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ON THE ORR-SOMMERFELD TYPE EQUATIONS, I; W. K. B. APPROXIMATION

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§ 1. Introduction.

We consider in this paper the 4-th order ordinary differential equations of the form

$$-\varepsilon^2 y^{(4)}+p_3(x,\varepsilon)y^{(2)}+p_2(x,\varepsilon)y'+p_1(x,\varepsilon)y=0, \tag{1.1}$$

where ε is a small positive parameter, y is a function of ε and a complex independent variable x, and the differentiation is taken with respect to x. The coefficients $p_i(x,\varepsilon)$ $(i=1,2,3)$ are holomorphic functions in the region

$$D:\ |x|<\infty, \qquad 0<\varepsilon\leqq\varepsilon_0<1 \tag{1.2}$$

and as ε tends to zero they have uniformly asymptotic expansions in power series of ε with holomorphic coefficients such that

$$p_i(x,\varepsilon)\cong\sum_{\nu=0}^{\infty}p_{i\nu}(x)\varepsilon^{\nu} \qquad (i=1,2,3), \tag{1.3}$$

in a region

$$D_M:\ |x|\leqq M, \qquad 0<\varepsilon\leqq\varepsilon_0<1,$$

for arbitrarily large M.

The above equation (1.1) is one of the generalization of the so-called Orr-Sommerfeld equation which plays a fundamental role in the problem of the stability of parallel flow of a viscous fluid and is written as

$$\frac{1}{\alpha R}\{\phi^{(4)}-2\alpha^2\phi^{(2)}+\alpha^4\phi\}=i\{[w(x)-c][\phi^{(2)}-\alpha^2\phi]-w^{(2)}(x)\phi\}. \tag{1.4}$$

Here $w(x)$ is a given function of x, α and R are real parameters and c is a complex parameter. Physically speaking the function $w(x)$ is the velocity for the basic parallel flow, α the wave number of disturbance, R the Raynolds number and c the complex wave speed. The deriviation of this equation from the Navier-Stokes equation is explained in, for example, Lin [7]. The function $w(x)$ can only be a quadratic function of x from the consideration of the exact solution of the

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Navier-Stokes equation, but for our present purpose, we shall not restrict ourselves to a quadratic function of x, so that our treatments of the equation (1.1) could be applied to other problems. From the physical reasoning, it is usually studied the asymptotic expansion of the fundamental system of solutions of (1.4) as $(\alpha R)^{-1}=\varepsilon$ tends to zero, but at the turning points where $w(x)-c=0$ it becomes quite complex.

There have been many contributions to the equation (1.4) and its variants, in particular by Langer [6], Lin [7], Wasow [10], Lin and Labenstein [8], Graebel [3] and Kuentum [5]. Most of them are concerned with the construction of asymptotic expansions of solutions in a small neighborhood of the turning point by using the related equation method, the matched asymptotic expansion technique or the method of multiple scales. On the other hand, asymptotic expansions in an arbitrarily large but bounded region were considered by Lin and Foote [2] when $w(x)-c$ is a quadratic function of x, and by Wasow [11] in the whole x-plane when $w(x)$ is a linear function of x.

The purpose of this paper is to construct systematically asymptotic expansions of fundamental system of solutions of the equation (1.1) in an arbitrarily large bounded region. The main idea is to define unbounded regions called canonical regions which were introduced firstly by Evgrafov and Fedoryuk [1] in the asymptotic theory of the second order differential equations containing turning points. Through certain deformations of these regions, we can obtain the regions of admissibility in which asymptotic expansions are valid. These regions are arbitrarily large, bounded and exclude certain neighborhoods of turning points called domains of influences which shrink to turning points as ε tends to zero. Therefore our theory does not give any informations about the asymptotic nature of solutions at turning points and at infinity. The lack of the former is overcome by combining the local theory at turning point and our theory, that is, by using the matching procedure between an inner asymptotic expansion which is valid in a small direct neighborhood of the turning point and our outer asymptotic expansions. On the other hand, the asymptotic expansions at infinity could not be obtained in this paper because of the fact that we have no enough global theory of the second order differential equations containing irregular singular point at infinity for our present studies.

In section 2, we state the assumptions on the holomorphic coefficients, and then a few transformations are made so that the coefficient matrix of the transformed equation has a formal expansion with diagonal form to the order of ε. In section 3, it is defined the domain of influence which is a small neighborhood of the turning point and shrinks to the turning point as ε tends to zero. This is done by considering the singularity which appears in the coefficients of transformations and the transformed equation at each turning point. In section 4, we calculate the first formal approximation of the fundamental system of solutions, and a region of admissibility is defined by small deformations of canonical region. And finally in section 5 the existence theorem is demonstrated, that is, for the first formal approximation there exists a fundamental system of solutions whose

asymptotic expansion coincides with it in the region of admissibility.

The connection formulas between the different regions of admissibility, or the error bounds for asymptotic expansions will be considered in future.

§ 2. Assumptions and preliminary transformations.

At first, we introduce the vector variable Y by

$$Y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}=\begin{bmatrix} y \\ y' \\ y^{(2)} \\ \varepsilon y^{(3)} \end{bmatrix}=\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \varepsilon \end{bmatrix}\begin{bmatrix} y \\ y' \\ y^{(2)} \\ y^{(3)} \end{bmatrix},$$

then the given 4-th order differential equation becomes the system of differential equation such that

$$\varepsilon Y'=P(x,\varepsilon)Y,$$

$$P(x,\varepsilon)=\begin{bmatrix} 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 1 \\ p_1(x,\varepsilon) & p_2(x,\varepsilon) & p_3(x,\varepsilon) & 0 \end{bmatrix}. \tag{2.1}$$

Here the matrix function $P(x,\varepsilon)$ is holomorphic in the region D, and when ε tends to zero it can be represented uniformly by an asymptotic power series of ε with holomorphic coefficients in D_M:

$$P(x,\varepsilon)\cong\sum_{\nu=0}^{\infty}P_\nu(x)\varepsilon^\nu,$$

$$P_0(x)=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ p_{10}(x) & p_{20}(x) & p_{30}(x) & 0 \end{bmatrix},\quad P_1(x)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ p_{11}(x) & p_{21}(x) & p_{31}(x) & 0 \end{bmatrix},$$

$$P_\nu(x)=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p_{1\nu}(x) & p_{2\nu}(x) & p_{3\nu}(x) & 0 \end{bmatrix}\quad(\nu\geqq 2). \tag{2.2}$$

Here we make following assumptions on the functions $p_{i\nu}(x)$.

(1) The function $p_3(x, 0)$ is not identically zero, and then has a finite number of zeros which we denote by $a_1, a_2, \cdots, a_s$, and call them turning points of the equation (1.1) or (2.1).

(2) Let us expand the function $p_{i\nu}(x)$ in the neighborhood of turning point a_j by the convergent power series of $x-a_j$:

$$p_{i\nu}(x)=\sum_{\mu=m_{i\nu j}}^{\infty} p_{i\nu\mu}(x-a_j)^{\mu}, \qquad p_{i\nu}, m_{i\nu j}\neq 0, \qquad m_{i\nu j}\geqq 0,$$

then we assume

$$m_{i\nu j}+\frac{2+q}{2}\nu+3-(i+q)\geqq 0 \qquad (i=1,2,3;\ \nu=0,1;\ j=1,2,\cdots,s), \tag{2.3}$$

where $q=m_{30j}$.

It is to be remarked that q is the order of zero of $p_{30}(x)$ at a_j, and the assumption (2.3) implies that the characteristic polygon (see Iwano-Sibuya [4]) associated with the turning point a_j consists of only one segment, and that the reduced equation

$$p_{30}(x)y^{(2)}+p_{20}(x)y'+p_{10}(x)y=0$$

has a regular singular point at a_j.

Now we state transformations which make the equation convenient to calculate formal solutions.

Transformation 1. By a linear transformation

$$Y=\Omega(x)Y_1 \quad \text{with} \quad \Omega(x)=\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \sqrt{p_{30}(x)} \end{bmatrix}, \tag{2.4}$$

the equation (2.1) becomes

$$\varepsilon Y_1'=P^{(1)}(x,\varepsilon)Y_1,$$

$$P^{(1)}(x,\varepsilon)=\Omega(x)^{-1}P(x,\varepsilon)\Omega(x)-\varepsilon\Omega(x)^{-1}\Omega'(x),$$

where $P^{(1)}(x,\varepsilon)$ has an formal asymptotic expansion of the form

$$P^{(1)}(x,\varepsilon)\sim\sum_{\nu=0}^{\infty}P_{\nu}^{(1)}(x)\varepsilon^{\nu},$$

$$P_0^{(1)}(x)=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{p_{30}(x)} \\ \dfrac{p_{10}(x)}{\sqrt{p_{30}(x)}} & \dfrac{p_{20}(x)}{\sqrt{p_{30}(x)}} & \sqrt{p_{30}(x)} & 0 \end{bmatrix},$$

$$P_1^{(1)}(x)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{p_{11}(x)}{\sqrt{p_{30}(x)}} & \dfrac{p_{21}(x)}{\sqrt{p_{30}(x)}} & \dfrac{p_{31}(x)}{\sqrt{p_{30}(x)}} & \dfrac{-p'_{30}(x)}{2p_{30}(x)} \end{bmatrix},$$

$$P_\nu^{(1)}(x)=\frac{1}{\sqrt{p_{30}(x)}}\begin{bmatrix} & & 0 & \\ p_{1\nu}(x) & p_{2\nu}(x) & p_{3\nu}(x) & 0 \end{bmatrix} \qquad (\nu\geqq 2).$$

