\left(-\frac{3369}{640}p^2+\frac{7}{8}x'^2\right)x'^2+\left(\frac{555}{128\cdot 8}p^3-\frac{2271}{320}p\right)x'y'-\frac{407}{32\cdot 32}p^2y'^2$$

$$-\frac{1}{2}x'\eta'+\frac{37}{64}px'\xi',$$

$$V_1=-\frac{29}{8}px'^2+\left(\frac{37}{64}p^2-\frac{3}{2}\right)x'y'-\frac{37}{64}px'\eta'-\frac{37}{64}py'^2.$$

Here we put $A=3/2$, $B=9/8$, $C=1$. Then we have

$$\Re a_8\leqq P_1+\frac{1}{640\cdot 16}(897p^4+1794p^3+6400p^2)xy+\frac{473}{640}p^3x\left(\eta+\frac{9}{8}py\right)+\frac{7}{640}p^3\xi$$

$$-\frac{1}{96\cdot 32}Q_1-\frac{1}{96\cdot 32}R_1-\frac{1}{96\cdot 32}S_1'y-\frac{1}{96\cdot 32}T_1'\eta-\frac{1}{96\cdot 32}V_1'\xi-\frac{5}{4}x'^2\varphi$$

$$+\left(\frac{37}{32\cdot 8}p^3-\frac{3}{8}p\right)y^3+\left(\frac{407}{32\cdot 32}p^2-\frac{9}{4}\right)y^2\eta+\frac{37}{64}py^2\xi-\frac{15}{4}y^2x'y'+\frac{17}{8}x'y'^3.$$

$$P_1=8-\frac{143463}{256\cdot 300}x-\frac{3086247}{256\cdot 600}x^2+\frac{9027800}{256\cdot 1200}x^3-\frac{2412463}{192\cdot 640}x^4+\frac{596416}{128\cdot 640}x^5$$

$$-\left(\frac{133}{128}+\frac{25}{96}+\frac{7}{40}-\frac{18599}{192\cdot 6400}\right)x^6+\left(\frac{9}{112}+\frac{25}{64\cdot 12}+\frac{1}{80}-\frac{8173}{192\cdot 25600}\right)x^7,$$

$$Q_1=(111p^5+6549.2625p^3-7776p^2+355.2p-4608-222p^3x'^2+12362.4px'^2)y^2$$

$$+2(305.25p^4+4188.6p^2-6912p-305.25p^2x'^2+6336x'^2)y\eta$$

$$+(839.4375p^3+3369.6p-6144)\eta^2+2(444p^3-1152p-444px'^2)y\xi$$

$$+2(1221p^2-3072)\eta\xi+1776p\xi^2,$$

$$R_1=(3226.2p^5+500.8875p^3-6265.8p^3x'^2+1303.4px'^4)x'^2$$

$$+2(4055.55p^4+421.8p^2-10178.25p^2x'^2+1056x'^4)x'y'$$

$$+(111p^5+6071.6625p^3+355.2p-222p^3x'^2-15861.6px'^2)y'^2$$

$$+2(6026.4p^3-5848.8px'^2)x'\eta'$$

$$+2(305.25p^4+7836.6p^2-305.25p^2x'^2-7488x'^2)y'\eta'$$

$$+(839.4375p^3+7977.6p)\eta'^2+2(6528p^2-2112x'^2)x'\xi'$$

$$+2(444p^3+6912p-444px'^2)y'\xi'+2(1221p^2+6144)\eta'\xi'$$

$$+1776p\xi'^2+2.5376px'\varphi'+2.4608y'\varphi'+2.3072x'\tau',$$

$$S_1'=(16935.3p^3-10087.2px'^2)x'^2+2(11424p^2-4608x'^2)x'y'+(-444p^3+12672p)y'^2$$
$$+2(832.5p^3+9835.2p)x'\eta'+2(-1221p^2+4608)y'\eta'+2(888p^2+4608)x'\xi'$$
$$-2\cdot 1776py'\xi',$$
$$T_1'=(16171.2p^2-2688x'^2)x'^2+2(-832.5p^3+10900\cdot 8p)x'y'+1221p^2y'^2$$
$$+2\cdot 768x'\eta'-2\cdot 888px'\xi',$$
$$V_1'=11136px'^2+2(-888p^2+2304)x'y'+2.888px'\eta'+1776py'^2.$$

We remark the following facts: for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$\frac{7}{640}p^3\xi\leqq\frac{1}{96\cdot 32}\cdot 13440px'^2\xi,$$

$$\frac{1}{640\cdot 16}(897p^4+1794p^3+6400p^2)xy$$
$$\leqq\frac{\alpha_1}{96\cdot 32}(8145.6x^2-11374.8x^3+5803.8x^4-1345.5x^5+134.55x^6)$$
$$+\frac{1}{96\cdot 32}\cdot\frac{1}{\alpha_1}(134.55p^4+269.1p^3+960p^2)y^2,$$

$$-\frac{5}{4}x'^2\varphi\leqq\frac{1}{96\cdot 32}1920\gamma_1\varphi^2+\frac{1}{96\cdot 32}\cdot\frac{1920}{\gamma_1}x'^4,$$

$$\left(\frac{407}{32\cdot 32}p^2-\frac{9}{4}\right)y^2\eta\leqq 0,$$

$$\frac{37}{64}py^2\xi\leqq 0.$$

Further by Lemma 2

$$\frac{473}{640}p^3x\left(\eta+\frac{9}{8}py\right)\leqq\frac{330627}{32\cdot 480}x^2-\frac{1562319}{30\cdot 1024}x^3+\frac{1486639}{30\cdot 1024}x^4-\frac{1426095}{60\cdot 1024}x^5$$
$$+\frac{1380687}{240\cdot 1024}x^6-\frac{270083}{480\cdot 1024}x^7$$
$$+\left(\frac{1135.2}{96\cdot 32}p^4-\frac{2270.4}{96\cdot 32}p^3\right)x'y'-\left(\frac{567.6}{96\cdot 32}p^5-\frac{1135.2}{96\cdot 32}p^4\right)x'^2.$$

By Lemma 1 for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$\left(\frac{37}{32\cdot 8}p^3-\frac{3}{8}p\right)y^3\leqq\left(\frac{37}{32\cdot 8}p^3-\frac{3}{8}p\right)\sqrt{\frac{4-p^2}{3}}\,y^2\leqq\frac{1}{96\cdot 32}(160.1064p^3-415.4112p)y^2,$$

$$-\frac{15}{4}\,y^2x'y' \leqq \frac{12362.4}{96\cdot 32}\,px'^2y^2 + \frac{1412.501}{96\cdot 32}\,y^2y'^2$$

$$\leqq \frac{12362.4}{96\cdot 32}\,px'^2y^2 + \frac{\gamma_2}{96\cdot 32}\,(-235.417p^2+941.668)y^2$$

$$+\frac{1}{96\cdot 32}\cdot\frac{1}{\gamma_2}(-235.417p^2+941.668)y'^2,$$

$$\frac{17}{8}\,x'y'^3 \leqq \frac{17}{8}\cdot\frac{1}{20}\,p\cdot\sqrt{\frac{4-p^2}{3}}\cdot y'^2 \leqq \frac{1}{96\cdot 32}\cdot 117.7024py'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-5504x/96\cdot 32$ we have, with $\alpha_1=0.6$, $\gamma_1=6.45$, $\gamma_2=1$,

$$\Re a_8 \leqq 8 - \frac{x}{96\cdot 32}\hat{P}_1(x) - \frac{1}{96\cdot 32}\,Q_1' - \frac{1}{96\cdot 32}\,R_1' - \frac{1}{96\cdot 32}\,S_1'y - \frac{1}{96\cdot 32}\,T_1'\eta - \frac{1}{96\cdot 32}\,V_1''\xi,$$

$$\hat{P}_1(x) = 234.52 - 7911.82x + 72778.78x^2 - 91834.605x^3 + 49746.45x^4 - 12856.215x^5,$$

$$\begin{aligned}
Q_1' = {} & (111p^5 - 224.25p^4 + 5940.656p^3 - 9140.583p^2 + 770.6112p \\
& -1421.668 - 222p^3x'^2)y^2 \\
& +2(305.25p^4 + 4188.6p^2 - 6912p - 305.25p^2x'^2 + 6336x'^2)y\eta \\
& +(839.4375p^3 + 3369.6p + 736)\eta^2 + 2(444p^3 - 1152p - 444px'^2)y\xi \\
& +2(1221p^2 - 3072)\eta\xi + (1776p + 9632)\xi^2,
\end{aligned}$$

$$\begin{aligned}
R_1' = {} & (3793.8p^5 - 1135.2p^4 + 500.8875p^3 + 1376 - 6265.8p^3x'^2 - 297.68x'^2 \\
& +1303.4px'^4)x'^2 \\
& +2(3487.95p^4 + 1135.2p^3 + 421.8p^2 - 10178.25p^2x'^2 + 1056x'^4)x'y' \\
& +(111p^5 + 6071.6625p^3 + 235.417p^2 + 237.49p + 3186.332 - 222p^3x'^2 \\
& -15861.6px'^2)y'^2 \\
& +2(6026.4p^3 - 5848.8px'^2)x'\eta' + (839.4375p^3 + 7977.6p + 6880)\eta'^2 \\
& +2(305.25p^4 + 7836.6p^2 - 305.25p^2x'^2 - 7488x'^2)y'\eta' \\
& +2(6528p^2 - 2112x'^2)x'\xi' + 2(444p^3 + 6912p - 444px'^2)y'\xi' + 2(1221p^2 \\
& +6144)\eta'\xi' + (1776p + 9632)\xi'^2 \\
& +2\cdot 5376px'\varphi' + 2\cdot 4608y'\varphi' + 12384\varphi'^2 + 2\cdot 3072x'\tau' + 15136\tau'^2,
\end{aligned}$$

$$V_1'' = -2304px'^2 + 2(-888p^2 + 2304)x'y' + 2\cdot 888px'\eta' + 1776py'^2.$$

Since $y\geqq 0$, $\eta\geqq 0$, $\xi\leqq 0$, we have, for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$-S_1'y\leqq -2(832.5p^3+9835.2p)x'\eta'y-2(-1221p^2+4608)y'\eta'y-2(888p^2+4608)x'\xi'y$$
$$+2\cdot 1776py'\xi'y$$
$$\leqq(832.5\alpha_2p^3+9835.2\alpha_2p)y^2+(832.5\alpha_2^{-1}p^3+9835.2\alpha_2^{-1}p)x'^2\eta'^2$$
$$+(1221\alpha_3p^2-4207.62\alpha_3)y^2+(-407\alpha_3^{-1}p^4+3030.54\alpha_3^{-1}p^2-5610.16\alpha_3^{-1})\eta'^2$$
$$+(888\alpha_4p^2+4608\alpha_4)y^2+(888\alpha_4^{-1}p^2+4608\alpha_4^{-1})x'^2\xi'^2$$
$$+1776\alpha_5py^2+(-592\alpha_5^{-1}p^3+2368\alpha_5^{-1}p)\xi'^2,$$

$$-T_1'\eta\leqq -2\cdot 768x'\eta'\eta+2\cdot 888px'\xi'\eta$$
$$\leqq 768\beta_1\eta^2+768\beta_1^{-1}x'^2\eta'^2+888\beta_2p\eta^2+888\beta_2^{-1}px'^2\xi'^2,$$

$$-V_1''\xi\leqq -2(-888p^2+2304)x'y'\xi-2\cdot 888px'\eta'\xi-1776py'^2\xi$$
$$\leqq(888\gamma_3p^2-2304\gamma_3)\xi^2+(888\gamma_3^{-1}p^2-2304\gamma_3^{-1})x'^2y'^2+888\gamma_4p\xi^2$$
$$+888\gamma_4^{-1}px'^2\eta'^2+888\gamma_5p\xi^2+(-296\gamma_5^{-1}p^3+1184\gamma_5^{-1}p)y'^2.$$

Hence we have, putting $\alpha_2=\alpha_3=\alpha_4=\alpha_5=0.2$, $\beta_1=\beta_2=2$, $\gamma_3=\gamma_4=\gamma_5=2$,

$$\Re a_8\leqq 8-\frac{x}{96\cdot 32}\hat{P}_1(x)-\frac{1}{96\cdot 32}\hat{Q}_1-\frac{1}{96\cdot 32}\hat{R}_1,$$

$$\hat{Q}_1=(110.445p^5-224.25p^4+5774.156p^3-9562.383p^2-1551.6288p$$
$$-1501.744)y^2+2(305.25p^4+4188.6p^2-6912p-305.25p^2x'^2+6336x'^2)y\eta$$
$$+(839.4375p^3+1593.6p-800)\eta^2+2(444p^3-1152p-444px'^2)y\xi$$
$$+2(1221p^2-3072)\eta\xi+(-1776p^2-1776p+14240)\xi^2,$$

$$\hat{R}_1=(3793.8p^5-1135.2p^4+500.8875p^3+1376-6265.8p^3x'^2-297.68x'^2+1303.4px'^4)x'^2$$
$$+2(3487.95p^4+1135.2p^3+421.8p^2-10178.25p^2x'^2+1056x'^4)x'y'$$
$$+(111p^5+6219.6625p^3+235.417p^2-354.51p+3131.932-222p^3x'^2-444p^2x'^2$$
$$-16364.1px'^2)y'^2+2(6026.4p^3-5848.8px'^2)x'\eta'+2(305.25p^4+7836.6p^2$$
$$-305.25p^2x'^2-7488x'^2)y'\eta'+(2035p^4+839.4375p^3-15152.7p^2+7977.6p+34930.8$$
$$-4162.5p^3x'^2-49620px'^2-384x'^2)\eta'^2$$
$$+2(6528p^2-2112x'^2)x'\xi'+2(444p^3+6912p-444px'^2)y'\xi'+2(1221p^2+6144)\eta'\xi'$$
$$+(2960p^3-10064p+9632-4440p^2x'^2-444px'^2-23040x'^2)\xi'^2$$
$$+2\cdot 5376px'\varphi'+2\cdot 4608y'\varphi'+12384\varphi'^2+2\cdot 3072x'\tau'+15136\tau'^2.$$

It is very easy to prove that $\hat{P}_1'(x)$ is monotone increasing for $0 \leqq x \leqq 0.1$ and $\hat{P}_1'(0) < 0$, $\hat{P}_1'(0.063) > 0$. Let λ be the root of $\hat{P}_1'(x) = 0$, $0 < \lambda < 0.063$. Construct $N(x) = 5\hat{P}_1(x) - x\hat{P}_1'(x)$. Then $N(x)$ is monotone decreasing for $0 \leqq x \leqq 0.063$. Further $N(0.063) > 0$. Hence $N(x) > 0$ for $0 \leqq x \leqq 0.063$. Especially $N(\lambda) > 0$, which implies $\hat{P}_1(\lambda) > 0$. Therefore $\hat{P}_1(x) > 0$ for $0 \leqq x \leqq 0.1$.

Next we prove the non-negativity of $\hat{Q}_1$. Since $y\eta \geqq 0$, we may consider $Q^* = \hat{Q}_1 - 2(305.25p^4 + 4188.6p^2 - 6912p - 305.25p^2x'^2 + 6336x'^2)y\eta$. We can prove the positive definiteness of the symmetric matrix associated with Q^* for $1.9 \leqq p \leqq 2$, $|x'/p| \leqq 1/20$ by taking its principal diagonal minor determinants. Hence $\hat{Q}_1$ is non-negative for $1.9 \leqq p \leqq 2$, $|x'/p| \leqq 1/20$.

We prove the positive definiteness of $\hat{R}_1$. Since

$$297.68x'^2 + 2\cdot 2112x'\xi' + 14984.36\xi'^2 \geqq 0,$$

$$15.12py'^2 + 2\cdot 444py'\xi' + (4440p^2 + 444p + 8055.64)\xi'^2 \geqq 0,$$

$$270.75p^3x'^2 + 2\cdot 5848.8px'\eta' + 35000p\eta'^2 \geqq 0,$$

$$686.98py'^2 + 2(305.25p^2 + 7488)y'\eta' + (4162.5p^3 + 14620p + 384)\eta'^2 \geqq 0,$$

$$5995.05p^3x'^2 + 2(10178.25p^2 - 1056x'^2)x'y' + (222p^3 + 444p^2 + 15662p)y'^2 \geqq 0$$

and

$$623.5x'^2 + 2\cdot 3072x'\tau' + 15136\tau'^2 \geqq 0,$$

we may consider $32^{-1}[\hat{R}_1 - (623.5x'^2 + 2\cdot 3072x'\tau' + 15136\tau'^2) - \{(6265.8p^3 + 297.68)x'^2 + 2(10178.25p^2 - 1056x'^2)x'y' + (222p^3 + 444p^2 + 16364.1p)y'^2 + 2\cdot 5848.8px'\eta' + 2(305.25p^2 + 7488)y'\eta' + (4162.5p^3 + 49620p + 384)\eta'^2 + 2\cdot 2112x'\xi' + 2\cdot 444py'\xi' + (4440p^2 + 444p + 23040)\xi'^2\}(0.0025p^2 - x'^2)]$. We prove the positive definiteness of the following matrix:

$$(a_{ij}),\ a_{ij} = a_{ji},$$

$$a_{11} = 118.0667p^5 - 35.475p^4 + 15.64p^3 + 23.515,$$

$$a_{12} = 108.2034p^4 + 35.475p^3 + 13.181p^2,$$

$$a_{22} = 3.4331p^5 + 193.086p^3 + 7.356p^2 - 11.079p + 97.872,$$

$$a_{13} = 187.868p^3,\ a_{23} = 9.5152p^4 + 244.308p^2,$$

$$a_{33} = 62.9433p^4 + 22.355p^3 - 473.552p^2 + 249.3p + 1091.587,$$

$$a_{14} = 203.835p^2,\ a_{24} = 13.8403p^3 + 216p,\ a_{34} = 38.156p^2 + 192,$$

$$a_{44} = 90.824p^3 - 314.5p + 301,\ a_{15} = 168p,\ a_{25} = 144,\ a_{35} = 0,$$

$$a_{45} = 0,\ a_{55} = 387.$$

Its principal diagonal minor determinants are larger than

$118p^5-36p^4+15p^3+23,$

$405p^{10}-122p^9+11142p^8-13659p^7-2661p^6+11209p^5-3820p^4$

$\quad +6071p^3+172p^2-261p+2301,$

$25513p^{14}-9295p^{13}+509903p^{12}-736538p^{11}-5045575p^{10}+5804052p^9$

$\quad +15140214p^8-19916150p^7-1717846p^6+8396026p^5-4001664p^4$

$\quad +6845115p^3-965991p^2+289370p+2512243,$

$2317202p^{17}-844137p^{16}+36864050p^{15}-55774235p^{14}-625910945p^{13}$

$\quad +929907081p^{12}+2580304222p^{11}-4991510927p^{10}-1583674912p^9$

$\quad +9814340033p^8-10824942904p^7+4337888815p^6+4678775591p^5$

$\quad -7268929772p^4+2575360931p^3-1619465056p^2-693396183p+671344390,$

$896757415p^{17}-326680831p^{16}+14266387424p^{15}-22138559044p^{14}$

$\quad -2421921806555p^{13}+351078613547p^{12}+1005891183900p^{11}$

$\quad -1780803925056p^{10}-832648863524p^9+3182190537594p^8$

$\quad -2903510866077p^7+2002721065755p^6-414107589033p^5$

$\quad -1639174860222p^4+1796104317650p^3-1317639321028p^2$

$\quad -137536511828p+117573507871,$

respectively. All of them are positive for $1.9\leqq p\leqq 2$. Therefore $\hat{R}_1$ is positive definite for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$.