In the following we make use of the block diagonalization technique. To do this, the vector Y_1 is splitted into two parts and the matrix $P^{(1)}(x,\varepsilon)$ into four blocks such that the equation for Y_1 is written

$$\varepsilon\begin{bmatrix} U_1 \\ V_1 \end{bmatrix}'=\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}\begin{bmatrix} U_1 \\ V_1 \end{bmatrix}, \tag{2.5}$$

where

$$Y_1=\begin{bmatrix} U_1 \\ V_1 \end{bmatrix}, \qquad U_1=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad V_1=\begin{bmatrix} y_3 \\ \sqrt{p_{30}(x)}y_4 \end{bmatrix},$$

$$A_1=\begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \end{bmatrix}, \qquad B_1=\begin{bmatrix} 0 & 0 \\ \varepsilon & 0 \end{bmatrix}, \qquad C_1=\frac{1}{\sqrt{p_{30}(x)}}\begin{bmatrix} 0 & 0 \\ p_1(x,\varepsilon) & p_2(x,\varepsilon) \end{bmatrix},$$

$$D_1=\begin{bmatrix} 0 & \sqrt{p_{30}(x)} \\ \dfrac{p_3(x,\varepsilon)}{\sqrt{p_{30}(x)}} & \dfrac{-p'_{30}(x)}{2p_{30}(x)} \end{bmatrix},$$

and

$$\begin{aligned} C_1 &\sim \sum_{\nu=0}^{\infty} C_{1\nu}(x)\varepsilon^\nu = \sum_{\nu=0}^{\infty}\frac{1}{\sqrt{p_{30}(x)}}\begin{bmatrix} 0 & 0 \\ p_{1\nu}(x) & p_{2\nu}(x) \end{bmatrix}\varepsilon^\nu, \\ D_1 &\sim \sum_{\nu=0}^{\infty} D_{1\nu}(x)\varepsilon^\nu = \begin{bmatrix} 0 & \sqrt{p_{30}(x)} \\ \sqrt{p_{30}(x)} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \dfrac{p_{31}(x)}{\sqrt{p_{30}(x)}} & \dfrac{-p'_{30}(x)}{2p_{30}(x)} \end{bmatrix}\varepsilon \\ &\quad + \sum_{\nu=2}^{\infty}\frac{1}{\sqrt{p_{30}(x)}}\begin{bmatrix} 0 & 0 \\ p_{3\nu}(x) & 0 \end{bmatrix}\varepsilon^\nu. \end{aligned} \tag{2.6}$$

Transformation 2. Let the equation (2.5) be transformed by the relation

$$\begin{bmatrix} U_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} E+\varepsilon QR & \varepsilon Q \\ R & E \end{bmatrix} \begin{bmatrix} U_2 \\ V_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} U_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} E & -\varepsilon Q \\ -R & E+\varepsilon RQ \end{bmatrix} \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} \tag{2.7}$$

into the form

$$\varepsilon \begin{bmatrix} U_2 \\ V_2 \end{bmatrix}' = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} U_2 \\ V_2 \end{bmatrix},$$

where E is the 2-2 unit matrix, Q and R are 2-2 matrices chosen appropriately in later, and the matrices A_2, B_2, C_2, D_2 become

$$\begin{aligned} A_2 &= A_1 + B_1 R - \varepsilon Q(C_1 + D_1 R) + \{\varepsilon A_1 QR - \varepsilon^2 QC_1 QR - \varepsilon^2 Q'R\}, \\ B_2 &= B_1 - \varepsilon QD_1 + \{\varepsilon A_1 Q - \varepsilon^2 QC_1 Q - \varepsilon^2 Q_1'\}, \\ C_2 &= C_1 + D_1 R + \{-RA_1 + \varepsilon C_1 QR - R(B_1 - \varepsilon QD_1)R + \varepsilon RQC_1 - \varepsilon R'\} \\ &\quad - \{\varepsilon RAQR - \varepsilon^2 RQC_1 QR - \varepsilon^2 RQ'R\}, \\ D_2 &= D_1 + \{-R(B_1 - \varepsilon QD_1) + \varepsilon C_1 Q\} - \{\varepsilon RA_1 Q - \varepsilon^2 RQC_1 Q - \varepsilon^2 RQ'\}. \end{aligned} \tag{2.8}$$

Here we determine the matrices $Q = Q_0(x)$ and $R = R_0(x) + R_1(x)\varepsilon$ by

$$\begin{aligned} &B_1 - \varepsilon Q_0(x) D_{10}(x) = 0, \\ &C_{10}(x) + D_{10}(x) R_0(x) = 0, \\ &\varepsilon\{C_{11}(x) + D_{11}(x) R_0(x) + D_{10}(x) R_1(x)\} \\ &\qquad + \{-R_0(x) A_1 + \varepsilon C_{10}(x) Q_0(x) R_0(x) - \varepsilon R_0(x) Q_0(x) C_{10}(x) - \varepsilon R_0'(x)\} = 0 \end{aligned} \tag{2.9}$$

so that we can write

$$\begin{aligned} A_2 &\sim \sum_{i=1}^{\infty} A_{2i}(x)\varepsilon^i, \qquad B_2 \sim \sum_{i=2}^{\infty} B_{2i}(x)\varepsilon^i, \\ C_2 &\sim \sum_{i=2}^{\infty} C_{2i}(x)\varepsilon^i, \qquad D_2 \sim \sum_{i=0}^{\infty} D_{2i}(x)\varepsilon^i, \end{aligned} \tag{2.10}$$

with

$$\begin{aligned} A_{21}(x) &= \{A_1 + B_1 R_0\}\varepsilon^{-1}, \\ A_{22}(x) &= B_1 R_1 \varepsilon^{-1} - Q\{C_{11} + D_{11} R_0 + D_{10} R_1\}, \\ A_{23}(x) &= -Q\{C_{12} + D_{12} R_0 + D_{11} R_1\} + A_1 \varepsilon^{-1} QR_1 - QC_{10} QR_1 - Q'R_1, \\ A_{2\nu}(x) &= -Q\{C_{1,\nu-1} + D_{1,\nu-1} R_0 + D_{1,\nu-2} R_1\} - QC_{1,\nu-3} QR_1, \\ B_{22}(x) &= A_1 Q\varepsilon^{-1} - (QD_{11} + QC_{10} Q + Q'), \\ B_{2\nu}(x) &= -[QD_{1,\nu-1} + QC_{1,\nu-2} Q] \qquad (\nu \geqq 3), \end{aligned}$$

$$C_{22}(x)=C_{12}+D_{12}R_0+D_{11}R_1-R_1A_1\varepsilon^{-1}+C_{10}QR_1-R_1B_1R_0\varepsilon^{-1}-R_0B_1R_1\varepsilon^{-1}$$
$$+R_0QD_{11}R_0+R_1QD_{10}R_0+R_0QD_{10}R_1+R_0QC_{11}+R_1QC_{10}-R_1',$$
$$C_{23}(x)=-R_1B_1R_1\varepsilon^{-1}+R_0Q'R_1-R_0A_1QR_1\varepsilon^{-1}+C_{13}+D_{13}R_0+D_{12}R_1+C_{11}QR_1$$
$$+R_0QD_{12}R_0+R_1QD_{11}R_0+R_0QD_{11}R_1+R_1QD_{10}R_1+R_0QC_{12}+R_1QC_{11}$$
$$+R_0QC_{10}QR_1,$$
$$C_{24}(x)=R_1Q'R_1-R_1A_1QR_1\varepsilon^{-1}+C_{14}+D_{14}R_0+D_{13}R_1+C_{12}QR_1+R_0QD_{13}R_0$$
$$+R_1QD_{12}R_0+R_0QD_{12}R_1+R_1QD_{11}R_1+R_0QC_{13}+R_1QC_{12}+R_0QC_{11}QR_1$$
$$+R_1QC_{10}QR_1,$$
$$C_{2\nu}(x)=C_{1\nu}+D_{1\nu}R_0+D_{1,\nu-1}R_1+C_{1,\nu-2}QR_1+R_0QD_{1,\nu-1}R_0+R_1QD_{1,\nu-2}R_0$$
$$+R_0QD_{1,\nu-2}R_1+R_1QD_{1,\nu-3}R_1+R_0QC_{1,\nu-1}+R_1QC_{1,\nu-2}+R_0QC_{1,\nu-3}QR_1$$
$$+R_1QC_{1,\nu-4}QR_1 \qquad (\nu\geqq 5),$$
$$D_{20}(x)=D_{10}(x),$$
$$D_{21}(x)=D_{11}+C_{10}Q,$$
$$D_{22}(x)=R_0Q'-(R_1B_1+R_0A_1Q)\varepsilon^{-1}+D_{12}+R_0QD_{11}+R_1QD_{10}+C_{11}Q+R_0QC_{10}Q,$$
$$D_{23}(x)=-R_1A_1Q\varepsilon^{-1}+R_1Q'+D_{13}+R_0QD_{12}+R_1QD_{11}+C_{12}Q+R_0QC_{11}Q+R_1QC_{10}Q,$$
$$D_{2\nu}(x)=D_{1\nu}+R_0QD_{1,\nu-1}+R_1QD_{1,\nu-2}+C_{1,\nu-1}Q+R_0QC_{1,\nu-2}Q+R_1QC_{1,\nu-3}Q \qquad (\nu\geqq 4).$$