Thus we have $\Re a_8\leqq 8$ for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$, $y\geqq 0$, $\eta\geqq 0$, $\xi\leqq 0$. Equality occurs only for $x=0$, that is, for the Koebe function.

Case 2. $-2py/3\leqq\eta\leqq 0$.

We start from (B_1) with $A=3/2$, $B=1$, $C=1/4$. We remark the following facts: for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$\frac{7}{640}p^3\xi\leqq\frac{1}{96\cdot 32}13440px'^2\xi,$$

$$\frac{1}{640\cdot 16}(897p^4+1794p^3+6400p^2)xy$$

$$\leqq\frac{\alpha_1}{96\cdot 32}(8145.6x^2-11374.8x^3+5803.8x^4-1345.5x^5+134.55x^6)$$

$$+\frac{1}{96\cdot 32}\cdot\frac{1}{\alpha_1}\cdot(134.55p^4+269.1p^3+960p^2)y^2,$$

$$\frac{473}{640}p^4xy \leqq \frac{\beta_1}{96\cdot 32}(18163.2x^2-36326.4x^3+27244.8x^4-9081.6x^5+1135.2x^6)$$

$$+\frac{1}{96\cdot 32}\cdot\frac{1}{\beta_1}\cdot 1135.2p^4y^2,$$

$$-\frac{5}{4}x'^2\varphi \leqq \frac{\gamma_1}{96\cdot 32}\,1920\varphi^2+\frac{1}{96\cdot 32}\cdot\frac{1}{\gamma_1}\cdot 1920x'^4,$$

$$\frac{37}{64}py^2\xi \leqq 0.$$

Further by Lemma 2

$$\frac{37}{32\cdot 8}p^3y^3+\frac{407}{32\cdot 32}p^2y^2\eta=\frac{407}{32\cdot 32}p^2y^2\left(\eta+\frac{4}{11}py\right)$$

$$\leqq \frac{1}{96\cdot 32}(-329.426p^5+455.353p^4+814p^2)y^2-\frac{407}{64\cdot 32}p^2x'y'y^2+\frac{407}{128\cdot 32}p^3x'^2y^2.$$

By Lemma 1 for $1.9 \leqq p \leqq 2$, $|x'/p| \leqq 1/20$

$$\frac{473}{640\cdot 4}p^3x\eta \leqq \frac{1}{96\cdot 32}(141.9p^3x'^2+425.7p^3y'^2+709.5p^3\eta'^2+993.3p^3\xi'^2)\eta,$$

$$-\frac{407}{64\cdot 32}p^2x'y'y^2 \leqq \frac{407}{64\cdot 32}p^2\cdot\frac{1}{20}p\sqrt{\frac{4-p^2}{3}}\,y^2 \leqq \frac{11.02}{96\cdot 32}p^3y^2,$$

$$-\frac{15}{4}y^2x'y' \leqq \frac{12362.4}{96\cdot 32}px'^2y^2+\frac{\gamma_2}{96\cdot 32}(-235.417p^2+941.668)y^2$$

$$+\frac{1}{96\cdot 32}\cdot\frac{1}{\gamma_2}(-235.417p^2+941.668)y'^2,$$

$$\frac{17}{8}x'y'^3 \leqq \frac{17}{8}\cdot\frac{1}{20}p\cdot\sqrt{\frac{4-p^2}{3}}\,y'^2 \leqq \frac{1}{96\cdot 32}\,117.7024py'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-(5738.52x+18.44x^2+22909.24x^3)/96\cdot 32$ we have, with $\alpha_1=\beta_1=2.4$, $\gamma_1=6.72$, $\gamma_2=1$,

$$\Re a_8 \leqq 8-\frac{x^2}{96\cdot 32}\hat{P}_2(x)-\frac{1}{96\cdot 32}Q_2-\frac{1}{96\cdot 32}R_2-\frac{1}{96\cdot 32}S_2y-\frac{1}{96\cdot 32}T_2\eta-\frac{1}{96\cdot 32}V_2\xi,$$

$$\hat{P}_2(x)=0.01+1300.25x-13277.755x^2+2659.44x^3+1435.7025x^4-380.15x^5,$$

$$Q_2=(440.426p^5-984.4155p^4+5130.1175p^3+10059.347p^2-68386.35p$$

$$+67509.602-527.25p^3x'^2)y^2$$

$$+2(305.25p^4-323.4p^2-1536p-305.25p^2x'^2+6336x'^2)y\eta$$

$$+(839.4375p^3+28636.55p^2-118687.65p+121381.45)\eta^2$$

$$+2(444p^3-1536p-444px'^2)y\xi+2(1221p^2-5376)\eta\xi$$

$$+(40091.17p^2-158620.95p+170471.63)\xi^2,$$

$$R_2=(3226.2p^5+500.8875p^3+5727.31p^2-22913.85p+24353.09$$

$$-6265.8p^3x'^2-288x'^2+1303.4px'^4)x'^2$$

$$+2(4055.55p^4+421.8p^2-10178.25p^2x'^2+1056x'^4)x'y'$$

$$+(111p^5+6071.6625p^3+17417.347p^2-68504.0524p+72117.602$$

$$-222p^3x'^2-15861.6px'^2)y'^2$$

$$+2(6026.4p^3-5848.8px'^2)x'\eta'+2(305.25p^4+7836.6p^2-305.25p^2x'^2-7488x'^2)y'\eta'$$

$$+(839.4375p^3+28636.55p^2-106591.65p+121765.45)\eta'^2$$

$$+2(6528p^2-2112x'^2)x'\xi'+2(444p^3+6912p-444px'^2)y'\xi'+2(1221p^2+6144)\eta'\xi'$$

$$+(40091.17p^2-158620.95p+170471.63)\xi'^2+2\cdot 5376px'\varphi'+2\cdot 4608y'\varphi'$$

$$+(51545.79p^2-206224.65p+219177.81)\varphi'^2+2\cdot 3072x'\tau'$$

$$+(63000.41p^2-252052.35p+267883.99)\tau'^2,$$

$$S_2=(16935.3p^3-10087.2px'^2)x'^2+2(11424p^2-4608x'^2)x'y'$$

$$+(-444p^3+12672p)y'^2+2(832.5p^3+9835.2p)x'\eta'+2(-1221p^2+4608)y'\eta'$$

$$+2(888p^2+4608)x'\xi'-2\cdot 1776py'\xi',$$

$$T_2=(-141.9p^3+16171.2p^2-2688x'^2)x'^2+2(-832.5p^3+10900.8p)x'y'$$

$$+(-425.7p^3+1221p^2)y'^2+2\cdot 768x'\eta'-709.5p^3\eta'^2-2\cdot 888px'\xi'-993.3p^3\xi'^2,$$

$$V_2=-2304px'^2+2(-888p^2+2304)x'y'+2\cdot 888px'\eta'+1776py'^2.$$

Since $y\geqq 0$, $0\geqq\eta\geqq -2py/3$, $\xi\leqq 0$, we have, for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$-V_2\xi\leqq -2(-888p^2+2304)x'y'\xi-2\cdot 888px'\eta'\xi-1776py'^2\xi$$

$$\leqq(888\gamma_3p^2-2304\gamma_3)\xi^2+(888\gamma_3^{-1}p^2-2304\gamma_3^{-1})x'^2y'^2$$

$$+888\gamma_4p\xi^2+888\gamma_4^{-1}px'^2\eta'^2+888\gamma_5p\xi^2+(-296\gamma_5^{-1}p^3+1184\gamma_5^{-1}p)y'^2,$$

$$-T_2\eta\leqq -T^*\eta-2(-832.5p^3+6031.5p)x'y'\eta$$

$$\leqq -T^*\eta+(-832.5\beta_2p^3+6031.5\beta_2p)\eta^2+(-832.5\beta_2^{-1}p^3+6031.5\beta_2^{-1}p)x'^2y'^2,$$

$$T^*=(-63.3p^3+16171.2p^2-2688x'^2)x'^2+2\cdot 4869.3px'y'+(-425.7p^3$$

$$+1221p^2)y'^2 \geqq 0,$$

$$-\left(S_2-\frac{2}{3}pT^*\right)y \leqq -2\cdot 327.7p^2x'y'y-2(832.5p^3+9835.2p)x'\eta'y$$

$$-2(-1221p^2+4608)y'\eta'y-2(888p^2+4608)x'\xi'y+2\cdot 1776py'\xi'y$$

$$\leqq 327.7\alpha_2p^2y^2+327.7\alpha_2^{-1}p^2x'^2y'^2+(832.5\alpha_3p^3+9835.2\alpha_3p)y^2$$

$$+(832.5\alpha_3^{-1}p^3+9835.2\alpha_3^{-1}p)x'^2\eta'^2+(1221\alpha_4p^2-4207.62\alpha_4)y^2$$

$$+(-407\alpha_4^{-1}p^4+3030.54\alpha_4^{-1}p^2-5610.16\alpha_4^{-1})\eta'^2+(888\alpha_5p^2+4608\alpha_5)y^2$$

$$+(888\alpha_5^{-1}p^2+4608\alpha_5^{-1})x'^2\xi'^2+1776\alpha_6py^2+(-592\alpha_6^{-1}p^3+2368\alpha_6^{-1}p)\xi'^2.$$

Hence we have, putting $\alpha_2=\alpha_3=\alpha_4=\alpha_5=\alpha_6=0.191$, $\beta_2=0.5$, $\gamma_3=\gamma_4=\gamma_5=2$,

$$\Re a_8 \leqq 8-\frac{x^2}{96\cdot 32}\hat{P}_2(x)-\frac{1}{96\cdot 32}\hat{Q}_2-\frac{1}{96\cdot 32}\hat{R}_2,$$

$$\hat{Q}_2=(439.107p^5-984.4155p^4+4971.11p^3+9593.9373p^2-70604.0892p$$

$$+67433.1294)y^2$$

$$+2(305.25p^4-323.4p^2-1536p-305.25p^2x'^2+6336x'^2)y\eta$$

$$+(1255.6875p^3+28636.55p^2-121703.4p+121381.45)\eta^2$$

$$+2(444p^3-1536p-444px'^2)y\xi$$

$$+2(1221p^2-5376)\eta\xi+(38315.17p^2-162172.95p+175079.63)\xi^2,$$

$$\hat{R}_2=(3226.2p^5+500.8875p^3+5727.31p^2-22913.85p+24353.09$$

$$-6265.8p^3x'^2-288x'^2+1303.4px'^4)x'^2$$

$$+2(4055.55p^4+421.8p^2-10178.25p^2x'^2+1056x'^4)x'y'$$

$$+(111p^5+6219.6625p^3+17417.347p^2-69096.0524p+72117.602$$

$$+1443p^3x'^2-2159.8372p^2x'^2-27924.6px'^2+1152x'^2)y'^2$$

$$+2(6026.4p^3-5848.8px'^2)x'\eta'$$

$$+2(305.25p^4+7836.6p^2-305.25p^2x'^2-7488x'^2)y'\eta'$$

$$+(2131.052p^4+839.4375p^3+12768.642p^2-106591.65p+151140.247$$

$$-4358.97p^3x'^2-51941.108px'^2)\eta'^2$$

$$+2(6528p^2-2112x'^2)x'\xi'+2(444p^3+6912p-444px'^2)y'\xi'$$

$$+2(1221p^2+6144)\eta'\xi'$$

$$+(3099.712p^3+40091.17p^2-171019.798p+170471.63-4649.568p^2x'^2$$

$$-24127.488x'^2)\xi'^2$$

$$+2\cdot 5376px'\varphi'+2\cdot 4608y'\varphi'+(51545.79p^2-206224.65p+219177.81)\varphi'^2$$

$$+2\cdot 3072x'\tau'+(63000.41p^2-252052.35p+267883.99)\tau'^2.$$

It is easy to prove that $\hat{P}_2(x)>0$ for $0\leqq x\leqq 0.1$.

We can prove the positive definiteness of the symmetric matrix associated with $\hat{Q}_2$ for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$ by taking its principal diagonal minor determinants. Hence $\hat{Q}_2$ is positive definite there.

Further it is easy to prove that $\hat{R}_2\geqq\hat{R}_1$ for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$. Here $\hat{R}_1$ was defined in Case 1 and was positive definite for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$. Hence $\hat{R}_2$ is positive definite there.

Therefore we have $\Re a_8\leqq 8$ for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$ in the present case with equality holding only for $x=0$.

Case 3. $-2py\leqq\eta\leqq -2py/3$.

We start from (B) with $\alpha=7/160$. Applying Lemma 3 to the term $(37\cdot 29p^4/640\cdot 32)(\eta+2py)$ we have

$$\Re a_8\leqq U+\frac{1073}{640\cdot 12}p+\frac{5365}{128\cdot 64}p^3-\frac{5365}{128\cdot 256}p^5-\frac{1073}{640\cdot 12\cdot 64}p^7-\frac{6671}{64\cdot 25600}p^6x$$

$$+\frac{5}{48}A(p^4+2p^3+4p^2)xy-\frac{1073}{128\cdot 128}(p^4+2p^3)xy+\frac{473}{640}Bp^4xy$$

$$+\frac{473}{640}Cp^3x\eta+\frac{37\cdot 3}{640\cdot 32}p^4\left(\eta+\frac{2}{3}py\right)+\frac{7}{640}p^3\xi$$

$$+\left\{-\frac{1073}{128\cdot 256}p^5+\left(\frac{267989}{640\cdot 256}-\frac{393}{640}A-\frac{5}{4}B\right)p^3+\frac{1}{12}A^2x(p^2+2p+4)+B^2xp^2\right.$$

$$\left.-\frac{1073}{640\cdot 16}p+\frac{1073}{128\cdot 128}p^3x'^2+\left(-\frac{218259}{640\cdot 64}+\frac{1}{2}A+\frac{1}{4}A^2\right)px'^2\right\}y^2$$

$$+\left\{-\frac{3219}{128\cdot 128}p^4+\left(\frac{85063}{640\cdot 32}-\frac{3}{4}A-2B-\frac{5}{4}C\right)p^2+2BCxp\right.$$

$$\left.+\frac{3219}{128\cdot 128}p^2x'^2+\left(\frac{A}{2}-\frac{39}{8}\right)x'^2\right\}y\eta$$

$$+\left\{-\frac{9657}{128\cdot 256}p^3+\left(\frac{19821}{640\cdot 16}-2C\right)p+C^2x\right\}\eta^2+\left\{-\frac{1073}{128\cdot 32}p^3+\left(\frac{9}{2}-2B-A\right)p\right.$$

$$\left.+\frac{1073}{128\cdot 32}px'^2\right\}y\xi$$

$$+\left\{-\frac{3219}{128\cdot 32}p^2+(4-2C)\right\}\eta\xi-\frac{1073}{64\cdot 32}p\xi^2+(3-2A)y\varphi$$

$$+\left(-\frac{171953}{640\cdot 256}p^5-\frac{5365}{128\cdot 256}p^3+\frac{334287}{640\cdot 256}p^3x'^2-\frac{208433}{640\cdot 768}px'^4\right)x'^2$$

$$+\left(-\frac{215963}{640\cdot 128}p^4-\frac{1073}{128\cdot 32}p^2+\frac{542137}{640\cdot 128}p^2x'^2-\frac{11}{16}x'^4\right)x'y'$$

$$+\left(-\frac{1073}{128\cdot 256}p^5-\frac{325931}{640\cdot 256}p^3-\frac{1073}{640\cdot 16}p+\frac{1073}{128\cdot 128}p^3x'^2+\frac{211821}{640\cdot 64}px'^2\right)y'^2$$

$$+\left(-\frac{80019}{640\cdot 32}p^3+\frac{77873}{640\cdot 32}px'^2\right)x'\eta'$$

$$+\left(-\frac{3219}{128\cdot 128}p^4-\frac{104377}{640\cdot 32}p^2+\frac{3219}{128\cdot 128}p^2x'^2+\frac{39}{8}x'^2\right)y'\eta'$$

$$+\left(-\frac{9657}{128\cdot 256}p^3-\frac{26259}{640\cdot 16}p\right)\eta'^2$$

$$+\left(-\frac{34}{8}p^2+\frac{11}{8}x'^2\right)x'\xi'+\left(-\frac{1073}{128\cdot 32}p^3-\frac{9}{2}p+\frac{1073}{128\cdot 32}px'^2\right)y'\xi'$$

$$+\left(-\frac{3219}{128\cdot 32}p^2-4\right)\eta'\xi'-\frac{1073}{64\cdot 32}p\xi'^2-\frac{7}{2}px'\varphi'-3y'\varphi'-2x'\tau'$$

$$+S_3y+T_3\eta+V_3\xi-\frac{5}{4}x'^2\varphi+\left\{\frac{1073}{64\cdot 128}p^3+\left(\frac{9}{8}+\frac{1}{2}A+\frac{1}{2}A^2-2AB\right)p\right\}y^3$$

$$+\left(\frac{3219}{128\cdot 64}p^2+2A-A^2-2AC\right)y^2\eta+\frac{1073}{64\cdot 32}py^2\xi+\left(-\frac{51}{8}+A+\frac{1}{2}A^2\right)y^2x'y'$$

$$+\frac{17}{8}x'y'^3,$$

$$S_3=\left\{\left(-\frac{98157}{64\cdot 256}+\frac{5}{16}A\right)p^3+\frac{134607}{640\cdot 64}px'^2\right\}x'^2+\left\{\left(-\frac{79}{8}+\frac{13}{8}A\right)p^2+3x'^2\right\}x'y'$$

$$+\left\{\frac{1073}{64\cdot 128}p^3-\left(\frac{27}{8}+\frac{1}{2}A\right)p\right\}y'^2+\left(-\frac{1073}{64\cdot 32}p^3-\frac{65901}{640\cdot 16}p\right)x'\eta'$$

$$+\left(\frac{3219}{128\cdot 32}p^2-2A\right)y'\eta'+\left\{-\frac{1073}{64\cdot 32}p^2-\left(\frac{3}{2}+A\right)\right\}x'\xi'+\frac{1073}{32\cdot 32}py'\xi',$$

$$T_3=\left(-\frac{108141}{640\cdot 32}p^2+\frac{7}{8}x'^2\right)x'^2+\left(\frac{1073}{64\cdot 32}p^3-\frac{72339}{640\cdot 16}p\right)x'y'-\frac{3219}{128\cdot 64}p^2y'^2$$

$$-\frac{1}{2}x'\eta'+\frac{1073}{64\cdot 32}px'\xi',$$

$$V_3=-\frac{29}{8}px'^2+\left(\frac{1073}{64\cdot 32}p^2-\frac{3}{2}\right)x'y'-\frac{1073}{64\cdot 32}px'\eta'-\frac{1073}{64\cdot 32}py'^2.$$

Here we put $A=3/2, B=1, C=12/25$. We remark the following facts: for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$\frac{7}{640}p^3\xi\leqq\frac{1}{96\cdot 32}\cdot 13440px'^2\xi,$$

$$\frac{1}{128\cdot 128}(1487p^4+2974p^3+10240p^2)xy$$

$$\leqq\frac{\alpha_1}{96\cdot 32}(8301x^2-11646.75x^3+5978.625x^4-1394.0625x^5+139.40625x^6)$$

$$+\frac{1}{96\cdot 32}\cdot\frac{1}{\alpha_1}(139.40625p^4+278.8125p^3+960p^2)y^2,$$

$$\frac{37\cdot 3}{640\cdot 32}p^4\left(\eta+\frac{2}{3}py\right)\leqq\frac{37\cdot 15}{8\cdot 32}p^2x'^2\left(\eta+\frac{2}{3}py\right)\leqq\frac{6660}{96\cdot 32}p^2x'^2\eta+\frac{4440}{96\cdot 32}p^3x'^2y,$$

$$-\frac{5}{4}x'^2\varphi\leqq\frac{\gamma_1}{96\cdot 32}\cdot 1920\varphi^2+\frac{1}{96\cdot 32}\cdot\frac{1}{\gamma_1}1920x'^4,$$

$$\frac{1073}{128\cdot 64}p^3y^3+\left(\frac{3219}{128\cdot 64}p^2-\frac{69}{100}\right)y^2\eta\leqq\frac{3219}{128\cdot 128}p^2y^2\left(\eta+\frac{2}{3}py\right)\leqq 0,$$

$$\frac{1073}{64\cdot 32}py^2\xi\leqq 0.$$

Further by Lemma 1 for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$\frac{473}{640}p^4xy+\frac{1419}{4000}p^3x\eta=\frac{500}{4000}p^3x\left(\eta+\frac{2}{3}py\right)+\frac{919}{4000}p^3x\eta+\frac{1259}{1920}p^4xy$$

$$\leqq\frac{919}{4000}p^3x\eta+\frac{1259}{1920}p^4xy$$

$$\leqq\frac{1}{96\cdot 32}(176.448p^3x'^2+529.344p^3y'^2+882.24p^3\eta'^2+1235.136p^3\xi'^2)\eta$$

$$+\frac{\beta_1}{96\cdot 32}(16115.2x^2-32230.4x^3+24172.8x^4-8057.6x^5+1007.2x^6)$$

$$+\frac{1}{96\cdot 32}\cdot\frac{1}{\beta_1}1007.2p^4y^2,$$

$$-\frac{15}{4}y^2x'y'\leqq\frac{12362.4}{96\cdot 32}px'^2y^2+\frac{\gamma_2}{96\cdot 32}(-235.417p^2+941.668)y^2$$

$$+\frac{1}{96\cdot 32}\cdot\frac{1}{\gamma_2}(-235.417p^2+941.668)y'^2,$$

$$\frac{17}{8}x'y'^3\leqq\frac{1}{96\cdot 32}117.7024py'^2.$$

Making use of these remarks and applying Lemma 1 to the term $(-5938.32x-4260.96x^2-15575x^3)/96\cdot 32$ we have, with $\alpha_1=2.4$, $\beta_1=2.4$, $\gamma_1=6.95$, $\gamma_2=1$,

$$\Re a_8\leqq 8-\frac{x^3}{96\cdot 32}\hat{P}_3(x)-\frac{1}{96\cdot 32}Q_3-\frac{1}{96\cdot 32}R_3-\frac{1}{96\cdot 32}S_3'y-\frac{1}{96\cdot 32}T_3'\eta-\frac{1}{96\cdot 32}V_3'\xi,$$

$$\hat{P}_3(x)=830.975-8331.5325x+374.58375x^2+1721.535x^3-379.457x^4,$$

$$\begin{aligned}Q_3=&(100.59375p^5-477.753p^4+5176.634p^3+5372.667p^2-49598.82p\\&+52020.512-201.1875p^3x'^2-24.975px'^2)y^2\\&+2(301.78125p^4+816.435p^2-2949.12p-301.78125p^2x'^2+6336x'^2)y\eta\\&+(905.34375p^3+19468.75p^2-85490.5912p+94534.7224)\eta^2\\&+2(402.375p^3-1536p-402.375px'^2)y\xi+2(1207.125p^2-4669.44)\eta\xi\\&+(27256.25p^2-114872.18p+134330.42)\xi^2,\end{aligned}$$

$$\begin{aligned}R_3=&(3224.11875p^5+502.96875p^3+3893.75p^2-16640.24p+19190.06\\&-6267.88125p^3x'^2-276.48x'^2+1302.70625px'^4)x'^2\\&+2(4049.30625p^4+402.375p^2-10165.06875p^2x'^2+1056x'^4)x'y'\\&+(100.59375p^5+6111.20625p^3+11916.667p^2-49716.5224p\\&+56628.512-201.1875p^3x'^2-15886.575px'^2)y'^2\\&+2(6001.425p^3-5840.475px'^2)x'\eta'\\&+2(301.78125p^4+7828.275p^2-301.78125p^2x'^2-7488x'^2)y'\eta'\\&+(905.34375p^3+19468.75p^2-75323.5p+95950.3)\eta'^2\\&+2(6528p^2-2112x'^2)x'\xi'\\&+2(402.375p^3+6912p-402.375px'^2)y'\xi'+2(1207.125p^2+6144)\eta'\xi'\\&+(27256.25p^2-114872.18p+134330.42)\xi'^2+2\cdot 5376px'\varphi'+2\cdot 4608y'\varphi'\\&+(35043.75p^2-149762.16p+172710.54)\varphi'^2+2\cdot 3072x'\tau'\\&+(42831.25p^2-183042.64p+211090.66)\tau'^2,\end{aligned}$$

$$S_3'=(12524.4375p^3-10095.525px'^2)x'^2+2(11424p^2-4608x'^2)x'y'$$

$$+(-402.375p^3+12672p)y'^2+2(804.75p^3+9885.15p)x'\eta'$$

$$+2(-1207.125p^2+4608)y'\eta'+2(804.75p^2+4608)x'\xi'-2\cdot1609.5py'\xi',$$

$$T_3'=(-176.448p^3+9561.15p^2-2688x'^2)x'^2+2(-804.75p^3+10850.85p)x'y'$$

$$+(-529.344p^3+1207.125p^2)y'^2+2\cdot768x'\eta'-882.24p^3\eta'^2$$

$$-2\cdot804.75px'\xi'-1235.136p^3\xi'^2,$$

$$V_3'=-2304px'^2+2(-804.75p^2+2304)x'y'+2\cdot804.75px'\eta'+1609.5py'^2,$$

Since $y\geqq0$, $-2py/3\geqq\eta\geqq-2py$, $\xi\leqq0$, we have, for $1.9\leqq p\leqq2$, $|x'/p|\leqq1/20$

$$-V_3'\xi\leqq-2(-804.75p^2+2304)x'y'\xi-2\cdot804.75px'\eta'\xi-1609.5py'^2\xi$$

$$\leqq(804.75\gamma_3p^2-2304\gamma_3)\xi^2+(804.75\gamma_3^{-1}p^2-2304\gamma_3^{-1})x'^2y'^2$$

$$+804.75\gamma_4p\xi^2+804.75\gamma_4^{-1}px'^2\eta'^2+804.75\gamma_5p\xi^2$$

$$+(-268.25\gamma_5^{-1}p^3+1073\gamma_5^{-1}p)y'^2,$$

$$-T_3'\eta\leqq-T^*\eta-(54.5p^3+8826p^2)x'^2\eta-2(-804.75p^3+10378.85p)x'y'\eta$$

$$\leqq-T^*\eta+(27.25\beta_2p^3+4413\beta_2p^2)\eta^2+(27.25\beta_2^{-1}p^3+4413\beta_2^{-1}p^2)x'^4$$

$$+(-804.75\beta_3p^3+10378.85\beta_3p)\eta^2+(-804.75\beta_3^{-1}p^3$$

$$+10378.85\beta_3^{-1}p)x'^2y'^2,$$

$$T^*=(-176.448p^3+735.15p^2-2688x'^2)x'^2+2\cdot472px'y'$$

$$+(-529.344p^3+1207.125p^2)y'^2\geqq0,$$

$$-(S_3'-2pT^*)y\leqq-2(804.75p^3+9885.15p)x'\eta'y-2(-1207.125p^2+4608)y'\eta'y$$

$$-2(804.75p^2+4608)x'\xi'y+2\cdot1609.5py'\xi'y$$

$$\leqq(804.75\alpha_2p^3+9885.15\alpha_2p)y^2+(804.75\alpha_2^{-1}p^3+9885.15\alpha_2^{-1}p)x'^2\eta'^2$$

$$+(1207.125\alpha_3p^2-4107.442\alpha_3)y^2+(-402.375\alpha_3^{-1}p^4+2978.647\alpha_3^{-1}p^2$$

$$-5476.588\alpha_3^{-1})\eta'^2+(804.75\alpha_4p^2+4608\alpha_4)y^2+(804.75\alpha_4^{-1}p^2$$

$$+4608\alpha_4^{-1})x'^2\xi'^2+1609.5\alpha_5py^2+(-536.5\alpha_5^{-1}p^3+2146\alpha_5^{-1}p)\xi'^2.$$

Hence we have, putting $\alpha_2=\alpha_3=\alpha_4=\alpha_5=0.2$, $\beta_2=\beta_3=0.22$, $\gamma_3=\gamma_4=\gamma_5=1$,

$$\Re a_8\leqq8-\frac{x^3}{96\cdot32}\hat{P}_3(x)-\frac{1}{96\cdot32}\hat{Q}_3-\frac{1}{96\cdot32}\hat{R}_3,$$

$$\hat{Q}_3=(100.09078p^5-477.753p^4+5015.621p^3+4970.292p^2-51897.75p+51920.4004)y^2$$

$$+2(301.78125p^4+816.435p^2-2949.12p-301.78125p^2x'^2+6336x'^2)y\eta$$

$$+(1076.39375p^3+18497.89p^2-87773.9382p+94534.7224)\eta^2$$

$$+2(402.375p^3-1536p-402.375px'^2)y\xi+2(1207.125p^2-4669.44)\eta\xi$$

$$+(26451.5p^2-116481.68p+136634.42)\xi^2,$$

$$\hat{R}_3=(3224.11875p^5+502.96875p^3+3893.75p^2-16640.24p+19190.06$$

$$-6391.75975p^3x'^2-20061.498p^2x'^2-276.48x'^2+1302.70625px'^4)x'^2$$

$$+2(4049.30625p^4+402.375p^2-10165.06875p^2x'^2+1056x'^4)x'y'$$

$$+(100.59375p^5+6379.45625p^3+11916.667p^2-50789.5224p+56628.512$$

$$+3457.206p^3x'^2-804.75p^2x'^2-63068.8271px'^2+2304x'^2)y'^2$$

$$+2(6001.425p^3-5840.475px'^2)x'\eta'$$

$$+2(301.78125p^4+7828.275p^2-301.78125p^2x'^2-7488x'^2)y'\eta'$$

$$+(2011.875p^4+905.34375p^3+4575.5149p^2-75323.5p+123333.2399$$

$$-4023.75p^3x'^2-50230.5px'^2)\eta'^2$$

$$+2(6528p^2-2112x'^2)x'\xi'+2(402.375p^3+6912p-402.375px'^2)y'\xi'$$

$$+2(1207.125p^2+6144)\eta'\xi'$$

$$+(2682.5p^3+27256.25p^2-125602.18p+134330.42-4023.75p^2x'^2-23040x'^2)\xi'^2$$

$$+2\cdot 5376px'\varphi'+2\cdot 4608y'\varphi'+(35043.75p^2-149762.16p+172710.54)\varphi'^2$$

$$+2\cdot 3072x'\tau'+(42831.25p^2-183042.64p+211090.66)\tau'^2.$$

$\hat{P}_3(x)$ is monotone decreasing for $0\leqq x\leqq 0.1$ and $\hat{P}_3(0.1)>0$. Hence $\hat{P}_3(x)>0$ for $0\leqq x\leqq 0.1$.

We can prove the positive definiteness of the symmetric matrix associated with $\hat{Q}_3$ for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$ by taking its principal diagonal minor determinants. Hence $\hat{Q}_3$ is positive definite there.

We prove the positive definiteness of $\hat{R}_3$. As in Case 1 we may consider the modified quadratic form, whose associated symmetric matrix is

$$(a_{ij}),\ a_{ij}=a_{ji},$$

$$a_{11}=100.254p^5-1.568p^4+15.717p^3+121.658p^2-532.964p+607.542.$$

$$a_{12}=125.746p^4+12.574p^2,$$

$$a_{22}=3.413p^5+194.305p^3+372.575p^2-1587.173p+1769.641,$$

$a_{13}=187.088p^3$, $a_{23}=9.407p^4+244.048p^2$,

$a_{33}=62.242p^4+24.367p^3+142.984p^2-2353.86p+3854.163$,

$a_{14}=203.835p^2$, $a_{24}=12.542p^3+216p$, $a_{34}=37.722p^2+192$,

$a_{44}=83.199p^3+849.957p^2-3925.069p+4197.825$,

$a_{15}=168p$, $a_{25}=144$, $a_{35}=0$, $a_{45}=0$,

$a_{55}=1095.117p^2-4680.068p+5397.204$.

Its principal diagonal minor determinants are larger than

$100p^5-2p^4+15p^3+121p^2-533p+607$,

$342p^{10}-6p^9+3721p^8+37462p^7-161633p^6+211470p^5-86110p^4-245800p^3$
$+1287552p^2-1907430p+1075131$,

$21297p^{14}-868p^{13}+280562p^{12}+1477693p^{11}-7287543p^{10}+4529914p^9$
$-110716198p^8+595899356p^7-1130865789p^6+926979689p^5$
$+415052553p^4-4224602505p^3+9605983213p^2-9882251469p+4143731016$,

$1771901p^{17}+18029517p^{16}-61968877p^{15}+454250687p^{14}-460234356p^{13}$
$-10401601114p^{12}+29224646413p^{11}-93542848218p^{10}$
$+869799619492p^9-3691648883746p^8+7811287428702p^7$
$-8643635307631p^6-35095475976p^5+25258329338603p^4$
$-63281475888881p^3+82462462596647p^2-57678076956720p+17355024015098$,

$1940439756p^{19}+11451810131p^{18}-142679221873p^{17}+884785188599p^{16}$
$-2964393239876p^{15}-6785857334195p^{14}+78195545360091p^{13}$
$-295338125410299p^{12}+1547900190177387p^{11}-8618151231782623p^{10}$
$+30528784182024911p^9-65955089828508094p^8+82589808980320277p^7$
$-19029959813551431p^6-186756136437168850p^5+520783451111825107p^4$
$-788425611583634739p^3+732721439651747896p^2$
$-392029581083976976p+93465245269336384$,

respectively. All of them are positive for $1.9\leqq p\leqq 2$. Hence $\hat{R}_3$ is positive definite for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$.

Therefore we have $\Re a_8 \leqq 8$ for $1.9 \leqq p \leqq 2$, $|x'/p| \leqq 1/20$ in the present case with equality holding only for $x=0$.

Case 4. $\eta \leqq -2py$.

In this case we start from (B) with $\alpha=7/160, A=3/2, B=1, C=1/4$. We remark the following facts: for $1.9 \leqq p \leqq 2$, $|x'/p| \leqq 1/20$

$$\frac{7}{640} p^3 \xi \leqq \frac{1}{96} \cdot 420 p x'^2 \xi,$$

$$\frac{5}{32} p^2(p^2+2p+4)xy$$

$$\leqq \frac{\alpha_1}{96} (360x^2-540x^3+300x^4-75x^5+7.5x^6)$$

$$+\frac{1}{96} \cdot \frac{1}{\alpha_1} (7.5p^4+15p^3+30p^2)y^2,$$

$$\frac{37}{640} p^4\left(\eta+\frac{15}{8} py\right) \leqq \frac{37 \cdot 3}{640 \cdot 32} p^4\left(\eta+\frac{2}{3} py\right) \leqq \frac{37 \cdot 15}{8 \cdot 32} p^2 x'^2\left(\eta+\frac{2}{3} py\right)$$

$$\leqq \frac{1}{96} \cdot 208.125 p^2 x'^2 \eta+\frac{1}{96} \cdot 138.75 p^3 x'^2 y,$$

$$-\frac{5}{4} x'^2 \varphi \leqq \frac{1}{96} \cdot 60 \gamma_1 \varphi^2+\frac{1}{96} \cdot 60 \gamma_1^{-1} x'^4.$$

Further by Lemma 1 for $1.9 \leqq p \leqq 2$, $|x'/p| \leqq 1/20$

$$\frac{473}{640} p^4 xy+\frac{473}{640 \cdot 4} p^3 x \eta \leqq \frac{1}{8} p^3 x \eta+\frac{793}{640 \cdot 2} p^4 xy$$

$$\leqq \frac{1}{96} (3p^3 x'^2+9p^3 y'^2+15p^3 \eta'^2) \eta$$

$$+\frac{\beta_1}{96} (475.8x^2-951.6x^3+713.7x^4-237.9x^5+29.7375x^6)$$

$$+\frac{1}{96} \cdot \frac{1}{\beta_1} \cdot 29.7375 p^4 y^2,$$

$$-\frac{15}{4} y^2 x' y' \leqq \frac{1}{96} 386.325 p x'^2 y^2+\frac{\gamma_2}{96} (-7.357p^2+29.428) y^2$$

$$+\frac{1}{96} \cdot \frac{1}{\gamma_2} (-7.357p^2+29.428) y'^2,$$

$$\frac{17}{8} x' y'^3 \leqq \frac{1}{96} \cdot 3.679 p y'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-749.2x/96$ we have, with $\alpha_1=1$, $\beta_1=2$, $\gamma_1=28$, $\gamma_2=1$,

$$\Re a_8 \leqq 8-\frac{x}{96}\hat{P}_4(x)-\frac{1}{96}Q_4-\frac{1}{96}R_4-\frac{1}{96}S_4 y-\frac{1}{96}T_4\eta-\frac{1}{96}V_4\xi,$$

$$\hat{P}_4(x)=19.816-227.349x+315.01x^2-63.941x^3-113.048x^4+69.883x^5-11.65x^6,$$

$$Q_4=(-22.369p^4+133.425p^3-214.643p^2+388.472-8.325px'^2)y^2$$
$$+2(-33p^2-48p+198x'^2)y\eta+(-162p+924.5)\eta^2-2\cdot 48py\xi-2\cdot 168\eta\xi+1311.1\xi^2,$$

$$R_4=(100.125p^5+187.3-196.5p^3x'^2-2.16x'^2+40.5px'^4)x'^2$$
$$+2(119.625p^4-312p^2x'^2+33x'^4)x'y'$$
$$+(174p^3+7.357p^2-3.679p+532.472-504px'^2)y'^2$$
$$+2(180p^3-180px'^2)x'\eta'+2(222p^2-234x'^2)y'\eta'+(216p+936.5)\eta'^2$$
$$+2(204p^2-66x'^2)x'\xi'+2\cdot 216py'\xi'+2\cdot 192\eta'\xi'+1311.1\xi'^2$$
$$+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'+1685.7\varphi'^2+2\cdot 96x'\tau'+2060.3\tau'^2,$$

$$S_4=(410.25p^3-318px'^2)x'^2+2(357p^2-144x'^2)x'y'+396py'^2$$
$$+2\cdot 324px'\eta'+2\cdot 144y'\eta'+2\cdot 144x'\xi',$$

$$T_4=(-3p^3+313.875p^2-84x'^2)x'^2+2\cdot 324px'y'-9p^3y'^2+2\cdot 24x'\eta'-15p^3\eta'^2,$$

$$V_4=-72px'^2+2\cdot 72x'y'.$$

Since $y\geqq 0$, $\eta\leqq -2py$, $\xi\leqq 0$, we have, for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$-V_4\xi\leqq -144x'y'\xi\leqq 72\gamma_3\xi^2+72\gamma_3^{-1}x'^2y'^2,$$

$$-T_4\eta\leqq -313.875p^2x'^2\eta-2\cdot 308px'y'\eta$$
$$\leqq 156.9375\beta_2p^2\eta^2+156.9375\beta_2^{-1}p^2x'^4+308\beta_3p\eta^2+308\beta_3^{-1}px'^2y'^2$$

and

$$-S_4y\leqq -2\cdot 324px'\eta'y-2\cdot 144y'\eta'y-2\cdot 144x'\xi'y$$
$$\leqq 324\alpha_2py^2+324\alpha_2^{-1}px'^2\eta'^2+144\alpha_3y^2+(-48\alpha_3^{-1}p^2+192\alpha_3^{-1})\eta'^2$$
$$+144\alpha_4y^2+144\alpha_4^{-1}x'^2\xi'^2.$$