A short calculation using (2.6) and (2.9) gives

$$Q(x)=\begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1}{\sqrt{p_{30}(x)}} \end{bmatrix}, \qquad R_0(x)=\begin{bmatrix} \dfrac{-p_{10}(x)}{p_{30}(x)} & \dfrac{-p_{20}(x)}{p_{30}(x)} \\ 0 & 0 \end{bmatrix},$$

$$R_1(x)=\begin{bmatrix} \dfrac{p_{10}(x)p_{31}(x)}{\{p_{30}(x)\}^2}-\dfrac{p_{11}(x)}{p_{30}(x)}, & \dfrac{p_{20}(x)p_{31}(x)}{\{p_{30}(x)\}^2}-\dfrac{p_{21}(x)}{p_{30}(x)} \\ \dfrac{p_{10}(x)p_{30}(x)'-p_{30}(x)p_{10}'(x)-p_{10}(x)p_{20}(x)}{\{p_{30}(x)\}^{5/2}}, & \dfrac{p_{20}(x)p_{30}(x)'-p_{30}(x)p_{20}(x)'-p_{20}(x)^2}{\{p_{30}(x)\}^{5/2}}-\dfrac{p_{10}(x)}{\{p_{30}(x)\}^{3/2}} \end{bmatrix},$$

(2.11)

$$A_{21}(x)\varepsilon=\begin{bmatrix} 0 & 1 \\ \dfrac{-p_{10}(x)}{p_{30}(x)} & -\dfrac{p_{20}(x)}{p_{30}(x)} \end{bmatrix}\varepsilon,$$

$$D_{20}(x)+D_{21}(x)\varepsilon=\begin{bmatrix} 0 & \sqrt{p_{30}(x)} \\ \sqrt{p_{30}(x)} & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ \dfrac{p_{31}(x)}{\sqrt{p_{30}(x)}} & \dfrac{p_{20}(x)}{p_{30}(x)}-\dfrac{p_{30}(x)'}{2p_{30}(x)} \end{bmatrix}\varepsilon.$$

From the method of determining the matrices Q and R, we can choose them analogously so that the coefficients of formal power series of the matrices B_2 and C_2 are zero to as many powers of ε as we wish, but to avoid the complexities, it will be calculated only the first term of asymptotic expansions in this paper. Next, the expansion of the matrix $D_2(x,\varepsilon)$ will be diagonalized to the order of ε by the following transformation.

Transformation 3. Firstly the matrix $D_{20}(x)$ becomes diagonal by

$$\begin{cases} V_2=\tilde{T}_3\tilde{V}_3, \\ U_2=\tilde{U}_3, \end{cases} \qquad \tilde{T}_3=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad \tilde{T}_3^{-1}=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

thus the equation (2.7) becomes

$$\varepsilon\begin{bmatrix} \tilde{U}_3 \\ \tilde{V}_3 \end{bmatrix}'=\begin{bmatrix} \tilde{A}_3 & \tilde{B}_3 \\ \tilde{C}_3 & \tilde{D}_3 \end{bmatrix}\begin{bmatrix} \tilde{U}_3 \\ \tilde{V}_3 \end{bmatrix}, \tag{2.12}$$

where the matrices $\tilde{A}_3, \tilde{B}_3, \tilde{C}_3$ and $\tilde{D}_3$ have the same forms as (2.10) and

$$\begin{aligned} &\tilde{D}_{30}(x)+\tilde{D}_{31}(x)\varepsilon \\ &=\begin{bmatrix} \sqrt{p_{30}(x)} & 0 \\ 0 & -\sqrt{p_{30}(x)} \end{bmatrix}+\frac{\varepsilon}{2}\begin{bmatrix} \dfrac{p_{31}}{\sqrt{p_{30}}}+\dfrac{p_{20}}{p_{30}}-\dfrac{p'_{30}}{2p_{30}}, & \dfrac{-p_{31}}{\sqrt{p_{30}}}+\dfrac{p_{20}}{p_{30}}-\dfrac{p'_{30}}{2p_{30}} \\ \dfrac{p_{31}}{\sqrt{p_{30}}}+\dfrac{p_{20}}{p_{30}}-\dfrac{p'_{30}}{2p_{30}}, & \dfrac{-p_{31}}{\sqrt{p_{30}}}+\dfrac{p_{20}}{p_{30}}-\dfrac{p'_{30}}{2p_{30}} \end{bmatrix}. \end{aligned}$$

By the same idea as used in the transformation 2, we change the equation (2.12) by

$$\begin{cases} \tilde{U}_3=U_3, \\ \tilde{V}_3=T_3(x,\varepsilon)V_3, \end{cases} \qquad T_3(x,\varepsilon)=\begin{bmatrix} 1+\varepsilon^2q(x)r(x) & \varepsilon q(x) \\ \varepsilon r(x) & 1 \end{bmatrix},$$

$$T_3(x,\varepsilon)^{-1}=\begin{bmatrix} 1 & -\varepsilon q(x) \\ -\varepsilon r(x) & 1+\varepsilon^2q(x)r(x) \end{bmatrix},$$

where

$$q(x)=\frac{-1}{4\sqrt{p_{30}(x)}}\left\{-\frac{p_{31}(x)}{\sqrt{p_{30}(x)}}+\frac{p_{20}(x)}{p_{30}(x)}-\frac{p_{30}(x)'}{2p_{30}(x)}\right\},$$

$$r(x)=\frac{1}{4\sqrt{p_{30}(x)}}\left\{\frac{p_{31}(x)}{\sqrt{p_{30}(x)}}+\frac{p_{20}(x)}{p_{30}(x)}-\frac{p_{30}(x)'}{2p_{30}(x)}\right\},$$

and we obtain

$$\varepsilon\begin{bmatrix} U_3 \\ V_3 \end{bmatrix}' = \begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix}\begin{bmatrix} U_3 \\ V_3 \end{bmatrix}, \tag{2.13}$$

where the matrices A_3, B_3, C_3 and D_3 have the same forms of formal power series of ε as in (2.10), and

$$\begin{aligned} &A_3=\tilde{A}_3=A_2, && B_3=\tilde{B}_3T_3=B_2\tilde{T}_3T_3, \\ &C_3=T_3^{-1}\tilde{C}_3=T_3^{-1}\tilde{T}_3^{-1}C_2, && D_3=(\tilde{T}_3T_3)^{-1}D_2\tilde{T}_3T_3-\varepsilon T_3^{-1}(T_3)', \end{aligned} \tag{2.14}$$

$$\tilde{T}_3T_3(x,\varepsilon)=\frac{1}{\sqrt{2}}\begin{bmatrix} 1-\varepsilon r(x)+\varepsilon^2q(x)r(x) & -1+\varepsilon q(x) \\ 1+\varepsilon r(x)+\varepsilon^2q(x)r(x) & 1+\varepsilon q(x) \end{bmatrix},$$

$$\{\tilde{T}_3T_3(x,\varepsilon)\}^{-1}=\frac{1}{\sqrt{2}}\begin{bmatrix} 1+\varepsilon q(x) & 1-\varepsilon q(x) \\ -1-\varepsilon r(x)-\varepsilon^2q(x)r(x) & 1-\varepsilon r(x)+\varepsilon^2q(x)r(x) \end{bmatrix}.$$

The matrix D_3 can be written as before in power series of ε, and its first two terms are

$$\begin{aligned} D_{30}(x)+D_{31}(x)\varepsilon=&\begin{bmatrix} \sqrt{p_{30}(x)} & 0 \\ 0 & -\sqrt{p_{30}(x)} \end{bmatrix} \\ &+\frac{\varepsilon}{2}\begin{bmatrix} \dfrac{p_{31}(x)}{\sqrt{p_{30}(x)}}+\dfrac{p_{20}(x)}{p_{30}(x)}-\dfrac{p_{30}(x)'}{2p_{30}(x)} & 0 \\ 0 & -\dfrac{p_{31}(x)}{\sqrt{p_{30}(x)}}+\dfrac{p_{20}(x)}{p_{30}(x)}-\dfrac{p_{30}(x)'}{2p_{30}(x)} \end{bmatrix}. \end{aligned} \tag{2.15}$$

§ 3. The domain of influence.

We rewrite the equation (2.13) by neglecting the indices

$$\varepsilon\begin{bmatrix} U \\ V \end{bmatrix}' = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix}, \tag{3.1}$$

where the matrices A, B, C, and D are identical with the matrices A_3, B_3, C_3, and D_3 defined in (2.14) respectively, and let their formal power series of ε be

$$\begin{aligned} &A\sim\sum_{\nu=1}^{\infty}A_\nu(x)\varepsilon^\nu, && B\sim\sum_{\nu=2}^{\infty}B_\nu(x)\varepsilon^\nu, \\ &C\sim\sum_{\nu=2}^{\infty}C_\nu(x)\varepsilon^\nu, && D\sim\sum_{\nu=0}^{\infty}D_\nu(x)\varepsilon^\nu. \end{aligned} \tag{3.2}$$

Here we consider the asymptotic nature of the matrices $A(x,\varepsilon)$, $B(x,\varepsilon)$, $C(x,\varepsilon)$

and $D(x, \varepsilon)$ in the neighborhoods of turning points, that is, count the order of poles of each element of these matrices at turning points.

Let a_j be one of the turning points, $q=m_{30j}$ be the order of zero of $p_{30}(x)$ at the point a_j. Furthermore, let $\mathrm{Or}[p(x)]=\alpha$ denote a rational number which is the order of pole of the function $p(x)$ at the point a_j, but if the function $p(x)$ has no pole there α may be negative and we put $\alpha=-\infty$ if $p(x)=0$. For a matrix $M(x)=(m_{ij}(x))$, $\mathrm{Or}[M(x)]$ denotes a matrix $(\mathrm{Or}[m_{ij}(x)])$.

The followings are immadiate consequences of the assumption (2.3):

$$
\begin{aligned}
&m_{10j}\geqq q-2, \qquad m_{20j}\geqq q-1, \qquad m_{30j}=q,\\
&m_{11j}\geqq \frac{q}{2}-3, \qquad m_{21j}\geqq \frac{q}{2}-2, \qquad m_{30j}\geqq \frac{q}{2}-1.
\end{aligned}
\tag{3.3}
$$

From these, we have

$$
\mathrm{Or}\left[\frac{p_{10}}{p_{30}}\right]\leqq 2, \qquad \mathrm{Or}\left[\frac{p_{30}}{p_{30}}\right]\leqq 1,
$$

$$
\mathrm{Or}\left[\frac{p_{10}}{\sqrt{p_{30}}}\right]\leqq 2-\frac{q}{2}, \qquad \mathrm{Or}\left[\frac{p_{20}}{\sqrt{p_{30}}}\right]\leqq 1-\frac{q}{2},
$$

$$
\mathrm{Or}\left[\frac{p_{11}}{\sqrt{p_{30}}}\right]\leqq 3, \qquad \mathrm{Or}\left[\frac{p_{21}}{\sqrt{p_{30}}}\right]\leqq 2, \qquad \mathrm{Or}\left[\frac{p_{31}}{\sqrt{p_{30}}}\right]\leqq 1,
$$

$$
\mathrm{Or}\left[\frac{p_{1\nu}}{\sqrt{p_{30}}}\right]\leqq \frac{q}{2}, \qquad \mathrm{Or}\left[\frac{p_{2\nu}}{\sqrt{p_{30}}}\right]\leqq \frac{q}{2}, \qquad \mathrm{Or}\left[\frac{p_{3\nu}}{\sqrt{p_{30}}}\right]\leqq \frac{q}{2} \qquad (\nu\geqq 2),
$$

$$
\mathrm{Or}\left[\frac{p_{10}p_{31}}{\{p_{30}\}^2}-\frac{p_{11}}{p_{30}}\right]\leqq \frac{q}{2}+3, \qquad \mathrm{Or}\left[\frac{p_{20}p_{31}}{\{p_{30}\}^2}-\frac{p_{21}}{p_{30}}\right]\leqq \frac{q}{2}+2,
$$

$$
\mathrm{Or}\left[\frac{p_{10}p'_{30}-p_{30}p'_{10}-p_{10}p_{20}}{\{p_{30}\}^{5/2}}\right]\leqq \frac{q}{2}+3, \qquad \mathrm{Or}\left[\frac{p_{20}p'_{30}-p'_{20}p_{30}-p_{20}^2}{\{p_{30}\}^{5/2}}-\frac{p_{10}}{\{p_{30}\}^{3/2}}\right]\leqq \frac{q}{2}+2,
$$

$$
\mathrm{Or}[q(x)]=\mathrm{Or}[r(x)]\leqq \frac{q}{2}+1.
$$

Thus we have at most

$$
\mathrm{Or}[C_{10}(x)]=\begin{bmatrix} -\infty & -\infty \\ 2-\frac{q}{2} & 1-\frac{q}{2} \end{bmatrix}, \qquad \mathrm{Or}[C_{11}(x)]=\begin{bmatrix} -\infty & -\infty \\ 3 & 2 \end{bmatrix},
$$

$$
\mathrm{Or}[C_{1\nu}(x)]=\begin{bmatrix} -\infty & -\infty \\ \frac{q}{2} & \frac{q}{2} \end{bmatrix},
$$

$$\mathrm{Or}[D_{10}(x)]=\begin{bmatrix} -\infty & -\frac{q}{2} \\ -\frac{q}{2} & -\infty \end{bmatrix}, \qquad \mathrm{Or}[D_{11}(x)]=\begin{bmatrix} -\infty & -\infty \\ 1 & 1 \end{bmatrix},$$

$$\mathrm{Or}[D_{1\nu}(x)]=\begin{bmatrix} -\infty & -\infty \\ \frac{q}{2} & -\infty \end{bmatrix} \quad (\nu\geqq 2),$$

$$\mathrm{Or}[Q(x)]=\begin{bmatrix} -\infty & -\infty \\ -\infty & \frac{q}{2} \end{bmatrix}, \qquad \mathrm{Or}[R_0(x)]=\begin{bmatrix} 2 & 1 \\ -\infty & -\infty \end{bmatrix},$$

$$\mathrm{Or}[R_1(x)]=\begin{bmatrix} \frac{q}{2}+3 & \frac{q}{2}+2 \\ \frac{q}{2}+3 & \frac{q}{2}+2 \end{bmatrix}.$$

From the relations (2.10), (2.14) and the above, each matrix function of (3.3) has at most the following order of pole at the turning point a_j:

$$\mathrm{Or}[A_2(x)]=\begin{bmatrix} 3 & 2 \\ \frac{q}{2}+3 & \frac{q}{2}+2 \end{bmatrix}, \qquad \mathrm{Or}[A_3(x)]=\begin{bmatrix} q+3 & q+2 \\ q+4 & q+3 \end{bmatrix},$$

$$\mathrm{Or}[A_4(x)]=\begin{bmatrix} -\infty & -\infty \\ \frac{3}{2}q+5 & \frac{3}{2}q+4 \end{bmatrix}, \qquad \mathrm{Or}[A_\nu(x)]=\begin{bmatrix} -\infty & -\infty \\ 2q+3 & 2q+2 \end{bmatrix} \quad (\nu\geqq 5),$$

$$\mathrm{Or}[B_2(x)]=\begin{bmatrix} \frac{q}{2} & \frac{q}{2} \\ \frac{q}{2}+1 & \frac{q}{2}+1 \end{bmatrix}, \qquad \mathrm{Or}[B_3(x)]=\begin{bmatrix} q+1 & q+1 \\ q+2 & q+2 \end{bmatrix},$$

$$\mathrm{Or}[B_4(x)]=\begin{bmatrix} \frac{3}{2}q+2 & -\infty \\ \frac{3}{2}q+3 & \frac{3}{2}q+3 \end{bmatrix}, \qquad \mathrm{Or}[B_5(x)]=\begin{bmatrix} -\infty & -\infty \\ 2q+4 & 2q+1 \end{bmatrix},$$

$$\mathrm{Or}[B_\nu(x)]=\begin{bmatrix} -\infty & -\infty \\ \frac{5}{2}q+2 & 2q+1 \end{bmatrix} \quad (\nu\geqq 6),$$

(3.4)

$$\mathrm{Or}[C_2(x)]=\begin{bmatrix} \frac{q}{2}+4 & \frac{q}{2}+3 \\ \frac{q}{2}+4 & \frac{q}{2}+3 \end{bmatrix}, \qquad \mathrm{Or}[C_3(x)]=\begin{bmatrix} q+5 & q+4 \\ q+5 & q+4 \end{bmatrix},$$

$$\mathrm{Or}[C_4(x)]=\begin{bmatrix} \frac{3}{2}q+6 & \frac{3}{2}q+5 \\ \frac{3}{2}q+6 & \frac{3}{2}q+5 \end{bmatrix}, \qquad \mathrm{Or}[C_5(x)]=\begin{bmatrix} 2q+7 & 2q+6 \\ 2q+7 & 2q+6 \end{bmatrix},$$

$$\mathrm{Or}[C_6(x)]=\begin{bmatrix} \frac{5}{2}q+8 & \frac{5}{2}q+7 \\ \frac{5}{2}q+8 & \frac{5}{2}q+7 \end{bmatrix}, \qquad \mathrm{Or}[C_7(x)]=\begin{bmatrix} 3q+6 & 3q+5 \\ 3q+9 & 3q+8 \end{bmatrix},$$

$$\mathrm{Or}[C_\nu(x)]=\begin{bmatrix} 3q+6 & 3q+5 \\ \frac{7}{2}q+7 & \frac{7}{2}q+6 \end{bmatrix} \quad (\nu\geqq 8),$$

$$\mathrm{Or}[D_2(x)]=\begin{bmatrix} \frac{q}{2}+2 & \frac{q}{2}+2 \\ \frac{q}{2}+2 & \frac{q}{2}+2 \end{bmatrix}, \qquad \mathrm{Or}[D_3(x)]=\begin{bmatrix} q+3 & q+3 \\ q+3 & q+3 \end{bmatrix},$$

$$\mathrm{Or}[D_4(x)]=\begin{bmatrix} \frac{3}{2}q+4 & \frac{3}{2}q+4 \\ \frac{3}{2}q+4 & \frac{3}{2}q+4 \end{bmatrix}, \qquad \mathrm{Or}[D_5(x)]=\begin{bmatrix} 2q+5 & 2q+5 \\ 2q+5 & 2q+5 \end{bmatrix},$$

$$\mathrm{Or}[D_6(x)]=\begin{bmatrix} \frac{5}{2}q+6 & \frac{5}{2}q+6 \\ \frac{5}{2}q+6 & \frac{5}{2}q+6 \end{bmatrix}, \qquad \mathrm{Or}[D_7(x)]=\begin{bmatrix} 3q+7 & 3q+4 \\ 3q+7 & 3q+7 \end{bmatrix},$$

$$\mathrm{Or}[D_8(x)]=\begin{bmatrix} \frac{7}{2}q+5 & 3q+4 \\ \frac{7}{2}q+8 & \frac{7}{2}q+5 \end{bmatrix}, \qquad \mathrm{Or}[D_\nu(x)]=\begin{bmatrix} \frac{7}{2}q+5 & 3q+4 \\ 4q+8 & \frac{7}{2}q+5 \end{bmatrix} \quad (\nu\geqq 9).$$