Hence we have, with $\alpha_2=\alpha_3=\alpha_4=0.255$, $\beta_2=\beta_3=0.464$, $\gamma_3=1$,

$$\Re a_8\leqq 8-\frac{x}{96}\hat{P}_4(x)-\frac{1}{96}\hat{Q}_4-\frac{1}{96}\hat{R}_4,$$

$$\hat{Q}_4=(-22.369p^4+133.425p^3-214.643p^2-82.62p+315.032-8.325px'^2)y^2$$
$$+2(-33p^2-48p+198x'^2)y\eta+(-72.819p^2-304.912p+924.5)\eta^2-2\cdot 48py\xi$$
$$-2\cdot 168\eta\xi+1239.1\xi^2,$$
$$\hat{R}_4=(100.125p^5+187.3-196.5p^3x'^2-338.358p^2x'^2-2.16x'^2+40.5px'^4)x'^2$$
$$+2(119.625p^4-312p^2x'^2+33x'^4)x'y'$$
$$+(174p^3+7.357p^2-3.679p+532.472-1168.048px'^2-72x'^2)y'^2$$
$$+2(180p^3-180px'^2)x'\eta'+2(222p^2-234x'^2)y'\eta'$$
$$+(188.256p^2+216p+183.476-1270.728px'^2)\eta'^2$$
$$+2(204p^2-66x'^2)x'\xi'+2\cdot 216py'\xi'+2\cdot 192\eta'\xi'+(1311.1-564.768x'^2)\xi'^2$$
$$+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'+1685.7\varphi'^2+2\cdot 96x'\tau'+2060.3\tau'^2.$$

$\hat{P}_4(x)$ is monotone decreasing for $0\leqq x\leqq 0.1$ and $\hat{P}_4(0.1)>0$. Hence $\hat{P}_4(x)>0$ for $0\leqq x\leqq 0.1$.

$\hat{Q}_4$ is positive definite for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$. Indeed we have

$$-22.369p^4+133.425p^3-214.643p^2-82.62p+315.032-8.325px'^2\geqq 0,$$
$$(-33p^2-48p+198x'^2)y\eta\geqq 0,$$
$$-48py\xi\geqq 0$$

and

$$(-72.819p^2-304.912p+924.5)\eta^2-2\cdot 168\eta\xi+1239\cdot 1\xi^2\geqq 0.$$

These imply the positive definiteness of $\hat{Q}_4$ for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$.

Since $4.48x'^2+2\cdot 96x'\tau'+2060.3\tau'^2\geqq 0$, $144y'^2+2\cdot 144y'\varphi'+144\varphi'^2\geqq 0$, $18.31p^2x'^2+2\cdot 168px'\varphi'+1541.7\varphi'^2\geqq 0$ and $192\eta'^2+2\cdot 192\eta'\xi'+192\xi'^2\geqq 0$ we may consider $R^*=\hat{R}_4-\{(18.31p^2+4.48)x'^2+144y'^2+192\eta'^2+2\cdot 192\eta'\xi'+192\xi'^2+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'+1685.7\varphi'^2+2\cdot 96x'\tau'+2060.3\tau'^2\}$. It is easy to prove that the symmetric matrix associated with R^* is positive definite for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$. Hence $\hat{R}_4$ is positive definite there.

Thus we have $\Re a_8\leqq 8$ for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$ in the present case with equality holding only for $x=0$.

Summing up the results we have $\Re a_8\leqq 8$ for $1.9\leqq p\leqq 2$, $|x'p|\leqq 1/20$ and $y\geqq 0$, $\xi\leqq 0$ with equality holding only for $x=0$.

§ 3. Next we shall be concerned with the case $y\geqq 0$, $\xi\geqq 0$ and $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$. In this case we put $\alpha=0$ in (B). We divide this case into several subcases.

Case 1. $\eta\geqq 0$.

We start from (B) with $\alpha=0$. Applying Lemma 3 to the term $(3p^4/64)(\eta+13py/6)$

we have

$$\Re a_8 \leqq U+\frac{1}{8}p+\frac{2}{3}p^3-\frac{1}{6}p^5-\frac{1}{32\cdot 16}p^7$$

$$+\frac{5}{48}A(p^4+2p^3+4p^2)xy-\frac{1}{16}(p^4+2p^3)xy+\frac{3}{4}Bp^4xy+\frac{3}{4}Cp^3x\eta$$

$$+\left\{-\frac{15}{16\cdot 32}p^5+\left(\frac{207}{4\cdot 32}-\frac{5}{8}A-\frac{5}{4}B\right)p^3+\frac{1}{12}A^2x(p^2+2p+4)+B^2xp^2-\frac{3}{32}p\right.$$

$$\left.+\frac{15}{8\cdot 32}p^3x'^2+\left(\frac{1}{4}A^2+\frac{1}{2}A-\frac{681}{128}\right)px'^2\right\}y^2$$

$$+\left\{-\frac{25}{4\cdot 32}p^4+\left(\frac{133}{32}-\frac{3}{4}A-2B-\frac{5}{4}C\right)p^2+2BCxp+\frac{75}{96\cdot 4}p^2x'^2+\left(\frac{A}{2}-\frac{39}{8}\right)x'^2\right\}y\eta$$

$$+\left\{-\frac{125}{96\cdot 4}p^3+\left(\frac{63}{32}-2C\right)p+C^2x\right\}\eta^2+\left\{-\frac{15}{64}p^3+\left(\frac{9}{2}-2B-A\right)p+\frac{15}{64}px'^2\right\}y\xi$$

$$(\mathrm{B}_2)\quad +\left(-\frac{25}{32}p^2+4-2C\right)\eta\xi-\frac{15}{32}p\xi^2+(3-2A)y\varphi$$

$$+\left(-\frac{537}{64\cdot 8}p^5-\frac{1}{6}p^3+\frac{1045}{64\cdot 8}p^3x'^2-\frac{217}{64\cdot 8}px'^4\right)x'^2$$

$$+\left(-\frac{337}{64\cdot 2}p^4-\frac{1}{4}p^2+\frac{423}{64}p^2x'^2-\frac{11}{16}x'^4\right)x'y'$$

$$+\left(-\frac{15}{16\cdot 32}p^5-\frac{257}{4\cdot 32}p^3-\frac{3}{32}p+\frac{15}{8\cdot 32}p^3x'^2+\frac{1989}{4\cdot 96}px'^2\right)y'^2$$

$$+\left(-\frac{249}{64}p^3+\frac{243}{64}px'^2\right)x'\eta'+\left(-\frac{25}{4\cdot 32}p^4-\frac{163}{32}p^2+\frac{75}{4\cdot 96}p^2x'^2+\frac{39}{8}x'^2\right)y'\eta'$$

$$+\left(-\frac{125}{4\cdot 96}p^3-\frac{81}{32}p\right)\eta'^2+\left(-\frac{17}{4}p^2+\frac{11}{8}x'^2\right)x'\xi'+\left(-\frac{15}{64}p^3-\frac{9}{2}p+\frac{15}{64}px'^2\right)y'\xi'$$

$$+\left(-\frac{25}{32}p^2-4\right)\eta'\xi'-\frac{15}{32}p\xi'^2-\frac{7}{2}px'\varphi'-3y'\varphi'-2x'\tau'$$

$$+S_5y+T_5\eta+V_5\xi-\frac{5}{4}x'^2\varphi+\left\{\frac{15}{4\cdot 32}p^3+\left(\frac{9}{8}+\frac{1}{2}A+\frac{1}{2}A^2-2AB\right)p\right\}y^3$$

$$+\left(\frac{25}{64}p^2+2A-A^2-2AC\right)y^2\eta+\frac{15}{32}py^2\xi+\left(-\frac{51}{8}+A+\frac{1}{2}A^2\right)y^2x'y'+\frac{17}{8}x'y'^3,$$

$$S_5=\left\{\left(\frac{5}{16}A-6\right)p^3+\frac{421}{128}px'^2\right\}x'^2+\left\{\left(\frac{13}{8}A-\frac{79}{8}\right)p^2+3x'^2\right\}x'y'$$

$$+\left\{\frac{15}{4\cdot 32}p^3-\left(\frac{1}{2}A+\frac{27}{8}\right)p\right\}y'^2+\left(-\frac{65}{4\cdot 32}p^3-\frac{207}{32}p\right)x'\eta'+\left(\frac{25}{32}p^2-2A\right)y'\eta'$$

$$+\left(-\frac{15}{32}p^2-\frac{3}{2}-A\right)x'\xi'+\frac{15}{16}py'\xi',$$

$$T_5=\left(-\frac{339}{64}p^2+\frac{7}{8}x'^2\right)x'^2+\left(\frac{65}{4\cdot 32}p^3-\frac{225}{32}p\right)x'y'-\frac{25}{64}p^2y'^2-\frac{1}{2}x'\eta'+\frac{15}{32}px'\xi',$$

$$V_5=-\frac{29}{8}px'^2+\left(\frac{15}{32}p^2-\frac{3}{2}\right)x'y'-\frac{15}{32}py'^2-\frac{15}{32}px'\eta'.$$

Here we put $A=3/2$, $B=9/8$, $C=1$. We remark the following facts: for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$\frac{1}{96}(9p^4+18p^3+60p^2)xy\leqq\frac{\alpha_1}{192}p^2(9p^2+18p+60)x^2+\frac{1}{192}\cdot\frac{1}{\alpha_1}p^2(9p^2+18p+60)y^2,$$

$$-\frac{5}{4}x'^2\varphi\leqq\frac{\gamma_1}{96}60\varphi^2+\frac{1}{96}\cdot\frac{1}{\gamma_1}60x'^4,$$

$$\left(\frac{25}{64}p^2-\frac{9}{4}\right)y^2\eta\leqq 0.$$

Further by Lemma 2

$$\frac{3}{4}p^3x\eta+\frac{27}{32}p^4xy=\frac{3}{4}p^3x\left(\eta+\frac{9}{8}py\right)$$

$$\leqq\frac{571}{1024}p^5x^2+\frac{1}{8}p^4x^2+\frac{1}{4}p^3x^2-\frac{3}{8}p^3xx'y'+\frac{3}{16}p^4xx'^2.$$

By Lemma 1 for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$\left(\frac{15}{4\cdot 32}p^3-\frac{3}{8}p\right)y^3\leqq\left(\frac{15}{4\cdot 32}p^3-\frac{3}{8}p\right)\sqrt{\frac{4-p^2}{3}}\,y^2\leqq\frac{1}{96}(4.062p^3-12.996p)y^2,$$

$$\frac{15}{32}py^2\xi\leqq\frac{15p}{64}\beta_1y^4+\frac{15p}{64}\cdot\frac{1}{\beta_1}\xi^2\leqq\frac{\beta_1}{96}(-7.5p^3+30p-7.5px'^2)y^2+\frac{1}{96}\cdot\frac{1}{\beta_1}\cdot 22.5p\xi^2,$$

$$-\frac{15}{4}y^2x'y'\leqq\frac{400}{96}px'^2y^2+\frac{42.632}{96}y'^2y^2\leqq\frac{400}{96}px'^2y^2+\frac{1}{96}(-14.211p^2+56.844)y'^2,$$

$$\frac{17}{8}x'y'^3\leqq\frac{17}{8}\cdot\frac{1}{20}p\sqrt{\frac{4-p^2}{3}}\,y'^2\leqq\frac{1}{96}\cdot 3.679py'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-154x/96$ we have, with $\alpha_1=0.6$, $\beta_1=0.2$, $\gamma_1=5.7$,

$$\Re a_8 \leqq 8 - \frac{x}{96}\hat{P}_5(x) - \frac{1}{96}Q_5 - \frac{1}{96}R_5 - \frac{1}{96}S_5'y - \frac{1}{96}T_5'\eta - \frac{1}{96}V_5'\xi,$$

$$\hat{P}_5(x) = 6 - 222.9x + 2274.7x^2 - 2891.2x^3 + 1569.2x^4 - 405.8375x^5,$$

$$Q_5 = (2.8125p^5 - 7.5p^4 + 191.688p^3 - 293p^2 + 15.996p - 28.5 - 5.625p^3x'^2 - 13.75px'^2)y^2$$
$$+ 2(9.375p^4 + 130.5p^2 - 216p - 9.375p^2x'^2 + 198x'^2)y\eta$$
$$+ (31.25p^3 + 99p + 0.5)\eta^2 + 2(11.25p^3 - 36p - 11.25px'^2)y\xi$$
$$+ 2(37.5p^2 - 96)\eta\xi + (-67.5p + 269.5)\xi^2,$$

$$R_5 = (118.6875p^5 - 36p^4 + 16p^3 + 38.5 - 195.9375p^3x'^2 - 10.8x'^2 + 40.6875px'^4)x'^2$$
$$+ 2(108.375p^4 + 36p^3 + 12p^2 - 317.25p^2x'^2 + 33x'^4)x'y'$$
$$+ (2.8125p^5 + 192.75p^3 + 14.211p^2 + 5.321p + 58.656 - 5.625p^3x'^2 - 497.25px'^2)y'^2$$
$$+ 2(186.75p^3 - 182.25px'^2)x'\eta' + 2(9.375p^4 + 244.5p^2 - 9.375p^2x'^2 - 234x'^2)y'\eta'$$
$$+ (31.25p^3 + 243p + 192.5)\eta'^2 + 2(204p^2 - 66x'^2)x'\xi'$$
$$+ 2(11.25p^3 + 216p - 11.25px'^2)y'\xi'$$
$$+ 2(37.5p^2 + 192)\eta'\xi' + (45p + 269.5)\xi'^2 + 2\cdot 168px'\varphi' + 2\cdot 144y'\varphi'$$
$$+ 346.5\varphi'^2 + 2\cdot 96x'\tau' + 423.5\tau'^2,$$

$$S_5' = (531p^3 - 315.75px'^2)x'^2 + 2(357p^2 - 144x'^2)x'y' + (-11.25p^3 + 396p)y'^2$$
$$+ 2(24.375p^3 + 310.5p)x'\eta' + 2(-37.5p^2 + 144)y'\eta' + 2(22.5p^2 + 144)x'\xi' - 2\cdot 45py'\xi',$$

$$T_5' = (508.5p^2 - 84x'^2)x'^2 + 2(-24.375p^3 + 337.5p)x'y' + 37.5p^2y'^2 + 2\cdot 24x'\eta'$$
$$- 2\cdot 22.5px'\xi',$$

$$V_5' = 348px'^2 + 2(-22.5p^2 + 72)x'y' + 45py'^2 + 2\cdot 22.5p'x'\eta'.$$

Since $y \geqq 0$, $\eta \geqq 0$, $\xi \geqq 0$, we have, for $1.9 \leqq p \leqq 2$, $|x'/p| \leqq 1/20$

$$-S_5'y \leqq -2(24.375p^3 + 310.5p)x'\eta'y - 2(-37.5p^2 + 144)y'\eta'y$$
$$- 2(22.5p^2 + 144)x'\xi'y + 2\cdot 45py'\xi'y$$
$$\leqq (24.375\alpha_2 p^3 + 310.5\alpha_2 p)y^2 + (24.375\alpha_2^{-1}p^3 + 310.5\alpha_2^{-1}p)x'^2\eta'^2$$
$$+ (37.5\alpha_3 p^2 - 126.75\alpha_3)y^2 + (-12.5\alpha_3^{-1}p^4 + 92.25\alpha_3^{-1}p^2 - 169\alpha_3^{-1})\eta'^2$$
$$+ (22.5\alpha_4 p^2 + 144\alpha_4)y^2 + (22.5\alpha_4^{-1}p^2 + 144\alpha_4^{-1})x'^2\xi'^2$$
$$+ 45\alpha_5 py^2 + (-7.5\alpha_5^{-1}p^3 + 30\alpha_5^{-1}p)y'^2 + (-3.2143\alpha_5^{-1}p^3 + 12.8572\alpha_5^{-1}p)\xi'^2,$$

$$-T_5'\eta \leqq -2\cdot 24x'\eta'\eta + 2\cdot 22.5px'\xi'\eta$$

$$\leqq 24\beta_2\eta^2 + 24\beta_2^{-1}x'^2\eta'^2 + 22.5\beta_3 p\eta^2 + 22.5\beta_3^{-1}px'^2\xi'^2$$

and

$$-V_5'\xi \leqq -2\cdot 22.5px'\eta'\xi \leqq 22.5\gamma_3 p\xi^2 + 22.5\gamma_3^{-1}px'^2\eta'^2.$$

Hence we have, putting $\alpha_2=\alpha_3=\alpha_4=0.1727$, $\alpha_5=0.6908$, $\beta_2=\beta_3=6.03$, $\gamma_3=2.98$,

$$\Re a_8 \leqq 8 - \frac{x}{96}\hat{P}_5(x) - \frac{1}{96}Q_5 - \frac{1}{96}\hat{R}_5,$$

$$\begin{aligned}\hat{Q}_5 = {} & (2.8125p^5 - 7.5p^4 + 187.4784375p^3 - 303.362p^2 - 68.71335p - 31.479075 \\ & -5.625p^3x'^2 - 13.75px'^2)y^2 \\ & +2(9.375p^4 + 130.5p^2 - 216p - 9.375p^2x'^2 + 198x'^2)y\eta \\ & +(31.25p^3 - 36.675p - 144.22)\eta^2 + 2(11.25p^3 - 36p - 11.25px'^2)y\xi \\ & +2(37.5p^2 - 96)\eta\xi + (-134.55p + 269.5)\xi^2,\end{aligned}$$

$$\begin{aligned}\hat{R}_5 = {} & (118.6875p^5 - 36p^4 + 16p^3 + 38.5 - 195.9375p^3x'^2 - 10.8x'^2 + 40.6875px'^4)x'^2 \\ & +2(108.375p^4 + 36p^3 + 12p^2 - 317.25p^2x'^2 + 33x'^4)x'y' \\ & +(2.8125p^5 + 203.607p^3 + 14.211p^2 - 38.107p + 58.656 - 5.625p^3x'^2 \\ & -497.25px')y'^2 \\ & +2(186.75p^3 - 182.25px'^2)x'\eta' + 2(9.375p^4 + 244.5p^2 - 9.375p^2x'^2 - 234x'^2)y'\eta' \\ & +(72.37987p^4 + 31.25p^3 - 534.16348p^2 + 243p + 1171.0759 - 141.1408p^3x'^2 \\ & -1805.4761px'^2 - 3.99x'^2)\eta'^2 \\ & +2(204p^2 - 66x'^2)x'\xi' + 2(11.25p^3 + 216p - 11.25px'^2)y'\xi' + 2(37.5p^2 + 192)\eta'\xi' \\ & +(4.65302p^3 + 26.3879p + 269.5 - 130.28378p^2x'^2 - 3.735px'^2 - 833.8162x'^2)\xi'^2 \\ & +2\cdot 168px'\varphi' + 2\cdot 144y'\varphi' + 346.5\varphi'^2 + 2\cdot 96x'\tau' + 423.5\tau'^2.\end{aligned}$$

It is easy to prove that $\hat{P}_5'(x)$ is monotone increasing for $0\leqq x\leqq 0.1$ and $\hat{P}_5'(0)<0$, $\hat{P}_5'(0.055)>0$. Let λ be the root of $\hat{P}_5'(x)=0$, $0<\lambda<0.055$. Construct $N(x)=5\hat{P}_5(x)-x\hat{P}_5'(x)$. Then $N(x)$ is monotone decreasing for $0\leqq x\leqq 0.055$. Further $N(0.055)>0$. Hence $N(x)>0$ for $0\leqq x\leqq 0.055$. Especially $N(\lambda)>0$, which implies $\hat{P}_5(\lambda)>0$. Therefore $\hat{P}_5(x)>0$ for $0\leqq x\leqq 0.1$.