When $q=1$, which is the most cases in applications, the above estimates are rather superfluous. In this case the condition (2.2) is automatically satisfied for all $m_{i\nu j}\geqq 0$ $(i=1,2,3;\nu=0,1)$, $m_{30j}=q=1$. By the same procedures as before we have instead of (3.4) the following estimates at the turning point a_j:

$$\mathrm{Or}[A_2(x)]=\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix},\quad \mathrm{Or}[A_3(x)]=\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix},\quad \mathrm{Or}[A_\nu(x)]=\begin{bmatrix} -\infty & -\infty \\ 4 & 4 \end{bmatrix}\quad (\nu\geqq 4),$$

$$\mathrm{Or}[B_2(x)]=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix},\quad \mathrm{Or}[B_3(x)]=\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix},\quad \mathrm{Or}[B_4(x)]=\begin{bmatrix} -\infty & -\infty \\ \frac{9}{2} & 3 \end{bmatrix}\quad (\nu\geqq 4),$$

$$\mathrm{Or}[C_2(x)]=\begin{bmatrix} \frac{7}{2} & \frac{7}{2} \\ \frac{7}{2} & \frac{7}{2} \end{bmatrix},\quad \mathrm{Or}[C_3(x)]=\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix},\quad \mathrm{Or}[C_4(x)]=\begin{bmatrix} \frac{13}{2} & \frac{13}{2} \\ \frac{13}{2} & \frac{13}{2} \end{bmatrix},$$

(3.4)′

$$\mathrm{Or}[C_5(x)]=\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix},\quad \mathrm{Or}[C_\nu(x)]=\begin{bmatrix} 8 & 8 \\ \frac{19}{2} & \frac{19}{2} \end{bmatrix}\quad (\nu\geqq 6),$$

$$\mathrm{Or}[D_2(x)]=\begin{bmatrix} \frac{5}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{5}{2} \end{bmatrix},\quad \mathrm{Or}[D_3(x)]=\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix},\quad \mathrm{Or}[D_4(x)]=\begin{bmatrix} \frac{11}{2} & \frac{11}{2} \\ \frac{11}{2} & \frac{11}{2} \end{bmatrix},$$

$$\mathrm{Or}[D_5(x)]=\begin{bmatrix} 7 & 7 \\ 7 & 7 \end{bmatrix},\quad \mathrm{Or}[D_6(x)]=\begin{bmatrix} \frac{17}{2} & 7 \\ \frac{17}{2} & \frac{17}{2} \end{bmatrix},\quad \mathrm{Or}[D_\nu(x)]=\begin{bmatrix} \frac{17}{2} & 7 \\ 10 & \frac{17}{2} \end{bmatrix}\quad (\nu\geqq 7).$$

From (3.4) and (3.4)′, we obtain following lemmas.

LEMMA 3.1. *For each turning point a_j, there exists a neighborhood of the form*

$$N\varepsilon^\alpha \leqq |x-a_j| \leqq \rho,\qquad 0<\varepsilon\leqq\varepsilon_0,\qquad \alpha=2/(q+2) \tag{3.5}$$

for sufficiently large positive constant N, sufficiently small positive constants ρ and ε_0, for which there exists a positive constant L such that if $q\geqq 2$,

$$\begin{aligned}
&\left\|\begin{bmatrix} (x-a_j)^{-1} & 0 \\ 0 & 1 \end{bmatrix}\{A(x,\varepsilon)-\varepsilon A_1(x)\}\begin{bmatrix} (x-a_j) & 0 \\ 0 & 1 \end{bmatrix}\right\| \leqq L|x-a_j|^{-(q/2+2)}\varepsilon^2,\\
&\left\|\begin{bmatrix} (x-a_j)^{-1} & 0 \\ 0 & 1 \end{bmatrix}B(x,\varepsilon)\right\| \leqq L|x-a_j|^{-(q/2+1)}\varepsilon^2,\\
&\left\|C(x,\varepsilon)\begin{bmatrix} (x-a_j) & 0 \\ 0 & 1 \end{bmatrix}\right\| \leqq L|x-a_j|^{-(q/2+3)}\varepsilon^2,\\
&\|D(x,\varepsilon)-D_0(x)-\varepsilon D_1(x)\| \leqq L|x-a_j|^{-(q/2+2)}\varepsilon^2,
\end{aligned} \tag{3.6}$$

and if $q=1$, *we have*

$$
(3.6)' \qquad \begin{aligned}
\|A(x,\varepsilon)-\varepsilon A_1(x)\| &\leqq L|x-a_j|^{-5/2}\varepsilon^2,\\
\|B(x,\varepsilon)\| &\leqq L|x-a_j|^{-3/2}\varepsilon^2,\\
\|C(x,\varepsilon)\| &\leqq L|x-a_j|^{-7/2}\varepsilon^2,\\
\|D(x,\varepsilon)-D_0(x)-\varepsilon D_1(x)\| &\leqq L|x-a_j|^{-5/2}\varepsilon^2.
\end{aligned}
$$

Here the norm $\|A\|$ *of a matrix* $A=(a_{ij})$ *is defined by* $\sum_{ij}|a_{ij}|$.

Proof. We prove only for $A(x,\varepsilon)$ with $q\geqq 2$ and other cases can be proved analogously. From (3.4) there exists a positive constant L' such that

$$
\left\|\begin{bmatrix}(x-a_j)^{-1} & 0\\ 0 & 1\end{bmatrix}A_i(x)\begin{bmatrix}(x-a_j) & 0\\ 0 & 1\end{bmatrix}\right\| \leqq L'|x-a_j|^{-(q/2+2)-(q/2+1)(i-2)} \qquad (i=2,3,4),
$$

in the neighborhood of a_j, and also from the asymptotic properties of $p_i(x,\varepsilon)$ $(i=1,2,3)$ and (3.4) it is easily verified that

$$
\left\|\begin{bmatrix}(x-a_j)^{-1} & 0\\ 0 & 1\end{bmatrix}\sum_{\nu=5}^{\infty}A_\nu(x)\varepsilon^\nu\begin{bmatrix}(x-a_j) & 0\\ 0 & 1\end{bmatrix}\right\| \leqq L'|x-a_j|^{-(2q+2)}\varepsilon^5.
$$

Then we have

$$
\begin{aligned}
&\left\|\begin{bmatrix}(x-a_j)^{-1} & 0\\ 0 & 1\end{bmatrix}\{A(x,\varepsilon)-\varepsilon A_1(x)\}\begin{bmatrix}(x-a_j) & 0\\ 0 & 1\end{bmatrix}\right\|\\
&\leqq L'|x-a_j|^{-(q/2+2)}\varepsilon^2\{1+\varepsilon|x-a_j|^{-(q/2+1)}+\varepsilon^2|x-a_j|^{-2(q/2+1)}+\varepsilon^3|x-a_j|^{-(3/2)q}\}\\
&\leqq L|x-a_j|^{-(q/2+2)}\varepsilon^2.
\end{aligned}
$$

Each element of the transformations used in the section 2 is bounded in the domain D_M except for turning points. Then we have from the asymptotic property (2.2):

LEMMA 3.2. *There exists a positive constant* L *such that*

$$
(3.7) \qquad \begin{aligned}
\|A(x,\varepsilon)-\varepsilon A_1(x)\| &\leqq L\varepsilon^2,\\
\|B(x,\varepsilon)\| &\leqq L\varepsilon^2,\\
\|C(x,\varepsilon)\| &\leqq L\varepsilon^2,\\
\|D(x,\varepsilon)-D_0(x)-D_1(x)\varepsilon\| &\leqq L\varepsilon^2,
\end{aligned}
$$

for all (x,ε) *satisfying*

$$
(3.8) \qquad \operatorname*{Min}_j |x-a_j|\geqq\rho, \qquad 0<\varepsilon\leqq\varepsilon_0.
$$

Now we can define the domain of influence N_{a_j} at the turning point a_j, that is

$$N_{a_j}=\{x:\ |x-a_j|\leqq N\varepsilon^{\alpha_j},\ 0<\varepsilon\leqq\varepsilon_0\},\qquad \alpha_j=2/(q+2). \tag{3.9}$$

This region shrinks to the turning point a_j when ε tends to zero.

§ 4. Region of admissibility.