Next, it is very easy to prove that the coefficients of y^2, $y\eta$, η^2, $y\xi$, $\eta\xi$ and ξ^2 are non-negative for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$. This implies that $\hat{Q}_5$ is non-negative there.

As in the above cases by a large amount of calculation we have the positive definiteness of $\hat{R}_5$ for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$.

Therefore we have $\Re a_8\leqq 8$ for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$ in the present case with equality holding only for $x=0$.

Case 2. $-2py/3\leqq \eta\leqq 0$.

We start from (B_2) with $A=3/2, B=1, C=1/4$. We remark the following facts: for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$\frac{1}{32}(3p^4+6p^3+20p^2)xy\leqq\frac{\alpha_1}{64}(3p^4+6p^3+20p^2)x^2+\frac{1}{64}\frac{1}{\alpha_1}(3p^4+6p^3+20p^2)y^2,$$

$$\frac{3}{4}p^4xy\leqq\frac{3}{8}\beta_1p^4x^2+\frac{3}{8}\cdot\frac{1}{\beta_1}p^4y^2,$$

$$-\frac{5}{4}x'^2\varphi\leqq\frac{\gamma_1}{96}60\varphi^2+\frac{1}{96}\cdot\frac{1}{\gamma_1}60x'^4.$$

Further by Lemma 1 for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$\frac{3}{16}xp^3\eta\leqq\frac{1}{96}(4.5p^3x'^2+13.5p^3y'^2+22.5p^3\eta'^2+31.5p^3\xi'^2)\eta,$$

$$\frac{15}{32}py^2\xi\leqq\frac{15}{64}\beta_2py^4+\frac{15}{64}\cdot\frac{1}{\beta_2}p\xi^2$$

$$\leqq\frac{\beta_2}{96}(-7.5p^3+30p-7.5px'^2)y^2+\frac{1}{96}\cdot\frac{1}{\beta_2}22.5p\xi^2,$$

$$\frac{17}{8}x'y'^3\leqq\frac{1}{96}\cdot 3.679py'^2.$$

By Lemma 2 and Lemma 1 for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$\frac{25}{64}p^2y^2\eta+\frac{15}{128}p^3y^3-\frac{15}{4}y^2x'y'=\frac{25}{64}p^2\left(\eta+\frac{3}{10}py\right)y^2-\frac{15}{4}y^2x'y'$$

$$\leqq\frac{1}{96}(-9.125p^5+12p^4+25p^2)y^2-\frac{25}{128}p^2y^2x'y'+\frac{25}{256}p^3x'^2y^2-\frac{15}{4}y^2x'y'$$

$$\leqq\frac{1}{96}(-9.125p^5+12p^4+25p^2-5.625p^3x'^2+399.75px'^2)y^2+\frac{97.742}{256}py^2y'^2$$

$$\leqq\frac{1}{96}(-9.125p^5+12p^4+25p^2-5.625p^3x'^2+399.75px'^2)y^2$$

$$+\frac{\gamma_2}{96}(-6.109p^3+24.436p)y^2+\frac{1}{96}\frac{1}{\gamma_2}(-6.109p^3+24.436p)y'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-(160x+18x^2$

$+716x^3)/96$ we have, with $\alpha_1=2.4$, $\beta_1=2.4$, $\gamma_1=6$, $\beta_2=0.5$, $\gamma_2=1$,

$$\Re a_8 \leqq 8-\frac{x^3}{96}\hat{P}_6(x)-\frac{1}{96}Q_6-\frac{1}{96}R_6-\frac{1}{96}S_6y-\frac{1}{96}T_6\eta-\frac{1}{96}V_6\xi,$$

$$\hat{P}_6(x)=43.1-416.9x+80.15x^2+46.975x^3-12.2268x^4,$$

$$\begin{aligned}Q_6=&(11.9375p^5-28.875p^4+174.859p^3+307.5p^2-2191.936p+2151-11.25px'^2)y^2\\&+2(9.375p^4-10.5p^2-48p-9.375p^2x'^2+198x'^2)y\eta\\&+(31.25p^3+895p^2-3737.5p+3813)\eta^2+2(11.25p^3-48p-11.25px'^2)y\xi\\&+2(37.5p^2-168)\eta\xi+(1253p^2-5043.5p+5355)\xi^2,\end{aligned}$$

$$\begin{aligned}R_6=&(100.6875p^5+16p^3+179p^2-720.5p+765-195.9375p^3x'^2-10x'^2)x'^2\\&+2(126.375p^4+12p^2-317.25p^2x'^2+33x'^4)x'y'\\&+(2.8125p^5+198.859p^3+537p^2-2180.615p+2295-5.625p^3x'^2-497.25px'^2)y'^2\\&+2(186.75p^3-182.25px'^2)x'\eta'+2(9.375p^4+244.5p^2-9.375p^2x'^2-234x'^2)y'\eta'\\&+(31.25p^3+895p^2-3359.5p+3825)\eta'^2+2(204p^2-66x'^2)x'\xi'\\&+2(11.25p^3+216p-11.25px'^2)y'\xi'+2(37.5p^2+192)\eta'\xi'\\&+(1253p^2-4998.5p+5355)\xi'^2+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'\\&+(1611p^2-6484.5p+6885)\varphi'^2+2\cdot 96x'\tau'+(1969p^2-7925.5p+8415)\tau'^2,\end{aligned}$$

$$\begin{aligned}S_6=&(531p^3-315.75px'^2)x'^2+2(357p^2-144x'^2)x'y'+(-11.25p^3+396p)y'^2\\&+2(24.375p^3+310.5p)x'\eta'+2(-37.5p^2+144)y'\eta'+2(22.5p^2+144)x'\xi'\\&-2\cdot 45py'\xi',\end{aligned}$$

$$\begin{aligned}T_6=&(-4.5p^3+508.5p^2-84x'^2)x'^2+2(-24.375p^3+337.5p)x'y'\\&+(-13.5p^3+37.5p^2)y'^2+2\cdot 24x'\eta'-22.5p^3\eta'^2-2\cdot 22.5px'\xi'-31.5p^3\xi'^2,\end{aligned}$$

$$V_6=348px'^2-2(22.5p^2-72)x'y'+45py'^2+45px'\eta'.$$

Now, since $y\geqq 0$, $0\geqq\eta\geqq -2py/3$, $\xi\geqq 0$, we have

$$-V_6\xi\leqq -45px'\eta'\xi\leqq 22.5\gamma_3p\xi^2+22.5\gamma_3^{-1}px'^2\eta'^2,$$

$$\begin{aligned}-T_6\eta&\leqq -T^*\eta-2(-24.375p^3+192.2p)x'y'\eta\\&\leqq -T^*\eta+(-24.375\beta_3p^3+192.2\beta_3p)\eta^2+(-24.375\beta_3^{-1}p^3+192.2\beta_3^{-1}p)x'^2y'^2,\end{aligned}$$

$$T^*=(-2.7p^3+508.5p^2-84x'^2)x'^2+2\cdot 145.3px'y'+(-13.5p^3+37.5p^2)y'^2\geqq 0,$$

$$-\left(S_6-\frac{2}{3}pT^*\right)y\leqq -2\cdot 9.3p^2x'y'y-2(24.375p^3+310.5p)x'\eta'y-2(-37.5p^2+144)y'\eta'y$$
$$-2(22.5p^2+144)x'\xi'y+2\cdot 45py'\xi'y$$
$$\leqq 9.3\alpha_2p^2y^2+9.3\alpha_2^{-1}p^2x'^2y'^2+(24.375\alpha_3p^3+310.5\alpha_3p)y^2$$
$$+(24.375\alpha_3^{-1}p^3+310.5\alpha_3^{-1}p)x'^2\eta'^2+(37.5\alpha_4p^2-126.75\alpha_4)y^2$$
$$+(-12.5\alpha_4^{-1}p^4+92.25\alpha_4^{-1}p^2-169\alpha_4^{-1})\eta'^2$$
$$+(22.5\alpha_5p^2+144\alpha_5)y^2+(22.5\alpha_5^{-1}p^2+144\alpha_5^{-1})x'^2\xi'^2$$
$$+45\alpha_6py^2+(-15\alpha_6^{-1}p^3+60\alpha_6^{-1}p)\xi'^2$$

for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$.

Hence we have, putting $\alpha_2=\alpha_3=\alpha_4=\alpha_5=\alpha_6=0.192$, $\beta_3=0.4$, $\gamma_3=0.5$

$$\Re a_8\leqq 8-\frac{x^3}{96}\hat{P}_6(x)-\frac{1}{96}\hat{Q}_6-\frac{1}{96}\hat{R}_6,$$

$$\hat{Q}_6=(11.9375p^5-28.875p^4+170.179p^3+294.194p^2-2260.192p+2147.688)y^2$$
$$+2(9.375p^4-10.5p^2-48p-9.375p^2x'^2+198x'^2)y\eta$$
$$+(41p^3+895p^2-3814.38p+3813)\eta^2+2(11.25p^3-48p-11.25px'^2)y\xi$$
$$+2(37.5p^2-168)\eta\xi+(1253p^2-5054.75p+5355)\xi^2,$$

$$\hat{R}_6=(100.6875p^5+16p^3+179p^2-720.5p+765-195.9375p^3x'^2-10x'^2)x'^2$$
$$+2(126.375p^4+12p^2-317.25p^2x'^2+33x'^4)x'y'$$
$$+(2.8125p^5+198.859p^3+537p^2-2180.615p+2295$$
$$+55.3125p^3x'^2-48.4437p^2x'^2-977.75px'^2)y'^2$$
$$+2(186.75p^3-182.25px'^2)x'\eta'$$
$$+2(9.375p^4+244.5p^2-9.375p^2x'^2-234x'^2)y'\eta'$$
$$+(65.1125p^4+31.25p^3+414.46975p^2-3359.5p+4705.321$$
$$-126.97p^3x'^2-1662.395px'^2)\eta'^2$$
$$+2(204p^2-66x'^2)x'\xi'+2(11.25p^3+216p-11.25px'^2)y'\xi'+2(37.5p^2+192)\eta'\xi'$$
$$+(78.135p^3+1253p^2-5311.04p+5355-117.2025p^2x'^2-750.096x'^2)\xi'^2$$
$$+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'+(1611p^2-6484.5p+6885)\varphi'^2+2\cdot 96x'\tau'$$
$$+(1969p^2-7925.5p+8415)\tau'^2.$$

$\hat{P}_6(x)$ is monotone decreasing for $0\leqq x\leqq 0.1$ and $\hat{P}_6(0.1)>0$. Hence $\hat{P}_6(x)>0$ for $0\leqq x\leqq 0.1$.

Since $\eta\xi\leqq 0$ we may consider $Q^*=\hat{Q}_6-2(37.5p^2-168)\eta\xi$. We can prove the positive definiteness of the symmetric matrix associated with Q^* for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$ by taking its principal diagonal minor determinants. Hence $\hat{Q}_6$ is non-negative for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$.

By a large amount of calculation we have the positive definiteness of $\hat{R}_6$ for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$.

Thus we have $\Re a_8\leqq 8$ for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$ in the present case with equality holding only for $x=0$.

Case 3. $-py\leqq\eta\leqq -2py/3$.

We start from (B) with $\alpha=0$. Applying Lemma 3 to the term $(27p^4/640)(\eta+7py/3)$ we have

$$\Re a_8\leqq U+\frac{9}{80}p-\frac{9}{64\cdot 80}p^7+\frac{867}{64\cdot 20}p^3-\frac{867}{64\cdot 80}p^5+\frac{3}{640}p^4\left(\eta+\frac{2}{3}py\right)$$

$$+\frac{5}{48}A(p^4+2p^3+4p^2)xy-\frac{153}{64\cdot 40}(p^4+2p^3)xy+\frac{3}{4}Bp^4xy+\frac{3}{4}Cp^3x\eta$$

$$+\left\{-\frac{27}{16\cdot 64}p^5+\left(\frac{8191}{64\cdot 80}-\frac{5}{8}A-\frac{5}{4}B\right)p^3+\frac{1}{12}A^2x(p^2+2p+4)+B^2xp^2-\frac{27}{320}p\right.$$

$$\left.+\frac{27}{64\cdot 8}p^3x'^2+\left(-\frac{6801}{64\cdot 20}+\frac{1}{2}A+\frac{1}{4}A^2\right)px'^2\right\}y^2$$

$$+\left\{-\frac{99}{64\cdot 8}p^4+\left(\frac{2663}{640}-\frac{3}{4}A-2B-\frac{5}{4}C\right)p^2+2BCxp+\frac{99}{64\cdot 8}p^2x'^2\right.$$

$$\left.+\left(\frac{A}{2}-\frac{39}{8}\right)x'^2\right\}y\eta$$

$$+\left\{-\frac{363}{64\cdot 16}p^3+\left(\frac{639}{320}-2C\right)p+C^2x\right\}\eta^2+\left\{-\frac{27}{128}p^3+\left(\frac{9}{2}-2B-A\right)p\right.$$

$$\left.+\frac{27}{128}px'^2\right\}y\xi$$

$$+\left\{-\frac{99}{128}p^2+(4-2C)\right\}\eta\xi-\frac{27}{64}p\xi^2+(3-2A)y\varphi$$

$$+\left(-\frac{5367}{64\cdot 80}p^5-\frac{867}{64\cdot 80}p^3+\frac{10453}{64\cdot 80}p^3x'^2-\frac{2169}{64\cdot 80}px'^4\right)x'^2$$

$$+\left(-\frac{6731}{64\cdot 40}p^4-\frac{153}{640}p^2+\frac{16901}{64\cdot 40}p^2x'^2-\frac{11}{16}x'^4\right)x'y'$$

$$+\left(-\frac{27}{64\cdot 16}p^5-\frac{10369}{64\cdot 80}p^3-\frac{27}{320}p+\frac{27}{64\cdot 8}p^3x'^2+\frac{6639}{64\cdot 20}px'^2\right)y'^2$$

$$+\left(-\frac{2481}{640}p^3+\frac{2427}{640}px'^2\right)x'\eta'+\left(-\frac{99}{64\cdot 8}p^4-\frac{3257}{640}p^2+\frac{99}{64\cdot 8}p^2x'^2+\frac{39}{8}x'^2\right)y'\eta'$$

$$+\left(-\frac{363}{64\cdot 16}p^3-\frac{801}{320}p\right)\eta'^2+\left(-\frac{17}{4}p^2+\frac{11}{8}x'^2\right)x'\xi'$$

$$+\left(-\frac{27}{128}p^3-\frac{9}{2}p+\frac{27}{128}px'^2\right)y'\xi'+\left(-\frac{99}{128}p^2-4\right)\eta'\xi'-\frac{27}{64}p\xi'^2-\frac{7}{2}px'\varphi'$$

$$-3y'\varphi'-2x'\tau'$$

$$+S_7y+T_7\eta+V_7\xi-\frac{5}{4}x'^2\varphi+\left\{\frac{27}{64\cdot 4}p^3+\left(\frac{9}{8}+\frac{1}{2}A+\frac{1}{2}A^2-2AB\right)p\right\}y^3$$

$$+\left(\frac{99}{64\cdot 4}p^2+2A-A^2-2AC\right)y^2\eta+\frac{27}{64}py^2\xi+\left(-\frac{51}{8}+A+\frac{1}{2}A^2\right)y^2x'y'+\frac{17}{8}x'y'^3,$$

$$S_7=\left\{\left(\frac{5}{16}A-\frac{15381}{64\cdot 40}\right)p^3+\frac{4213}{64\cdot 20}px'^2\right\}x'^2+\left\{\left(\frac{13}{8}A-\frac{79}{8}\right)p^2+3x'^2\right\}x'y'$$

$$+\left\{\frac{27}{64\cdot 4}p^3-\left(\frac{1}{2}A+\frac{27}{8}\right)p\right\}y'^2+\left(-\frac{63}{128}p^3-\frac{2079}{320}p\right)x'\eta'+\left(\frac{99}{128}p^2-2A\right)y'\eta'$$

$$+\left(-\frac{27}{64}p^2-\frac{3}{2}-A\right)x'\xi'+\frac{54}{64}py'\xi',$$

$$T_7=\left(-\frac{3399}{640}p^2+\frac{7}{8}x'^2\right)x'^2+\left(\frac{63}{128}p^3-\frac{2241}{320}p\right)x'y'-\frac{99}{64\cdot 4}p^2y'^2-\frac{1}{2}x'\eta'+\frac{27}{64}px'\xi',$$

$$V_7=-\frac{29}{8}px'^2+\left(\frac{27}{64}p^2-\frac{3}{2}\right)x'y'-\frac{27}{64}py'^2-\frac{27}{64}px'\eta'.$$

Here we put $A=3/2$, $B=1$, $C=1/2$. We remark the following facts: for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$\frac{3}{640}p^4\left(\eta+\frac{2}{3}py\right)\leqq\frac{15}{8}p^2x'^2\left(\eta+\frac{2}{3}py\right)=\frac{1}{96}120p^3x'^2y+\frac{1}{96}180p^2x'^2\eta,$$

$$\frac{1}{64\cdot 40}(247p^4+494p^3+1600p^2)xy$$

$$\leqq\frac{\alpha_1}{64\cdot 80}(247p^4+494p^3+1600p^2)x^2+\frac{1}{64\cdot 80}\frac{1}{\alpha_1}(247p^4+494p^3+1600p^2)y^2,$$

$$-\frac{5}{4}x'^2\varphi\leqq\frac{\gamma_1}{96}60\varphi^2+\frac{1}{96}\cdot\frac{1}{\gamma_1}60x'^4,$$

$$\frac{27}{64\cdot 4}p^3y^3+\left(\frac{99}{64\cdot 4}p^2-\frac{3}{4}\right)y^2\eta\leqq\frac{81}{64\cdot 8}p^2y^2\left(\eta+\frac{2}{3}py\right)\leqq 0.$$

Further by Lemma 1 for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$\frac{3}{4}p^4xy+\frac{3}{8}p^3x\eta=\frac{1}{8}p^3x\left\{\left(\eta+\frac{2}{3}py\right)+\left(2\eta+\frac{16}{3}py\right)\right\}\leqq\frac{1}{4}p^3x\eta+\frac{2}{3}p^4xy$$

$$\leqq\frac{1}{96}(6p^3x'^2+18p^3y'^2+30p^3\eta'^2+42p^3\xi'^2)\eta$$

$$+\frac{\beta_1}{96}(512x^2-1024x^3+768x^4-256x^5+32x^6)+\frac{1}{96}\cdot\frac{1}{\beta_1}32p^4y^2,$$

$$\frac{27}{64}py^2\xi\leqq\frac{\beta_2}{96}(-6.75p^3+27p)y^2+\frac{1}{96}\cdot\frac{1}{\beta_2}20.25p\xi^2,$$

$$-\frac{15}{4}y^2x'y'\leqq\frac{384.075}{96}px'^2y^2+\frac{\gamma_2}{96}(-7.4p^2+29.6)y^2+\frac{1}{96}\cdot\frac{1}{\gamma_2}(-7.4p^2+29.6)y'^2,$$

$$\frac{17}{8}x'y'^3\leqq\frac{1}{96}3.679py'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-(159x+163.02x^2+481.28x^3)/96$ we have, with $\alpha_1=2.4$, $\beta_1=2.4$, $\beta_2=0.5$, $\gamma_1=5.96$, $\gamma_2=1$,

$$\Re a_8\leqq 8-\frac{x^3}{96}\hat{P}_7(x)-\frac{1}{96}Q_7-\frac{1}{96}R_7-\frac{1}{96}S_7'y-\frac{1}{96}T_7'\eta-\frac{1}{96}V_7'\xi,$$

$$\hat{P}_7(x)=25.79-259.2075x+7.81875x^2+55.9975x^3-12.209x^4,$$

$$\begin{aligned}Q_7=&(2.53125p^5-15.26304p^4+169.93437p^3+163.86p^2-1571.505p\\&+1634.02-5.0625p^3x'^2)y^2\\&+2(9.28125p^4+28.275p^2-96p-9.28125p^2x'^2+198x'^2)y\eta\\&+(34.03125p^3+601.6p^2-2681.875p+2964.7)\eta^2\\&+2(10.125p^3-48p-10.125px'^2)y\xi+2(37.125p^2-144)\eta\xi\\&+(842.24p^2-3654.245p+4217.78)\xi^2,\end{aligned}$$

$$\begin{aligned}R_7=&(100.63125p^5+16.25625p^3+120.32p^2-522.035p+602.54\\&-195.99375p^3x'^2-10.08x'^2+40.66875px'^4)x'^2\\&+2(126.20625p^4+11.475p^2-316.89375p^2x'^2+33x'^4)x'y'\\&+(2.53125p^5+194.41875p^3+368.36p^2-1561.684p+1778.02\\&-5.0625p^3x'^2-497.925px'^2)y'^2\end{aligned}$$

$$+2(186.075p^3-182.025px'^2)x'\eta'$$
$$+2(9.28125p^4+244.275p^2-9.28125p^2x'^2-234x'^2)y'\eta'$$
$$+(34.03125p^3+601.6p^2-2369.875p+3012.7)\eta'^2+2(204p^2-66x'^2)x'\xi'$$
$$+2(10.125p^3+216p-10.125px'^2)y'\xi'+2(37.125p^2+192)\eta'\xi'$$
$$+(842.24p^2-3613.745p+4217.78)\xi'^2+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'$$
$$+(1082.88p^2-4698.315p+5422.86)\varphi'^2+2\cdot 96x'\tau'$$
$$+(1323.52p^2-5742.385p+6627.94)\tau'^2,$$

$$S_7'=(411.7875p^3-315.975px'^2)x'^2+2(357p^2-144x'^2)x'y'+(-10.125p^3+396p)y'^2$$
$$+2(23.625p^3+311.85p)x'\eta'+2(-37.125p^2+144)y'\eta'$$
$$+2(20.25p^2+144)x'\xi'-2\cdot 40.5py'\xi',$$

$$T_7'=(-6p^3+329.85p^2-84x'^2)x'^2+2(-23.625p^3+336.15p)x'y'$$
$$+(-18p^3+37.125p^2)y'^2+2\cdot 24x'\eta'-30p^3\eta'^2-2\cdot 20.25px'\xi'-42p^3\xi'^2,$$

$$V_7'=348px'^2-2(20.25p^2-72)x'y'+40.5py'^2+2\cdot 20.25px'\eta'.$$

Now, since $y\geqq 0$, $-2py/3\geqq\eta\geqq -py$, $\xi\geqq 0$ we have

$$-V_7'\xi\leqq -2\cdot 20.25px'\eta'\xi\leqq 20.25\gamma_3 p\xi^2+20.25\gamma_3^{-1}px'^2\eta'^2,$$

$$-T_7'\eta\leqq -T^*\eta-2(-23.625p^3+298.21p)x'y'\eta$$
$$\leqq -T^*\eta+(-23.625\beta_3p^3+298.21\beta_3p)\eta^2+(-23.625\beta_3^{-1}p^3+298.21\beta_3^{-1}p)x'^2y'^2,$$

$$T^*=(-4.84p^3+329.85p^2-84x'^2)x'^2+2\cdot 37.94px'y'+(-18p^3+37.125p^2)y'^2\geqq 0,$$

$$-(S_7'-pT^*)y\leqq -2\cdot 141.32p^2x'y'y-2(23.625p^3+311.85p)x'\eta'y$$
$$-2(-37.125p^2+144)y'\eta'y-2(20.25p^2+144)x'\xi'y+2\cdot 40.5py'\xi'y$$
$$\leqq 141.32\alpha_2p^2y^2+141.32\alpha_2^{-1}p^2x'^2y'^2+(23.625\alpha_3p^3+311.85\alpha_3p)y^2$$
$$+(23.625\alpha_3^{-1}p^3+311.85\alpha_3^{-1}p)x'^2\eta'^2+(37.125\alpha_4p^2-124.042\alpha_4)y^2$$
$$+(-12.375\alpha_4^{-1}p^4+90.847\alpha_4^{-1}p^2-165.388\alpha_4^{-1})\eta'^2$$
$$+(20.25\alpha_5p^2+144\alpha_5)y^2+(20.25\alpha_5^{-1}p^2+144\alpha_5^{-1})x'^2\xi'^2$$
$$+40.5\alpha_6py^2+(-13.5\alpha_6^{-1}p^3+54\alpha_6^{-1}p)\xi'^2.$$

Hence we have, putting $\alpha_2=\alpha_3=\alpha_4=\alpha_5=\alpha_6=0.16$, $\beta_3=0.4$, $\gamma_3=0.25$,

$$\Re a_8\leqq 8-\frac{x^3}{96}\hat{P}_7(x)-\frac{1}{96}\hat{Q}_7-\frac{1}{96}\hat{R}_7,$$

$$\hat{Q}_7=(2.53125p^5-15.26304p^4+166.15437p^3+132.0688p^2-1627.881p$$
$$+1630.82672-5.0625p^3x'^2)y^2$$
$$+2(9.28125p^4+28.275p^2-96p-9.28125p^2x'^2+198x'^2)y\eta$$
$$+(43.48125p^3+601.6p^2-2801.159p+2964.7)\eta^2$$
$$+2(10.125p^3-48p-10.125px'^2)y\xi+2(37.125p^2-144)\eta\xi$$
$$+(842.24p^2-3659.3075p+4217.78)\xi^2,$$

$$\hat{R}_7=(100.63125p^5+16.25625p^3+120.32p^2-522.035p+602.54$$
$$-195.99375p^3x'^2-10.08x'^2+40.66875px'^4)x'^2$$
$$+2(126.20625p^4+11.475p^2-316.89375p^2x'^2+33x'^4)x'y'$$
$$+(2.53125p^5+194.41875p^3+368.36p^2-1561.684p+1778.02$$
$$+54p^3x'^2-883.25p^2x'^2-1243.45px'^2)y'^2$$
$$+2(186.075p^3-182.025px'^2)x'\eta'$$
$$+2(9.28125p^4+244.275p^2-9.28125p^2x'^2-234x'^2)y'\eta'$$
$$+(77.34375p^4+34.03125p^3+33.80625p^2-2369.875p+4046.375$$
$$-147.657p^3x'^2-2030.0625px'^2)\eta'^2$$
$$+2(204p^2-66x'^2)x'\xi'+2(10.125p^3+216p-10.125px'^2)y'\xi'+2(37.125p^2+192)\eta'\xi'$$
$$+(84.375p^3+842.24p^2-3951.245p+4217.78-126.5625p^2x'^2-900x'^2)\xi'^2$$
$$+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'+(1082.88p^2-4698.315p+5422.86)\varphi'^2$$
$$+2\cdot 96x'\tau'+(1323.52p^2-5742.385p+6627.94)\tau'^2.$$

$\hat{P}_7(x)$ is monotone decreasing for $0\leqq x\leqq 0.1$ and $\hat{P}_7(0.1)>0$. Hence $\hat{P}_7(x)>0$ for $0\leqq x\leqq 0.1$.

By a large amount of calculation we can prove the positive definiteness of $\hat{Q}_7$ and $\hat{R}_7$ for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$.

Therefore in the present case we have $\Re a_8\leqq 8$ for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$ with equality holding only for $x=0$.

Case 4. $\eta\leqq -py$.

We start from (B) with $\alpha=0$. Applying Lemma 3 to the term $(9p^4/256)(\eta+23py/9)$ we have

$$\Re a_8\leqq U+\frac{3}{32}p-\frac{3}{32\cdot 64}p^7+\frac{121\cdot 25}{144\cdot 32}p^3-\frac{121\cdot 25}{576\cdot 32}p^5+\frac{3}{256}p^4(\eta+py)$$

$$+\frac{5}{48}A(p^4+2p^3+4p^2)xy-\frac{55}{256\cdot 4}(p^4+2p^3)xy+\frac{3}{4}Bp^4xy+\frac{3}{4}Cp^3x\eta$$

$$+\left\{-\frac{45}{64\cdot 32}p^5+\left(\frac{9767}{192\cdot 32}-\frac{5}{8}A-\frac{5}{4}B\right)p^3+\frac{1}{12}A^2x(p^2+2p+4)+B^2xp^2-\frac{9}{128}p\right.$$

$$\left.+\frac{45}{32\cdot 32}p^3x'^2+\left(-\frac{2715}{16\cdot 32}+\frac{1}{2}A+\frac{1}{4}A^2\right)px'^2\right\}y^2$$

$$+\left\{-\frac{185}{32\cdot 32}p^4+\left(\frac{1073}{256}-\frac{3}{4}A-2B-\frac{5}{4}C\right)p^2+2BCxp+\frac{185}{32\cdot 32}p^2x'^2\right.$$

$$\left.+\left(\frac{A}{2}-\frac{39}{8}\right)x'^2\right\}y\eta$$

$$+\left\{-\frac{6845}{32\cdot 576}p^3+\left(\frac{261}{128}-2C\right)p+C^2x\right\}\eta^2+\left\{-\frac{45}{256}p^3+\left(\frac{9}{2}-2B-A\right)p+\frac{45}{256}px'^2\right\}y\xi$$

$$+\left\{-\frac{185}{256}p^2+(4-2C)\right\}\eta\xi-\frac{45}{128}p\xi^2+(3-2A)y\varphi$$

$$+\left(-\frac{2145}{64\cdot 32}p^5-\frac{121\cdot 25}{576\cdot 32}p^3+\frac{4183}{64\cdot 32}p^3x'^2-\frac{867}{64\cdot 32}px'^4\right)x'^2$$

$$+\left(-\frac{2681}{32\cdot 32}p^4-\frac{55}{256}p^2+\frac{6747}{32\cdot 32}p^2x'^2-\frac{11}{16}x'^4\right)x'y'$$

$$+\left(-\frac{45}{64\cdot 32}p^5-\frac{12505}{32\cdot 192}p^3-\frac{9}{128}p+\frac{45}{32\cdot 32}p^3x'^2+\frac{2661}{16\cdot 32}px'^2\right)y'^2$$

$$+\left(-\frac{987}{8\cdot 32}p^3+\frac{969}{8\cdot 32}px'^2\right)x'\eta'+\left(-\frac{185}{32\cdot 32}p^4-\frac{1295}{8\cdot 32}p^2+\frac{185}{32\cdot 32}p^2x'^2+\frac{39}{8}x'^2\right)y'\eta'$$

$$+\left(-\frac{6845}{576\cdot 32}p^3-\frac{315}{4\cdot 32}p\right)\eta'^2+\left(-\frac{17}{4}p^2+\frac{11}{8}x'^2\right)x'\xi'+\left(-\frac{45}{32\cdot 8}p^3-\frac{9}{2}p\right.$$

$$\left.+\frac{45}{32\cdot 8}px'^2\right)y'\xi'$$

$$+\left(-\frac{185}{32\cdot 8}p^2-4\right)\eta'\xi'-\frac{45}{128}p\xi'^2-\frac{7}{2}px'\varphi'-3y'\varphi'-2x'\tau'$$

$$+S_8y+T_8\eta+V_8\xi-\frac{5}{4}x'^2\varphi+\left\{\frac{45}{16\cdot 32}p^3+\left(\frac{9}{8}+\frac{1}{2}A+\frac{1}{2}A^2-2AB\right)p\right\}y^3$$

$$+\left(\frac{185}{16\cdot 32}p^2+2A-A^2-2AC\right)y'^2\eta+\frac{45}{4\cdot 32}py^2\xi+\left(-\frac{51}{8}+A+\frac{1}{2}A^2\right)y^2x'y'+\frac{17}{8}x'y'^3,$$

$$S_8=\left\{\left(-\frac{6171}{32\cdot 32}+\frac{5}{16}A\right)p^3+\frac{1687}{16\cdot 32}px'^2\right\}x'^2+\left\{\left(\frac{13}{8}A-\frac{79}{8}\right)p^2+3x'^2\right\}x'y'$$

$$+\left\{\frac{45}{16\cdot 32}p^3-\left(\frac{1}{2}A+\frac{27}{8}\right)p\right\}y'^2+\left(-\frac{115}{8\cdot 32}p^3-\frac{837}{4\cdot 32}p\right)x'\eta'$$

$$+\left(\frac{185}{8\cdot 32}p^2-2A\right)y'\eta'+\left(-\frac{45}{4\cdot 32}p^2-\frac{3}{2}-A\right)x'\xi'+\frac{45}{64}py'\xi',$$

$$T_8=\left(-\frac{1365}{8\cdot 32}p^2+\frac{7}{8}x'^2\right)x'^2+\left(\frac{115}{8\cdot 32}p^3-\frac{891}{4\cdot 32}p\right)x'y'-\frac{185}{16\cdot 32}p^2y'^2-\frac{1}{2}x'\eta'+\frac{45}{4\cdot 32}px'\xi',$$

$$V_8=-\frac{29}{8}px'^2+\left(\frac{45}{4\cdot 32}p^2-\frac{3}{2}\right)x'y'-\frac{45}{4\cdot 32}py'^2-\frac{45}{4\cdot 32}px'\eta'.$$

Here we put $A=3/2$, $B=1$, $C=1/2$. We remark the following facts: for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$\frac{3}{256}p^4(\eta+py)\leqq\frac{75}{16}p^2x'^2(\eta+py)=\frac{1}{96}450p^3x'^2y+\frac{1}{96}450p^2x'^2\eta,$$

$$\frac{1}{256\cdot 4}(105p^4+210p^3+640p^2)xy\leqq\frac{\alpha_1}{256\cdot 8}(105p^4+210p^3+640p^2)x^2$$

$$+\frac{1}{256\cdot 8}\frac{1}{\alpha_1}(105p^4+210p^3+640p^2)y^2,$$

$$-\frac{5}{4}x'^2\varphi\leqq\frac{\gamma_1}{96}60\varphi^2+\frac{1}{96}\cdot\frac{1}{\gamma_1}60x'^4,$$

$$\frac{45}{16\cdot 32}p^3y^3+\left(\frac{185}{16\cdot 32}p^2-\frac{3}{4}\right)y^2\eta\leqq\frac{45}{16\cdot 32}p^2y^2(\eta+py)\leqq 0.$$

Further by Lemma 1 for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$\frac{3}{4}p^4xy+\frac{3}{8}p^3x\eta\leqq\frac{1}{4}p^3x\eta+\frac{5}{8}p^4xy$$

$$\leqq\frac{1}{96}(6p^3x'^2+18p^3y'^2+30p^3\eta'^2+42p^3\xi'^2)\eta$$

$$+\frac{\beta_1}{96}(480x^2-960x^3+720x^4-240x^5+30x^6)+\frac{1}{96}\cdot\frac{1}{\beta_1}30p^4y^2,$$

$$\frac{45}{4\cdot 32}py^2\xi\leqq\frac{\beta_2}{96}(-5.625p^3+22.5p)y^2+\frac{1}{96}\cdot\frac{1}{\beta_2}16.875p\xi^2,$$

$$-\frac{15}{4}y^2x'y'\leqq\frac{383.0625}{96}px'^2y^2+\frac{\gamma_2}{96}(-7.42p^2+29.68)y^2$$

$$+\frac{1}{96}\frac{1}{\gamma_2}(-7.42p^2+29.68)y'^2,$$

$$\frac{17}{8}x'y'^3 \leqq \frac{1}{96}\cdot 3.679py'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-(185.8x+149.48x^2+441.08x^3)/96$ we have, with $\alpha_1=2.5$, $\beta_1=2.4$, $\beta_2=0.5$, $\gamma_1=6.96$, $\gamma_2=1$,

$$\Re a_8 \leqq 8-\frac{x^3}{96}\hat{P}_8(x)-\frac{1}{96}Q_8-\frac{1}{96}R_8-\frac{1}{96}S_8'y-\frac{1}{96}T_8'\eta-\frac{1}{96}V_8'\xi,$$

$$\hat{P}_8(x)=21.41-212.78x-15.83x^2+59.21x^3-12.18x^4,$$

$$\begin{aligned}
Q_8=&(2.109375p^5-14.46875p^4+170.265625p^3+134.23p^2-1439.85p\\
&+1513.13-4.21875p^3x'^2)y^2\\
&+2(8.671875p^4+26.8125p^2-96p-8.671875p^2x'^2+198x'^2)y\eta\\
&+(35.6510416p^3+551.35p^2-2468p+2763.35)\eta^2\\
&+2(8.4375p^3-48p-8.4375px'^2)y\xi+2(34.6875p^2-144)\eta\xi\\
&+(771.89p^2-3349.15p+3935.89)\xi^2,
\end{aligned}$$

$$\begin{aligned}
R_8=&(100.546875p^5+15.7552083p^3+110.27p^2-478.45p+562.27\\
&-196.078125p^3x'^2-8.64x'^2+40.64px'^4)x'^2\\
&+2(125.671875p^4+10.3125p^2-316.265625p^2x'^2+33x'^4)x'y'\\
&+(2.109375p^5+195.390625p^3+338.23p^2-1432.279p+1657.13\\
&-4.21875p^3x'^2-498.9375px'^2)y'^2\\
&+2(185.0625p^3-181.6875px'^2)x'\eta'\\
&+2(8.671875p^4+242.8125p^2-8.671875p^2x'^2-234x'^2)y'\eta'\\
&+(35.6510416p^3+551.35p^2-2156p+2811.35)\eta'^2+2(204p^2-66x'^2)x'\xi'\\
&+2(8.4375p^3+216p-8.4375px'^2)y'\xi'+2(34.6875p^2+192)\eta'\xi'\\
&+(771.89p^2-3315.4p+3935.89)\xi'^2+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'\\
&+(992.43p^2-4306.05p+5060.43)\varphi'^2+2\cdot 96x'\tau'\\
&+(1212.97p^2-5262.95p+6184.97)\tau'^2,
\end{aligned}$$

$$\begin{aligned}
S_8'=&(83.53125p^3-316.3125px'^2)x'^2+2(357p^2-144x'^2)x'y'\\
&+(-8.4375p^3+396p)y'^2+2(21.5625p^3+313.875p)x'\eta'\\
&+2(-34.6875p^2+144)y'\eta'+2(16.875p^2+144)x'\xi'-2\cdot 33.75py'\xi'.
\end{aligned}$$

$$T_8'=(-6p^3+61.875p^2-84x'^2)x'^2+2(-21.5625p^3+334.125p)x'y'$$
$$+(-18p^3+34.6875p^2)y'^2+2\cdot 24x'\eta'-30p^3\eta'^2-2\cdot 16.875px'\xi'-42p^3\xi'^2,$$
$$V_8'=348px'^2+2(-16.875p^2+72)x'y'+33.75py'^2+2\cdot 16.875px'\eta'.$$