For the first formal approximation of a fundamental system of solutions of (3.1), we put

$$W_0(x)=\begin{bmatrix} U_0(x) & 0 \\ 0 & V_0(x,\varepsilon)\end{bmatrix}, \tag{4.1}$$

where $U_0(x)$ is a 2-2 matrix which constitutes a fundamental system of solutions of the reduced equation

$$U_0'=A_1(x)U_0, \tag{4.2}$$

and $V_0(x,\varepsilon)$ is a 2-2 matrix such that

$$\begin{aligned} V_0(x,\varepsilon)&=\exp\left[\int^x \frac{D_0(x)+D_1(x)\varepsilon}{\varepsilon}dx\right] \\ &=p_{30}(x)^{-1/4}\begin{bmatrix} \exp\int^x\left\{\frac{\sqrt{p_{30}(x)}}{\varepsilon}+\frac{p_{31}(x)}{2\sqrt{p_{30}(x)}}+\frac{p_{20}(x)}{2p_{30}(x)}\right\}dx & 0 \\ 0 & \exp\int^x\left\{\frac{-\sqrt{p_{30}(x)}}{\varepsilon}-\frac{p_{31}(x)}{2\sqrt{p_{30}(x)}}+\frac{p_{20}(x)}{2p_{30}(x)}\right\}dx\end{bmatrix}. \end{aligned} \tag{4.3}$$

Let $Y(x,\varepsilon)$ be a fundamental system of solutions of (3.1) and let

$$Y(x,\varepsilon)=W_0(x,\varepsilon)+W(x,\varepsilon),$$

with

$$W(x,\varepsilon)=\begin{bmatrix} U^{(1)}(x,\varepsilon) & U^{(2)}(x,\varepsilon) \\ V^{(1)}(x,\varepsilon) & V^{(2)}(x,\varepsilon)\end{bmatrix},$$

where $U^{(i)}(x,\varepsilon)$, $V^{(i)}(x,\varepsilon)$ $(i=1,2)$ are 2-2 matrices and must satisfy

$$\begin{aligned} \varepsilon U^{(1)\prime}&=\varepsilon A_1U^{(1)}+A_2(U_0+U^{(1)})+B_2V^{(1)}, \\ \varepsilon V^{(1)\prime}&=(D_0+\varepsilon D_1)V^{(1)}+C_2(U_0+U^{(1)})+D_2V^{(1)}, \\ \varepsilon U^{(2)\prime}&=\varepsilon A_1U^{(2)}+A_2U^{(2)}+B_2(V_0+V^{(2)}), \\ \varepsilon V^{(2)\prime}&=(D_0+\varepsilon D_1)V^{(2)}+D_2(V^{(0)}+V^{(2)})+C_2U^{(2)}, \end{aligned} \tag{4.4}$$

where $A_2=A-\varepsilon A_1$, $B_2=B$, $C_2=C$ and $D_2=D-D_0-\varepsilon D_1$.

Here we write down the explicit form of the matrix $U_0(x)$. If $q\geqq 2$, it can be written as

$$(4.5)\quad U_0(x)=\begin{bmatrix}(x-a_k) & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\varphi_{11}^{(k)}(x-a_k), & \varphi_{12}^{(k)}(x-a_k)\\ \varphi_{21}^{(k)}(x-a_k), & \varphi_{22}^{(k)}(x-a_k)\end{bmatrix}\begin{bmatrix}(x-a_k)^{\lambda_k} & c(x-a_k)^{\lambda_k}\log(x-a_k)\\ 0 & (x-a_k)^{\mu_k}\end{bmatrix},$$

where λ_k, μ_k are roots of the characteristic equation

$$\lambda^2+(a+1)\lambda+a+b=0$$

with

$$a=\lim_{x\to a_k}(x-a_k)\frac{p_{20}(x)}{p_{30}(x)},\qquad b=\lim_{x\to a_k}(x-a_k)^2\frac{p_{10}(x)}{p_{30}(x)}.$$

When $\lambda_k-\mu_k\neq$integer, c equals zero, but when $\lambda_k-\mu_k$ is an integer c may be 1 or 0, and we assume $\lambda_k-\mu_k\geqq 0$. The functions $\varphi_{ij}^{(k)}(x)$ $(i,j=1,2)$ are all convergent power series of $(x-a_k)$ in the neighborhood of a_k and

$$\det\begin{bmatrix}\varphi_{11}^{(k)}(0) & \varphi_{12}^{(k)}(0)\\ \varphi_{21}^{(k)}(0) & \varphi_{22}^{(k)}(0)\end{bmatrix}\neq 0.$$

If $q=1$, $U_0(x)$ has the same form as (4.5) except it does not take the first matrix along and λ_k, μ_k are roots of

$$\lambda(\lambda-1)+a\lambda=0.$$

The system of differential equation (4.4) is converted into the system of integral equations of the form

$$(4.7)\quad \begin{aligned}
U^{(1)}(x,\varepsilon)&=\varepsilon^{-1}U_0(x)\int^x U_0(\tau)^{-1}\{A_2(\tau,\varepsilon)(U_0(\tau)+U^{(1)}(\tau,\varepsilon))+B_2(\tau,\varepsilon)V^{(1)}(\tau,\varepsilon)\}d\tau,\\
V^{(1)}(x,\varepsilon)&=\varepsilon^{-1}V_0(x,\varepsilon)^{-1}\int^x V_0(\tau,\varepsilon)^{-1}\{C_2(\tau,\varepsilon)(U_0(\tau)+U^{(1)}(\tau,\varepsilon))+D_2(\tau,\varepsilon)V^{(1)}(\tau,\varepsilon)\}d\tau,\\
U^{(2)}(x,\varepsilon)&=\varepsilon^{-1}U_0(x)\int^x U_0(\tau)^{-1}\{A_2(\tau,\varepsilon)U^{(2)}(\tau,\varepsilon)+B_2(\tau,\varepsilon)(V_0(\tau,\varepsilon)+V^{(2)}(\tau,\varepsilon))\}d\tau,\\
V^{(2)}(x,\varepsilon)&=\varepsilon^{-1}V_0(x,\varepsilon)^{-1}\int^x V_0(\tau,\varepsilon)^{-1}\{D_2(\tau,\varepsilon)(V_0(\tau,\varepsilon)+V^{(2)}(\tau,\varepsilon))+C_2(\tau,\varepsilon)U^{(2)}(\tau,\varepsilon)\}d\tau.
\end{aligned}$$

If we put

$$(4.8)\quad \begin{aligned}
\Lambda(x,a_k)&=\begin{bmatrix}(x-a_k)^{\lambda_k} & c(x-a_k)^{\lambda_k}\log(x-a_k)\\ 0 & (x-a_k)^{\mu_k}\end{bmatrix},\\
U_0(x)&=\tilde{U}_0(x)\Lambda(x,a_k),\\
U^{(1)}(x,\varepsilon)&=\tilde{U}^{(1)}(x,\varepsilon)\Lambda(x,a_k),\\
V^{(1)}(x,\varepsilon)&=\tilde{V}^{(1)}(x,\varepsilon)\Lambda(x,a_k),\\
U^{(2)}(x,\varepsilon)&=\tilde{U}^{(2)}(x,\varepsilon)V_0(x,\varepsilon),\\
V^{(2)}(x,\varepsilon)&=\tilde{V}^{(2)}(x,\varepsilon)V_0(x,\varepsilon).
\end{aligned}$$

then (4.7) becomes

$$
\begin{aligned}
\tilde{U}^{(1)}(x,\varepsilon)&=\tilde{U}_0(x)\int^x \Lambda(x,a_k)\Lambda(\tau,a_k)^{-1}\tilde{U}_0(\tau)^{-1}\varepsilon^{-1}\\
&\quad\times\{A_2(\tau,\varepsilon)(\tilde{U}_0(\tau)+\tilde{U}^{(1)}(\tau,\varepsilon))+B_2(\tau,\varepsilon)\tilde{V}^{(1)}(\tau,\varepsilon)\}\Lambda(\tau,a_k)\Lambda(x,a_k)^{-1}d\tau,\\
\tilde{V}^{(1)}(x,\varepsilon)&=\int^x V_0(x,\varepsilon)V_0(\tau,\varepsilon)\varepsilon^{-1}\\
&\quad\times\{C_2(\tau,\varepsilon)(\tilde{U}_0(\tau)+\tilde{U}^{(1)}(\tau,\varepsilon))+D_2(\tau,\varepsilon)\tilde{V}^{(1)}(\tau,\varepsilon)\}\Lambda(\tau,a_k)\Lambda(x,a_k)^{-1}d\tau,\\
\tilde{U}^{(2)}(x,\varepsilon)&=\tilde{U}_0(x)\int^x \Lambda(x,a_k)\Lambda(\tau,a_k)^{-1}\varepsilon^{-1}\\
&\quad\times\{A_2(\tau,\varepsilon)\tilde{U}^{(2)}(\tau,\varepsilon)+B_2(\tau,\varepsilon)(E+\tilde{V}^{(2)}(\tau,\varepsilon))\}V_0(\tau,\varepsilon)V_0(x,\varepsilon)^{-1}d\tau,\\
\tilde{V}^{(2)}(x,\varepsilon)&=\int^x V_0(x,\varepsilon)V_0(\tau,\varepsilon)^{-1}\varepsilon^{-1}\\
&\quad\times\{D_2(\tau,\varepsilon)+D_2(\tau,\varepsilon)\tilde{V}^{(2)}(\tau,\varepsilon)+C_2(\tau,\varepsilon)\tilde{U}^{(2)}(\tau,\varepsilon)\}V_0(\tau,\varepsilon)V_0(x,\varepsilon)^{-1}d\tau,
\end{aligned}
\tag{4.9}
$$

where each integral is to be taken along an appropriate curve described in later.

We prove in the next section the existence of solution of the equation (4.9). To do so, the notion of canonical region with respect to $\xi(x,x_0)$ will be introduced following [1], where

$$\xi(x,x_0)=\int_{x_0}^x \sqrt{p_{30}(x)}\,dx.$$

The family of curves S: $\operatorname{Re}\xi(x,x_0)=\text{const.}$ does not depend on the initial point x_0, the choice of path of the integral in the complex x-plane or the determination of the square root of $p_{30}(x)$, and has branch points at turning points. We call the curves passing through turning points the Stokes curves, and these curves divide the x-plane into a finite or infinite number of simply connected unbounded regions: Stokes regions. The function $\xi(x,x_0)$ can be considered as the mapping of the x-plane into the ξ-plane which is univalent at all points of x except turning points, and each Stokes curve is mapped onto a straight segment or a ray parallel to the imaginary ξ-axis. Then the image of Stokes region is vertical strip or half plane.