Further we remark the following facts: for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$

$$-S_8'y\leqq -2\cdot 183.88p^2x'y'y-2(21.5625p^3+313.875p)x'\eta'y$$
$$-2(-34.6875p^2+144)y'\eta'y-2(16.875p^2+144)x'\xi'y+2\cdot 33.75py'\xi'y$$
$$\leqq 183.88\alpha_2p^2y^2+183.88\alpha_2^{-1}p^2x'^2y'^2+(21.5625\alpha_3p^3+313.875\alpha_3p)y^2$$
$$+(21.5625\alpha_3^{-1}p^3+313.875\alpha_3^{-1}p)x'^2\eta'^2+(-34.6875\alpha_4p^2+144\alpha_4)y^2$$
$$+(11.5625\alpha_4^{-1}p^4-94.25\alpha_4^{-1}p^2+192\alpha_4^{-1})\eta'^2+(16.875\alpha_5p^2+144\alpha_5)y^2$$
$$+(16.875\alpha_5^{-1}p^2+144\alpha_5^{-1})x'^2\xi'^2+33.75\alpha_6py^2+(-11.25\alpha_6^{-1}p^3+45\alpha_6^{-1}p)\xi'^2,$$
$$-T_8'\eta\leqq -(52.239p^2-84x'^2)x'^2\eta-2(-21.5625p^3+334.125p)x'y'\eta-0.4875p^2y'^2\eta$$
$$\leqq(26.1195\beta_3p^2-42\beta_3x'^2)\eta^2+(26.1195\beta_3^{-1}p^2-42\beta_3^{-1}x'^2)x'^4$$
$$+(-21.5625\beta_4p^3+334.125\beta_4p)\eta^2+(-21.5625\beta_4^{-1}p^3+334.125\beta_4^{-1}p)x'^2y'^2$$
$$+0.24375\beta_5p^2\eta^2+(-0.08125\beta_5^{-1}p^4+0.325\beta_5^{-1}p^2)y'^2,$$
$$-V_8'\xi\leqq -2\cdot 16.875px'\eta'\xi$$
$$\leqq 16.875\gamma_3p\xi^2+16.875\gamma_3^{-1}px'^2\eta'^2.$$

Hence we have, putting $\alpha_2=\alpha_3=\alpha_4=\alpha_5=\alpha_6=0.168$, $\beta_3=\beta_4=\beta_5=0.25$, $\gamma_3=0.2$,

$$\Re a_8\leqq 8-\frac{x^3}{96}\hat{P}_8(x)-\frac{1}{96}\hat{Q}_8-\frac{1}{96}\hat{R}_8,$$
$$\hat{Q}_8=(2.098828p^5-14.46875p^4+166.643125p^3+106.33066p^2-1498.251p+1464.746)y^2$$
$$+2(8.671875p^4+26.8125p^2-96p-8.671875p^2x'^2+198x'^2)y\eta$$
$$+(41.0416p^3+544.7591p^2-2551.53125p+2763.35)\eta^2$$
$$+2(8.4375p^3-48p-8.4375px'^2)y\xi+2(34.6875p^2-144)\eta\xi$$
$$+(771.89p^2-3352.525p+3935.89)\xi^2,$$
$$\hat{R}_8=(100.546875p^5+15.7552083p^3+110.27p^2-478.45p+562.27$$
$$-196.078125p^3x'^2-104.478p^2x'^2-8.64x'^2)x'^2$$
$$+2(125.671875p^4+10.3125p^2-316.265625p^2x'^2+33x'^4)x'y'$$

$$+(2.109375p^5+0.325p^4+195.390625p^3+336.93p^2-1432.279p$$

$$+1657.13+82.03125p^3x'^2-1094.63764p^2x'^2-1835.4375px'^2)y'^2$$

$$+2(185.0625p^3-181.6875px'^2)x'\eta'$$

$$+2(8.671875p^4+242.8125p^2-8.671875p^2x'^2-234x'^2)y'\eta'$$

$$+(-68.8316p^4+35.6510416p^3+1112.4202p^2-2156p+1668.374$$

$$-128.3616p^3x'^2-1952.873px'^2)\eta'^2$$

$$+2(204p^2-66x'^2)x'\xi'+2(8.4375p^3+216p-8.4375px'^2)y'\xi'+2(34.6875p^2+192)\eta'\xi'$$

$$+(66.97125p^3+771.89p^2-3583.285p+3935.89-100.4569p^2x'^2-857.232x'^2)\xi'^2$$

$$+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'+(992.43p^2-4306.05p+5060.43)\varphi'^2+2\cdot 96x'\tau'$$

$$+(1212.97p^2-5262.95p+6184.97)\tau'^2.$$

$\hat{P}_8(x)$ is monotone decreasing for $0\leqq x\leqq 0.1$ and $\hat{P}_8(0.1)>0$. Hence $\hat{P}_8(x)>0$ for $0\leqq x\leqq 0.1$.

By a large amount of calculation we can prove the positive definiteness of $\hat{Q}_8$ and $\hat{R}_8$ for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$.

Therefore we have $\Re a_8\leqq 8$ for $1.9\leqq p\leqq 2$, $|x'/p|\leqq 1/20$ in the present case with equality holding only for $x=0$.

§ **4.** In this section we shall be concerned with the case $y\leqq 0$ and $1.8\leqq p\leqq 2$, $|x'/p|\leqq 1/10$. We divide this case into several subcases.

Case 1. $\eta\geqq 0$, $\xi\leqq 0$.

In this case we put

$$\delta=\frac{5}{8}p^2+Ay,\qquad \gamma=\frac{3}{8}p^3+Bpy+C\eta+G\xi$$

in (A). Then we have

$$\Re a_8\leqq U+\frac{13}{128}p^5y+\frac{3}{64}p^4\eta+\frac{3}{4}Bp^4xy+\frac{3}{4}Cp^3x\eta$$

$$+\frac{5}{48}A(p^4+2p^3+4p^2)xy+\frac{3}{4}Gp^3x\xi$$

$$+\left\{\left(\frac{29}{16}-\frac{5}{8}A-\frac{5}{4}B\right)p^3+\frac{1}{12}A^2x(p^2+2p+4)+B^2xp^2\right.$$

$$\left.+\left(-\frac{21}{4}+\frac{1}{2}A+\frac{1}{4}A^2\right)px'^2\right\}y^2$$

$$+\left\{\left(\frac{37}{8}-\frac{3}{4}A-2B-\frac{5}{4}C\right)p^2+2BCxp+\left(\frac{A}{2}-\frac{39}{8}\right)x'^2\right\}y\eta$$

$$+\left\{\left(\frac{9}{4}-2C\right)p+C^2x\right\}\eta^2+\left\{\left(\frac{9}{2}-2B-A\right)p+2BGxp-\frac{5}{4}Gp^2\right\}y\xi$$

$$+(4-2C+2CGx-2Gp)\eta\xi+(3-2A)y\varphi+(G^2x-2G)\xi^2$$

$$+\left(-\frac{267}{64\cdot 4}p^5+\frac{131}{64}p^3x'^2-\frac{27}{64}px'^4\right)x'^2+\left(-\frac{319}{64\cdot 2}p^4+\frac{13}{2}p^2x'^2-\frac{11}{16}x'^4\right)x'y'$$

$$+\left(-\frac{29}{16}p^3+\frac{21}{3}px'^2\right)y'^2+\left(-\frac{15}{4}p^3+\frac{15}{4}px'^2\right)x'\eta'+\left(-\frac{37}{8}p^2+\frac{39}{8}x'^2\right)y'\eta'$$

$$-\frac{9}{4}p\eta'^2+\left(-\frac{17}{4}p^2+\frac{11}{8}x'^2\right)x'\xi'-\frac{9}{2}py'\xi'-4\eta'\xi'-\frac{7}{2}px'\varphi'-3y'\varphi'-2x'\tau'$$

$$+\left[\left\{\left(-\frac{99}{16}+\frac{5}{16}A\right)p^3+\frac{53}{16}px'^2\right\}x'^2+\left\{\left(-\frac{79}{8}+\frac{13}{8}A\right)p^2+3x'^2\right\}x'y'\right.$$

$$\left.+\left(-\frac{27}{8}-\frac{1}{2}A\right)py'^2-\frac{27}{4}px'\eta'-2Ay'\eta'-\left(\frac{3}{2}+A\right)x'\xi'\right]y$$

$$+\left[\left(-\frac{87}{16}p^2+\frac{7}{8}x'^2\right)x'^2-\frac{27}{4}px'y'-\frac{1}{2}x'\eta'\right]\eta$$

$$+\left(-\frac{29}{8}px'^2-\frac{3}{2}x'y'\right)\xi-\frac{5}{4}x'^2\varphi-2AGy^2\xi+\left(\frac{9}{8}+\frac{1}{2}A+\frac{1}{2}A^2-2AB\right)py^3$$

$$+(2A-A^2-2AC)y^2\eta+\left(-\frac{51}{8}+A+\frac{1}{2}A^2\right)y^2x'y'+\frac{17}{8}x'y'^3.$$

Here we put $A=3/2$, $B=3/2$, $C=1$, $G=2$. By Lemma 1 and Lemma 2, for $1.8\leqq p\leqq 2$, $|x'/p|\leqq 1/10$

$$\frac{13}{128}p^5y+\frac{3}{64}p^4\eta=\frac{3}{64}p^4\left(\eta-\frac{1}{2}py\right)+\frac{1}{8}p^5y$$

$$\leqq-\frac{1}{64\cdot 4}p^7+\frac{1}{32}p^4+\frac{3}{64\cdot 4}p^5x'^2-\frac{3}{128}p^4x'y'+\frac{25}{2}p^3x'^2y,$$

$$\frac{9}{8}p^4xy+\frac{3}{4}p^3x\eta=\frac{3}{4}p^3x\left(\eta-\frac{1}{2}py\right)+\frac{3}{2}p^4xy$$

$$\leqq\frac{1}{2}xp^3-\frac{1}{16}xp^6+\frac{3}{16}xp^4x'^2-\frac{3}{8}xp^3x'y'$$

$$+\frac{3}{8}p^4(x'^2+3y'^2+5\eta'^2+7\xi'^2+3y^2)y,$$

$$\frac{5}{32}(p^4+2p^3+4p^2)xy \leqq \frac{5}{128}(p^4+2p^3+4p^2)(x'^2+3y'^2+5\eta'^2+7\xi'^2)y,$$

$$\frac{3}{2}p^3x\xi \leqq \frac{3}{8}p^3(x'^2+3y'^2+3y^2)\xi,$$

$$-\frac{63}{16}px'^2y^2-\frac{15}{4}x'y'y^2 \leqq \frac{7.94}{16}y'^2y^2 \leqq \frac{\gamma_2}{96}7.94(4-p^2)y^2+\frac{1}{96}\cdot\frac{1}{\gamma_2}7.94(4-p^2)y'^2,$$

$$\frac{17}{8}x'y'^3 \leqq \frac{\delta_1}{16}17x'^2y'^2+\frac{1}{48}\cdot\frac{1}{\delta_1}17(4-p^2)y'^2.$$

Further we remark the following facts:

$$-\frac{9}{4}y^2\eta \leqq 0,$$

$$-\frac{5}{4}x'^2\varphi \leqq \frac{5\gamma_1}{8}\varphi^2+\frac{5}{8}\frac{1}{\gamma_1}x'^4.$$

Making use of these remarks we have, with $\gamma_1=25.2$, $\gamma_2=20$, $\delta_1=1$

$$\Re a_8 \leqq 8-\frac{x}{96}P_1^*(x)-\frac{1}{96}Q_1^*-\frac{1}{96}R_1^*+S_1^*y+T_1^*\eta+V_1^*\xi$$

$$+\left(\frac{9}{8}p^3-6\right)y^2\xi+\left(\frac{9}{8}p^4-\frac{3}{2}p\right)y^3,$$

$$P_1^*(x)=672+576x-1256x^2+916x^3-358.8x^4+74.8x^5-6.41429x^6,$$

$$Q_1^*=(330p^3-273.2p^2-779.2)y^2+2(180p^2-288p+198x'^2)y\eta$$
$$+(72p-192)\eta^2+2(408p^2-576p)y\xi+2(384p-480)\eta\xi+(384p-384)\xi^2-1512\varphi^2,$$

$$R_1^*=(117p^5-36p^4-196.5p^3x'^2-2.4x'^2+40.5px'^4)x'^2$$
$$+2(102.75p^4+36p^3-312p^2x'^2+33x'^4)x'y'$$
$$+(174p^3+34.397p^2-137.588-504px'^2-102x'^2)y'^2$$
$$+2(180p^3-180px'^2)x'\eta'+2(222p^2-234x'^2)y'\eta'+216p\eta'^2+2(204p^2-66x'^2)x'\xi'$$
$$+2\cdot 216py'\xi'+2\cdot 192\eta'\xi'+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'+2\cdot 96x'\tau',$$

$$S_1^*=\left(\frac{53}{128}p^4+\frac{439}{64}p^3+\frac{5}{32}p^2+\frac{53}{16}px'^2\right)x'^2+\left(-\frac{119}{16}p^2+3x'^2\right)x'y'$$
$$+\left(\frac{159}{128}p^4+\frac{15}{64}p^3+\frac{15}{32}p^2-\frac{33}{8}p\right)y'^2-\frac{27}{4}px'\eta'-3y'\eta'$$
$$+\left(\frac{265}{128}p^4+\frac{25}{64}p^3+\frac{25}{32}p^2\right)\eta'^2$$

$$-3x'\xi'+\left(\frac{371}{128}p^4+\frac{35}{64}p^3+\frac{35}{32}p^2\right)\xi'^2,$$

$$T_1^*=\left(-\frac{87}{16}p^2+\frac{7}{8}x'^2\right)x'^2-\frac{27}{4}px'y'-\frac{1}{2}x'\eta',$$

$$V_1^*=\left(\frac{3}{8}p^3-\frac{29}{8}p\right)x'^2-\frac{3}{2}x'y'+\frac{9}{8}p^3y'^2.$$

Since $y\leqq 0$, $\eta\geqq 0$, $\xi\leqq 0$ we have

$$S_1^*y\leqq 0,$$

$$T_1^*\eta\leqq\frac{7}{8}x'^4\eta+\frac{33.52}{16}y'^2\eta-\frac{1}{2}x'\eta'\eta$$

$$\leqq\frac{7\alpha_1}{16}x'^2\eta^2+\frac{7}{16\alpha_1}x'^6+\frac{33.52\alpha_2}{32}\eta^2+\frac{33.52}{96\alpha_2}(4-p^2)y'^2+\frac{\alpha_3}{4}\eta^2+\frac{1}{4\alpha_3}x'^2\eta'^2,$$

$$V_1^*\xi\leqq\left(\frac{3}{8}p^3-\frac{29.382}{8}p\right)x'^2\xi$$

$$\leqq\left(-\frac{3\beta_1}{16}p^3+\frac{29.382\beta_1}{16}p\right)\xi^2+\left(-\frac{3}{16\beta_1}p^3+\frac{29.382}{16\beta_1}p\right)x'^4$$

and

$$\left(\frac{9}{8}p^3-6\right)y^2\xi\leqq 0,$$

$$\left(\frac{9}{8}p^4-\frac{3}{2}p\right)y^3\leqq 0.$$

Hence, applying Lemma 1 to the term $(-672x+552x^2)/96$ we have, putting $\alpha_1=1$, $\alpha_2=4$, $\alpha_3=2$, $\beta_1=2$,

$$\Re a_8\leqq 8-\frac{x^2}{96}\tilde{P}_1(x)-\frac{1}{96}\tilde{Q}_1-\frac{1}{96}\tilde{R}_1,$$

$$\tilde{P}_1(x)=192-1118x+916x^2-358.8x^3+74.8x^4-6.5x^5,$$

$$\tilde{Q}_1=(330p^3-273.2p^2-414p+552.8)y^2+2(180p^2-288p+198x'^2)y\eta$$

$$+(-618p+1577.76-42x'^2)\eta^2+2(408p^2-576p)y\xi+2(384p-480)\eta\xi$$

$$+(36p^3-934.584p+2724)\xi^2+(-1242p+2484)\varphi^2,$$

$$\tilde{R}_1=(117p^5-36p^4-138p+444-187.5p^3x'^2-88.146px'^2-2.4x'^2+40.5px'^4$$

$$-42x'^4)x'^2+2(102.75p^4+36p^3-312p^2x'^2+33x'^4)x'y'$$

$$+(174p^3+42.777p^2-414p+1160.892-504px'^2-102x'^2)y'^2$$

$$+2(180p^3-180px'^2)x'\eta'+2(222p^2-234x'^2)y'\eta'+(-474p+2220-12x'^2)\eta'^2$$

$$+2(204p^2-66x'^2)x'\xi'+2\cdot 216py'\xi'+2\cdot 192\eta'\xi'+(-966p+3108)\xi'^2$$

$$+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'+(-1242p+3996)\varphi'^2+2\cdot 96x'\tau'+(-1518p+4884)\tau'^2.$$

It is easy to prove that $\tilde{P}_1(x)>0$ for $0\leqq x\leqq 0.2$. Since $(408p^2-576p)y\xi\geqq 0$ we may consider $\tilde{Q}_1^1=\tilde{Q}_1-2(408p^2-576p)y\xi$. We can prove the positive definiteness of the symmetric matrix associated with $\tilde{Q}_1^1$ for $1.8\leqq p\leqq 2$, $|x/p|\leqq 1/10$ by taking its principal diagonal minor determinants. Hence $\tilde{Q}_1$ is non-negative there.