The canonical region with respect to $\xi(x,x_0)$ is a union of an appropriate number of adjacent Stokes regions bounded by the Stokes curves, contains no turning point in its interior, and is mapped by $\xi(x,x_0)$ onto the whole ξ-plane cut by unbounded verticals. Various properties of the canonical region or examples are given in [1] and Wasow [12].

We denote the complex x-plane by X and the complex ξ-plane by Σ. Let $D[\xi]$ be one of the canonical regions and $D_M[\xi]=D[\xi]\cap\{x\in X,\ |x|\leqq M\}$. Suppose that $a_1, a_2, \cdots, a_m, b_1, b_2, \cdots, b_n$ be the set of turning points which are on the boundary of $D_M[\xi]$ and a Stokes curve from each a_j is going into the interior of $D_M[\xi]$. We take up a turning point a_k from a_j $(j=1,2,\cdots,m)$ and fix it. Now

we define a few type of regions which are obtained by small deformations of $D_M[\xi]$. Firstly

$$D_M[\xi, a_k]=D_M[\xi]\cap(X-N_{a_k}),$$

where N_{a_k} is the domain of influence at a_k defined in (3.9).

If this region $D_M[\xi, a_k]$ is transformed into the ξ-plane by $\xi(x, x_0)$, the image $\mathcal{D}_M[\xi, a_k]$ is a region bounded by a curve which consists of a part of an image of the boundary $|x|=M$ and vertical cuts issuing from $\xi(a_k, x_0)$, and removed the small neighborhood of $\xi(a_k, x_0)$: $\mathcal{N}_{a_k}$ that is an image of N_{a_k}.

Now we change the region $\mathcal{D}_M[\xi, a_k]$ into $\mathcal{D}_M[\xi, a_k, \gamma]$ by a small deformation with the following conditions:

There exist two points η^+ and η^- on the boundary of $\mathcal{D}_M[\xi, a_k, \gamma]$ such that for every point ξ in $\mathcal{D}_M[\xi, a_k, \gamma]$ we can describe two piecewise smooth curves $c^{(+)}(s, \xi, \eta^+)$ and $c^{(-)}(s, \xi, \eta^-)$ connecting ξ and η^+, η^- respectively and they satisfy

(1) $c^{(\pm)}(s, \xi, \eta^\pm)$ are contained in $\mathcal{D}_M[\xi, a_k, \gamma]$,

where s denotes the arc length,

(2) for some small positive constant γ,

$$\text{(4.10)}\qquad \begin{aligned} &\operatorname{Re}\frac{d\xi}{ds}>\gamma \quad \text{on} \quad c^{(+)}(s, \xi, \eta^+),\\ &\operatorname{Re}\frac{d\xi}{ds}<-\gamma \quad \text{on} \quad c^{(-)}(s, \xi, \eta^-).\end{aligned}$$

As the region $\mathcal{D}_M[\xi, a_k, \gamma]$ and the points $\eta^\pm$, we take here as follow. Let us put $\tan\varphi=\gamma/\sqrt{1-\gamma^2}$.

(1) If there exists one point on the boundary of $\mathcal{D}_M[\xi, a_k]$ which is on the most righthand side, we take this point as η^+, and if there are many such points, we fix one of these points as η^+.

(2) We describe the two lines issuing from η^+ with argments $\pi/2+\varphi$ and $-\pi/2-\varphi$, and cut off from $\mathcal{D}_M[\xi, a_k]$ the right hand parts of two lines, and if there exist regions which are in the exterior of $\mathcal{D}_M[\xi, a_k]$ and bounded by the boundary of $\mathcal{D}_M[\xi, a_k]$ and two lines, we add them to $\mathcal{D}_M[\xi, a_k]$. For sufficiently small γ, the above two lines are almost vertical line starting from η^+.

(3) By the same methods as in (1), (2), we choose the point η^- and cut off or add some portions from $\mathcal{D}_M[\xi, a_k]$.

(4) In the neighborhoods of $\xi(a_k, x_0)$ for which we suppose that the vertical cut is directed downward, we firstly draw concentric circles C and C' around $\xi(a_k, x_0)$ whose radii are $N'\varepsilon$ and ρ' respectively, where N' is nearly $N^{(q+2)/2}$ and ρ' nearly $\rho^{(q+2)/2}$ of (3.5). Moreover write two segments L_1 and L_2 starting from $\xi(a_k, x_0)$ with arguments φ and $\pi-\varphi$ respectively. Let P_1, Q_1 be cross points of L_1 with C and C', and P_2 and Q_2 be cross points of L_2 with C and C' respectively. From P_1 we draw a segment of argument $-\pi/2+2\varphi$ to the cross point R_1 with C', and from R_1 continue a line of argument $-\pi/2+\varphi$ to the boundary of $\mathcal{D}_M[\xi, a_k]$, and denote this polygonal segment by l_1. Analogously we draw a poly-

gonal segment l_2. We delete from $\mathcal{D}_M[\xi, a_k]$ the neighborhood of vertical cut, that is, a region surrounded by l_1, upper circle P_1P_2, l_2 and the boundary of $\mathcal{D}_M[\xi, a_k]$ (Fig. 1).

(5) For each turning point $a_j (j \neq k)$, we draw in the ξ-plane a circle C' of radius ρ' around the $\xi(a_j, x_0)$ and let P_1, P_2 be the same points as in (4). From P_1 we draw a line l_1 of argument $-\pi/2+\varphi$ to the boundary of $\mathcal{D}_M[\xi, a_k]$, and also from P_2 a line l_2 of argument $-\pi/2-\varphi$. Then from $\mathcal{D}_M[\xi, a_k]$ we delete a region bounded by l_1, l_2 and the part of circle C'.

Thus we obtained the two points $\eta^{\pm}$ and region $\mathcal{D}_M[\xi, a_k, \gamma]$ by performing the above procedures. Next we define the curve $c^{(+)}(s, \xi, \eta^+)$ for every $\xi \in \mathcal{D}_M[\xi, a_k, \gamma]$. Clearly the curve $c^{(-)}(s, \xi, \eta^-)$ can be defined by the same method (Fig. 1).

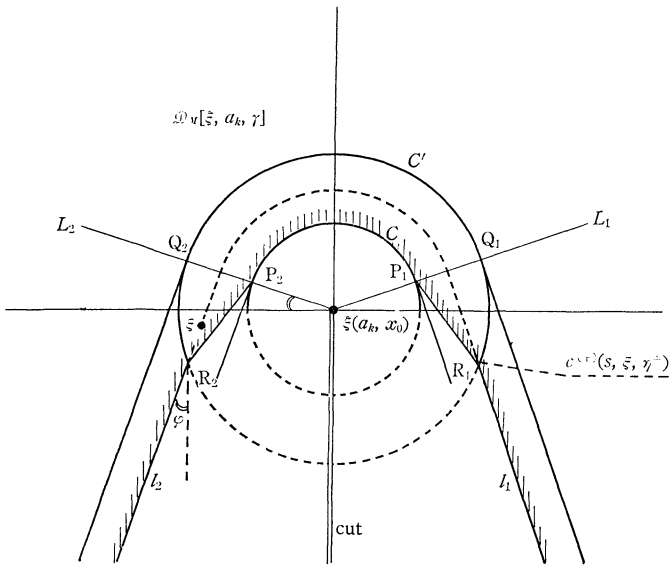

Fig. 1

(1) Let

$$\mathcal{D}[a_k, \gamma] = \{\xi\colon N'\varepsilon \leqq |\xi - \xi(a_k, x_0)| \leqq \rho'\} \cap \{\mathcal{D}_M[\xi, a_k, \gamma]\}$$

and if vertical cut issuing from $\xi(a_k, x_0)$ is directed downward, draw the line of argument $-\pi/2-\varphi$ from the point Q_2, and denote by $\mathcal{D}_M^{(+)}[\xi, a_k, \gamma]$ the region bounded by the above line, the circle C' and the boundary of $\mathcal{D}_M[\xi, a_k, \gamma]$. Then the region $\mathcal{D}_M[\xi, a_k, \gamma]$ is divided into the several subregions such that

$$\mathcal{D}_M[\xi, a_k, \gamma] = \mathcal{D}[a_k, \gamma] \cup \mathcal{D}_M^{(+)}[\xi, a_k, \gamma] \cup \mathcal{D}'_M[\xi, a_k, \gamma],$$

where

$$\mathcal{D}'_M[\xi, a_k, \gamma] = \mathcal{D}_M[\xi, a_k, \gamma] - \mathcal{D}[a_k, \gamma] - \mathcal{D}_M^{(+)}[\xi, a_k, \gamma].$$

(2) For ξ in $\mathcal{D}'_M[\xi, a_k, \gamma]$, we can easily draw a curve $c^{(+)}(s, \xi, \eta^+)$ contained in $\mathcal{D}'_M[\xi, a_k, \gamma]$ with the desired property by at least connecting several segments, owing to the shape of $\mathcal{D}_M[\xi, a_k, \gamma]$.