Since $5x'^2+2\cdot 96x'\tau'+(-1518p+4884)\tau'^2\geqq 0$, $38y'^2+2\cdot 144y'\varphi'+550\varphi'^2\geqq 0$, $59px'^2+2\cdot 168px'\varphi'+(-1242p+3446)\varphi'^2\geqq 0$ and $192\eta'^2+2\cdot 192\eta'\xi'+192\xi'^2\geqq 0$ we may consider $\tilde{R}_1^1=\tilde{R}_1-\{(59p+5)x'^2+38y'^2+192\eta'^2+2\cdot 192\eta'\xi'+192\xi'^2+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'+(-1242p+3996)\varphi'^2+2\cdot 96x'\tau'+(-1518p+4884)\tau'^2\}$. It is not so difficult to prove the positive definiteness of $\tilde{R}_1^1$ for $1.8\leqq p\leqq 2$, $|x'/p|\leqq 1/10$. Hence $\tilde{R}_1$ is positive definite there.

Therefore we have $\Re a_8\leqq 8$ for $1.8\leqq p\leqq 2$, $|x'/p|\leqq 1/10$ with equality holding only for $x=0$ in the present case.

Case 2. $\eta\geqq 0$, $\xi\geqq 0$.

We put

$$\delta=\frac{5}{8}p^2+Ay,\qquad \gamma=\frac{3}{8}p^3+Bpy+C\eta+Dy'^2+E\eta'^2$$

in (A). Then we have

$$\Re a_8\leqq U+\left(\frac{3}{64}p^4-\frac{1}{2}\mathfrak{A}p^2x'^2\right)\left(\eta+\frac{13}{6}py\right)+\frac{3}{4}Bp^4xy+\frac{3}{4}Cp^3x\eta+\frac{5}{48}A(p^4+2p^3+4p^2)xy$$

$$+\left\{\left(\frac{29}{16}-\frac{5}{8}A-\frac{5}{4}B\right)p^3+\frac{1}{12}A^2x(p^2+2p+4)+B^2xp^2\right.$$

$$\left.+\left(-\frac{21}{4}+\frac{1}{2}A+\frac{1}{4}A^2\right)px'^2\right\}y^2$$

$$+\left\{\left(\frac{37}{8}-\frac{3}{4}A-2B-\frac{5}{4}C\right)p^2+2BCxp+\left(\frac{A}{2}-\frac{39}{8}\right)x'^2\right\}y\eta$$

$$+\left\{\left(\frac{9}{4}-2C\right)p+C^2x\right\}\eta^2+\left(\frac{9}{2}-2B-A\right)py\xi+(4-2C)\eta\xi+(3-2A)y\varphi$$

$$+\left(-\frac{267}{64\cdot 4}p^5+\frac{131}{64}p^3x'^2-\frac{27}{64}px'^4\right)x'^2+\left(-\frac{319}{64\cdot 2}p^4+\frac{13}{2}p^2x'^2-\frac{11}{16}x'^4\right)x'y'$$

$$+\left(\frac{3}{4}Dxp^3-\frac{29}{16}p^3+\frac{21}{4}px'^2\right)y'^2+\left(-\frac{15}{4}p^3+\frac{15}{4}px'^2\right)x'\eta'$$

$$+\left(-\frac{37}{8}p^2+\frac{39}{8}x'^2\right)y'\eta'$$

$$+\left(\frac{3}{4}Exp^3-\frac{9}{4}p\right)\eta'^2+\left(-\frac{17}{4}p^2+\frac{11}{8}x'^2\right)x'\xi'-\frac{9}{2}py'\xi'-4\eta'\xi'-\frac{7}{2}px'\varphi'$$

$$\text{(C)}\qquad -3y'\varphi'-2x'\tau'$$

$$+\left[\left\{\left(\frac{13}{12}\mathfrak{A}-\frac{99}{16}+\frac{5}{16}A\right)p^3+\frac{53}{16}px'^2\right\}x'^2+\left\{\left(-\frac{79}{8}+\frac{13}{8}A\right)p^2+3x'^2\right\}x'y'\right.$$

$$+\left\{-\frac{5}{4}Dp^2+2BDxp+\left(-\frac{27}{8}-\frac{1}{2}A\right)p\right\}y'^2-\frac{27}{4}px'\eta'-2Ay'\eta'$$

$$\left.+\left(-\frac{5}{4}Ep^2+2BExp\right)\eta'^2-\left(\frac{3}{2}+A\right)x'\xi'\right]y$$

$$+\left[\left\{\left(\frac{1}{2}\mathfrak{A}-\frac{87}{16}\right)p^2+\frac{7}{8}x'^2\right\}x'^2-\frac{27}{4}px'y'+(-2Dp+2CDx)y'^2-\frac{1}{2}x'\eta'\right.$$

$$\left.+(-2Ep+2CEx)\eta'^2\right]\eta$$

$$+\left(-\frac{29}{8}px'^2-\frac{3}{2}x'y'-2Dy'^2-2E\eta'^2\right)\xi-\frac{5}{4}x'^2\varphi$$

$$+\left(\frac{9}{8}+\frac{1}{2}A+\frac{1}{2}A^2-2AB\right)py^3+(2A-A^2-2AC)y^2\eta-2ADy^2y'^2$$

$$-2AEy^2\eta'^2+\left(-\frac{51}{8}+A+\frac{1}{2}A^2\right)y^2x'y'+\frac{17}{8}x'y'^3+x(Dy'^2+E\eta'^2)^2.$$

Here we put $A=3/2$, $B=3/2$, $C=1$, $D=1/2$, $E=1/8$, $\mathfrak{A}=6$. By Lemma 1 and Lemma 2 we have

$$\left(\frac{3}{64}p^4-3p^2x'^2\right)\left(\eta+\frac{13}{6}py\right)=\left(\frac{3}{64}p^4-3p^2x'^2\right)\left(\eta-\frac{1}{2}py\right)+\left(\frac{1}{8}p^4-8p^2x'^2\right)py$$

$$\leqq\frac{1}{32}p^4-\frac{1}{64\cdot 4}p^7+\left(\frac{67}{64\cdot 4}p^5-2p^2-\frac{3}{4}p^3x'^2\right)x'^2+\left(-\frac{3}{128}p^4+\frac{3}{2}p^2x'^2\right)x'y'$$

$$+\left(\frac{25}{2}p^3-800px'^2\right)x'^2y,$$

$$\frac{3}{4}p^3x\eta+\frac{9}{8}p^4xy=\frac{3}{4}p^3x\left(\eta-\frac{1}{2}py\right)+\frac{3}{2}p^4xy$$

$$\leqq\frac{1}{2}xp^3-\frac{1}{16}xp^6+\frac{3}{16}xp^4x'^2-\frac{3}{8}xp^3x'y'$$

$$+\frac{3}{8}p^4(x'^2+3y'^2+5\eta'^2+7\xi'^2+3y^2)y,$$

$$\frac{5}{32}(p^4+2p^3+4p^2)xy\leqq\frac{5}{128}(p^4+2p^3+4p^2)(x'^2+3y'^2+5\eta'^2+7\xi'^2+3y^2)y,$$

$$\frac{17}{8}x'y'^3\leqq\frac{17\delta_1}{16}x'^2y'^2+\frac{17}{48\delta_1}(4-p^2)y'^2,$$

$$x\left(\frac{1}{2}y'^2+\frac{1}{8}\eta'^2\right)^2=\frac{x}{4}\left(y'^4+\frac{1}{2}y'^2\eta'^2+\frac{1}{16}\eta'^4\right)$$

$$\leqq\frac{1}{12}(p^3-2p^2-4p+8)y'^2+\frac{1}{960}(43p^3-86p^2-172p+344)\eta'^2$$

for $1.8\leqq p\leqq 2$, $|x'/p|\leqq 1/10$.
Further we remark the following facts:

$$-\frac{5}{4}x'^2\varphi\leqq\frac{5\gamma_1}{8}\varphi^2+\frac{5}{8\gamma_1}x'^4,$$

$$-\frac{9}{4}y^2\eta\leqq 0,$$

$$-\frac{21}{16}px'^2y^2-\frac{15}{4}y^2x'y'-\frac{3}{2}y^2y'^2-\frac{3}{8}y^2\eta'^2\leqq 0,$$

$$\left(-\frac{29}{8}px'^2-\frac{3}{2}x'y'-y'^2-\frac{1}{4}\eta'^2\right)\xi\leqq 0,$$

$$\left\{\left(-\frac{39}{16}p^2+\frac{7}{8}x'^2\right)x'^2-\frac{27}{4}px'y'+(-2p+2)y'^2-\frac{1}{2}x'\eta'+\left(-\frac{1}{2}p+\frac{1}{2}\right)\eta'^2\right\}\eta$$

$$\leqq-\frac{23}{8}px'y'\eta\leqq\frac{23\beta_1}{16}p\eta^2+\frac{23}{16\beta_1}px'^2y'^2$$

for $1.8\leqq p\leqq 2$, $|x'/p|\leqq 1/10$.

Hence, applying Lemma 1 to the term $-(672x+552x^2)/96$ we have, with $\beta_1=2.774$, $\gamma_1=25.2$, $\delta_1=2$,

$$\Re a_8\leqq 8-\frac{x^2}{96}\tilde{P}_2(x)-\frac{1}{96}\tilde{Q}_2-\frac{1}{96}\tilde{R}_2+\tilde{S}_2y+\left(\frac{159}{128}p^4+\frac{15}{64}p^3+\frac{15}{32}p^2-\frac{3}{2}p\right)y^3,$$

$$\tilde{P}_2(x)=\tilde{P}_1(x),$$

$$\tilde{Q}_2=(330p^3-432p^2-414p+1188+252px'^2)y^2+2(180p^2-288p+198x'^2)y\eta$$

$$+(-1000.812p+2028)\eta^2-2\cdot 96\eta\xi+(-966p+3108)\xi^2,$$

$$\tilde{R}_2=(93p^5-36p^4+192p^2-138p+444-124.5p^3x'^2-2.4x'^2+40.5px'^4)x'^2$$

$$+2(102.75p^4+36p^3-384p^2x'^2+33x'^4)x'y'$$

$$+(36p^4+94p^3+33p^2-382p+1200-553.75px'^2-204x'^2)y'^2$$

$$+2(180p^3-180px'^2)x'\eta'+2(222p^2-234x'^2)y'\eta'$$

$$+(9p^4-22.3p^3+8.6p^2-456.8p+2185.6)\eta'^2+2(204p^2-66x'^2)x'\xi'$$

$$+2\cdot 216py'\xi'+2\cdot 192\eta'\xi'+(-966p+3108)\xi'^2+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'$$

$$+(-1242p+3996)\varphi'^2+2\cdot 96x'\tau'+(-1518p+4884)\tau'^2,$$

$$\tilde{S}_2=\left(\frac{53}{128}p^4+\frac{855}{64}p^3+\frac{5}{32}p^2-\frac{12747}{16}px'^2\right)x'^2+\left(-\frac{119}{16}p^2+3x'^2\right)x'y'$$

$$+\left(\frac{159}{128}p^4+\frac{15}{64}p^3-\frac{53}{32}p^2-\frac{9}{8}p\right)y'^2-\frac{27}{4}px'\eta'-3y'\eta'$$

$$+\left(\frac{265}{128}p^4+\frac{25}{64}p^3+\frac{1}{4}p^2+\frac{3}{4}p\right)\eta'^2-3x'\xi'+\left(\frac{371}{128}p^4+\frac{35}{64}p^3+\frac{35}{32}p^2\right)\xi'^2.$$

Since $y\leqq 0$ we have

$$\tilde{S}_2 y\leqq 0,$$

$$\left(\frac{159}{128}p^4+\frac{15}{64}p^3+\frac{15}{32}p^2-\frac{3}{2}p\right)y^3\leqq 0$$

for $1.8\leqq p\leqq 2$, $|x'/p|\leqq 1/10$.

It is very easy to prove that $\tilde{Q}_2$ is positive definite for $1.8\leqq p\leqq 2$, $|x'p|\leqq 1/10$.

As in Case 1 we may consider $\tilde{R}_2^1=\tilde{R}_2-\{(59p+5)x'^2+38y'^2+192\eta'^2+2\cdot 192\eta'\xi'+192\xi'^2+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'+(-1242p+3996)\varphi'^2+2\cdot 96x'\tau'+(-1518p+4884)\tau'^2\}$. It is not so difficult to prove the positive definiteness of $\tilde{R}_2^1$ for $1.8\leqq p\leqq 2$, $|x'/p|\leqq 1/10$.

Therefore in the present case we have $\Re a_8\leqq 8$ for $1.8\leqq p\leqq 2$, $|x'/p|\leqq 1/10$ with equality holding only for $x=0$.

Case 3. $\eta\leqq 0$.

We start from (C) with $A=3$, $B=3/4$, $C=2$, $D=0$, $E=0$, $\mathfrak{A}=9$. By Lemma 1 we have

$$\frac{9}{16}p^4xy\leqq\frac{9}{64}p^4(x'^2+3y'^2+5\eta'^2+7\xi'^2+3y^2+5\eta^2)y,$$

$$\frac{3}{2}p^3x\eta\leqq\frac{3}{8}p^3(x'^2+3y'^2+5\eta'^2+3y^2)\eta,$$

$$\frac{5}{16}(p^4+2p^3+4p^2)xy\leqq\frac{5}{64}(p^4+2p^3+4p^2)(x'^2+3y'^2+5\eta'^2+7\xi'^2+3y^2+5\eta^2)y,$$

$$-\frac{3}{2}px'^2y^2+\frac{9}{8}y^2x'y'\leqq\frac{0.47}{4}y^2y'^2\leqq\frac{0.47\delta_1}{24}(4-p^2)y^2+\frac{0.47}{24\delta_1}(4-p^2)y'^2,$$

$$\frac{17}{8}x'y'^3\leqq\frac{17}{16}\delta_2x'^2y'^2+\frac{17}{48\delta_2}(4-p^2)y'^2$$

for $1.8\leqq p\leqq 2$. Further we remark the following facts: for $1.8\leqq p\leqq 2$, $|x'/p|\leqq 1/10$

$$\left(\frac{13}{128}p^5-\frac{39}{4}p^3x'^2\right)y\leqq\left(\frac{325}{32}p^3-975px'^2\right)x'^2y,$$

$$\left(\frac{3}{64}p^4-\frac{9}{2}p^2x'^2\right)\eta\leqq\left(\frac{75}{16}p^2-450x'^2\right)x'^2\eta,$$

$$\left\{\left(\frac{7}{32}p^4+\frac{237}{16}p^3+\frac{5}{16}p^2-\frac{15547}{16}px'^2\right)x'^2+(-5p^2+3x'^2)x'y'\right.$$

$$+\left(\frac{21}{32}p^4+\frac{15}{32}p^3+\frac{15}{16}p^2-\frac{39}{8}p\right)y'^2-\frac{27}{4}px'\eta'-6y'\eta'+\left(\frac{35}{32}p^4+\frac{25}{32}p^3+\frac{25}{16}p^2\right)\eta'^2$$

$$\left.-\frac{9}{2}x'\xi'+\left(\frac{49}{32}p^4+\frac{35}{32}p^3+\frac{35}{16}p^2\right)\xi'^2\right\}y\leqq 0,$$

$$\left\{\left(\frac{8}{3}p^3+\frac{15}{4}p^2-\frac{3593}{8}x'^2\right)x'^2-\frac{27}{4}px'y'+\frac{9}{8}p^3y'^2-\frac{1}{2}x'\eta'+\frac{15}{8}p^3\eta'^2\right\}\eta$$

$$\leqq-\frac{14.5}{8}p^2x'^2\eta\leqq\frac{14.5\beta_1}{16}p^2\eta^2+\frac{14.5}{16\beta_1}p^2x'^4,$$

$$\left(-\frac{29}{8}px'^2-\frac{3}{2}x'y'\right)\xi\leqq\frac{29\gamma_2}{16}p\xi^2+\frac{29}{16\gamma_2}px'^4+\frac{3\gamma_3}{4}\xi^2+\frac{3}{4\gamma_3}x'^2y'^2,$$

$$-\frac{5}{4}x'^2\varphi\leqq\frac{5}{8}\gamma_1\varphi^2+\frac{5}{8}\frac{1}{\gamma_1}x'^4,$$

$$\frac{21}{8}py^3-15y^2\eta+\frac{27}{64}p^4y^3+\frac{45}{64}p^4y\eta^2+\frac{9}{8}p^3y^2\eta$$

$$+\frac{15}{64}(p^4+2p^3+4p^2)y^3+\frac{25}{64}(p^4+2p^3+4p^2)y\eta^2\leqq 0.$$

Making use of these remarks and applying Lemma 1 to the term $-(744x+816x^2)/96$ we have, with $\beta_1=3.63$, $\gamma_1=20$, $\gamma_2=3.2$, $\gamma_3=2.5$, $\delta_1=600$, $\delta_2=2.5$,

$$\Re a_8\leqq 8-\frac{x^2}{96}\tilde{P}_3(x)-\frac{1}{96}\tilde{Q}_3-\frac{1}{96}\tilde{R}_3,$$

$$\tilde{P}_3(x)=342-2018x+1726x^2-687.3x^3+141.55x^4-12.05x^5,$$

$$\tilde{Q}_3=(222p^3+1020p^2-612p-3306)y^2+2(222p^2-288p+162x'^2)y\eta$$

$$+(-315.81p^2-468p+2202)\eta^2+(-1984.8p+3978)\xi^2$$

$$+2\cdot 144y\varphi+(-1836p+4146)\varphi^2,$$

$$\tilde{R}_3=(100.125p^5-204p+594-196.5p^3x'^2-24p^2x'^2-54.38px'^2-3x'^2$$

$$+40.5px'^4)x'^2+2(119.625p^4-312p^2x'^2+33x'^4)x'y'$$

$$+(174p^3+13.6p^2-612p+1727.58-504px'^2-283.8x'^2)y'^2$$

$$+2(180p^3-180px'^2)x'\eta'+2(222p^2-234x'^2)y'\eta'+(-804p+2970)\eta'^2$$

$$+2(204p^2-66x'^2)x'\xi'+2\cdot 216py'\xi'+2\cdot 192\eta'\xi'+(-1428p+4158)\xi'^2$$

$$+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'+(-1836p+5346)\varphi'^2+2\cdot 96x'\tau'+(-2244p+6534)\tau'^2.$$

It is easy to prove that $\tilde{P}_3(x)>0$ for $0\leqq x\leqq 0.2$. Q_3 is non-negative for $1.8\leqq p\leqq 2$, $|x'/p|\leqq 1/10$. Indeed we have

$$(222p^3+1020p^2-612p-3306)y^2+2\cdot 144y\varphi+(-1836p+4146)\varphi^2\geqq 0.$$

$$(222p^2-288p+162x'^2)y\eta\geqq 0,$$

$$(-315.81p^2-468p+2202)\eta^2\geqq 0$$

and

$$(-1984.8p+3978)\xi^2\geqq 0$$

for $1.8\leqq p\leqq 2$, $|x'/p|\leqq 1/10$.

As in Case 1 we may consider $\tilde{R}_3^1=\tilde{R}_3-\{(51p+5)x'^2+38y'^2+192\eta'^2+2\cdot 192\eta'\xi'+192\xi'^2+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'+(-1836p+5346)\varphi'^2+2\cdot 96x'\tau'+(-2244p+6534)\tau'^2\}$. It is not so difficult to prove the positive definiteness of $\tilde{R}_3^1$ for $1.8\leqq p\leqq 2$, $|x'/p|\leqq 1/10$.

Therefore in the present case we have $\Re a_8\leqq 8$ for $1.8\leqq p\leqq 2$, $|x'/p|\leqq 1/10$ with equality holding only for $x=0$.

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