(3) Let us divide $\mathcal{D}[a_k, \gamma]$ into three parts $\mathcal{D}^{(1)}[a_k, \gamma]$, $\mathcal{D}^{(2)}[a_k, \gamma]$ and $\mathcal{D}^{(3)}[a_k, \gamma]$, where $\mathcal{D}^{(1)}[a_k, \gamma]$ is a part of $\mathcal{D}[a_k, \gamma]$ below the segment L_1, $\mathcal{D}^{(2)}[a_k, \gamma]$ between L_1 and L_2, and $\mathcal{D}^{(3)}[a_k, \gamma]$ below L_2.

(4) For ξ in $\mathcal{D}^{(1)}[a_k, \gamma]$, we draw downward a segment of argment $-\pi/2+\varphi$ from ξ to the point of C' or to the point on the segment P_1R_1. In the former case we connect it with a curve defined in (2), and in the latter case we continue it along P_1R_1 and then combine it with one described in (2).

(5) For ξ in $\mathcal{D}^{(2)}[a_k, \gamma]$, $c^{(+)}(s, \xi, \eta^+)$ is a curve along the circle of radius $|\xi-\xi(a_k, x_0)|$ from ξ to the point on L_1 and connect it with a curve stated in (4).

(6) For ξ in $\mathcal{D}^{(3)}[a_k, \gamma]$ and $\mathcal{D}^{(+)}_M[\xi, a_k, \gamma]$, $c^{(+)}(s, \xi, \eta^+)$ consists of the segment of argument $\pi/2-\varphi$ from ξ to the point on L_2 and a connected curve described in (5).

As a region of admissibility we take the inverse image of $\mathcal{D}_M[\xi, a_k, \gamma]$ under the mapping $\xi=\xi(x, x_0)$ and denote it by $D_M[x, a_k, \gamma]$, and also we denote the inverse images of $\mathcal{D}[a_k, \gamma]$, two points $\eta^\pm$ and two curves $c^{(\pm)}(s, \xi, \eta)$ by $D[a_k, \gamma]$, $x_0^\pm$ and $c^{(\pm)}(s, x, x_0)$ respectively.

From the above construction, we can prove the following two lemmas.

LEMMA 4.1. *If we put*

$$\xi(x, x_0, \varepsilon)=\int_{x_0}^{x} \sqrt{p_{30}(x)}\left(1+\varepsilon\frac{p_{31}(x)}{p_{30}(x)}\right)dx,$$

then we have

$$\begin{aligned} &\frac{d\,\mathrm{Re}\,\xi(x, x_0, \varepsilon)}{ds} \geqq 0 \qquad \textit{along} \quad c^{(+)}(s, x, x_0), \\ &\frac{d\,\mathrm{Re}\,\xi(x, x_0, \varepsilon)}{ds} \leqq 0 \qquad \textit{along} \quad c^{(-)}(s, x, x_0). \end{aligned} \tag{4.11}$$

Proof. Since

$$\frac{d\xi(x, x_0)}{ds} = \sqrt{p_{30}(x)}\,\frac{dx}{ds},$$

$$\frac{d\xi(x, x_0, \varepsilon)}{ds} = \frac{d\xi(x, x_0)}{ds}\left(1+\varepsilon\frac{p_{31}(x)}{p_{30}(x)}\right),$$

$$\frac{d\,\mathrm{Re}\,\xi(x, x_0, \varepsilon)}{ds} = \frac{d\,\mathrm{Re}\,\xi(x, x_0)}{ds}\left(1+\mathrm{Re}\,\frac{\varepsilon p_{31}(x)}{p_{30}(x)}\right)-\mathrm{Im}\,\frac{d\xi(x, x_0)}{ds}\,\mathrm{Im}\,\frac{\varepsilon p_{31}(x)}{p_{30}(x)},$$

then if we take $|\varepsilon p_{31}(x)/p_{30}(x)|$ sufficiently small, and by using the condition (4.10), the desired inequality (4.11) clearly follows. The magnitude $|\varepsilon p_{31}(x)/p_{30}(x)|$ can be as small as we please by taking ε_0 sufficiently small and N sufficiently large in the definition (3.9).

LEMMA 4.2. *For all ξ in $\mathcal{D}'_M[\xi, a_k, \gamma]$ or $\mathcal{D}^{(+)}_M[\xi, a_k, \gamma]$, there exists a constant K such that*

$$\left|\int_{c^{(+)}(s,\xi,\eta^+)} d\eta\right| \leqq K, \tag{4.12}$$

and for ξ in $\mathcal{D}[a_k, \gamma]$,

$$\int_{c^{(+)}(s,\xi,\eta^+)} |\eta-\xi(a_k, x_0)|^{-r}|d\eta| \leqq K|\xi-\xi(a_k, x_0)|^{-r+1} \qquad (r \neq 1). \tag{4.13}$$

Analogous inequalities hold when the integrals are taken along $c^{(-)}(s, \xi, \eta^-)$.

Proof. (1) For ξ in $\mathcal{D}'_M[\xi, a_k, \gamma]$ the inequality (4.12) is clear since the region $\mathcal{D}_M[\xi, a_k, \gamma]$ is bounded.

(2) Let ξ be in $\mathcal{D}^{(1)}[a_k, \gamma]$ and $\xi-\xi(a_k, x_0)=|\xi-\xi(a_k, x_0)|e^{i\psi}$. The integral path $c^{(+)}(s, \xi, \eta^+)$ consists of possibly three parts, $c_1^{(+)}(s, \xi, \eta^+)$, $c_2^{(+)}(s, \xi, \eta^+)$ and $c_3^{(+)}(s, \xi, \eta^+)$. Here $c_1^{(+)}(s, \xi, \eta^+)$ is in the exterior of $\mathcal{D}[a_k, \gamma]$, $c_2^{(+)}(s, \xi, \eta^+)$ is a straight segment of argument $-\pi/2+\varphi$ from ξ to a point S_ξ which is on the C' or P_1R_1, and if S_ξ is on P_1R_1, the $c_3^{(+)}(s, \xi, \eta^+)$ is a segment $S_\xi R_1$. Now we estimate the integral (4.13) for each parts.

Firstly we have

$$\int_{c_1^{(+)}(s,\xi,\eta^+)} |\eta-\xi(a_k, x_0)|^{-r}|d\eta| \leqq \rho'^{-r}\int_{c_1^{(+)}(s,\xi,\eta^+)} |ds| \leqq K'$$

for some constant K'.

Next, $c_2^{(+)}(s, \xi, \eta^+)$ can be written as

$$\eta-\xi(a_k, x_0)=\xi-\xi(a_k, x_0)+se^{-i(\pi/2-\varphi)},$$

$$s=|\xi-\xi(a_k, x_0)|\{\cos(\varphi-\psi)\tan(\varphi-\psi+\theta)-\sin(\varphi-\psi)\},$$

where

$$-\frac{\pi}{2}+2\varphi \leqq \psi \leqq \varphi, \qquad 0 \leqq \varphi-\psi \leqq \frac{\pi}{2}-\varphi,$$

$$0 \leqq \theta \leqq \frac{\pi}{2}-\varphi-(\varphi-\psi), \qquad 0 \leqq \varphi-\psi+\theta \leqq \frac{\pi}{2}-\varphi.$$

Then we have

$$\begin{aligned} |\eta-\xi(a_k, x_0)| &\geqq |\xi-\xi(a_k, x_0)|\cos(\varphi-\psi) \\ &\geqq |\xi-\xi(a_k, x_0)|\sin\varphi=\gamma|\xi-\xi(a_k, x_0)|, \end{aligned}$$

$$ds=|\xi-\xi(a_k, x_0)|\frac{\cos(\varphi-\psi)}{\cos^2(\varphi-\psi+\theta)}d\theta.$$

Thus we have

$$\int_{c_2^{(+)}(s,\xi,\eta^+)} |\eta-\xi(a_k,x_0)|^{-r}|d\eta| \leqq \gamma^{-r}|\xi-\xi(a_k,x_0)|^{-r+1}\int \frac{|\cos(\varphi-\theta)|d\theta}{\cos^2(\varphi-\psi+\theta)}$$
$$\leqq K''|\xi-\xi(a_k,x_0)|^{-r+1}.$$

Lastly we consider the contribution of $c_3^{(+)}(s,\xi,\eta^+)$. On this segment

$$ds \leqq \frac{d|\eta-\xi(a_k,x_0)|}{\cos(\pi/2-\varphi)} = \gamma^{-1}d|\eta-\xi(a_k,x_0)|,$$

and we have

$$|S_\xi-\xi(a_k,x_0)| = |\xi-\xi(a_k,x_0)|\cos(\varphi-\psi)\tan\psi_\xi,$$

where ψ_ξ is a certain argument satisfying $0<\psi_\xi<\pi/2-2\varphi$. Then we have

$$\int_{c_3^{(+)}(s,\xi,\eta^+)} |\eta-\xi(a_k,x_0)|^{-r}|d\eta| \leqq \gamma\int_{|S_\xi-\xi(a_k,x_0)|}^{\rho'} |\eta-\xi(a_k,x_0)|^{-r}d|\eta-\xi(a_k,x_0)|$$
$$\leqq \frac{\gamma}{|1-r|}|S_\xi-\xi(a_k,x_0)|^{-r+1}+\frac{\gamma\rho'^{-r+1}}{|1-r|} \leqq K'''|\xi-\xi(a_k,x_0)|^{-r+1}.$$

Thus by adding the above three estimates, we obtained the desired inequality.

(3) For ξ in $\mathcal{D}^{(2)}[a_k,\gamma]$, we denote by $c_1^{(+)}(s,\xi,\eta^+)$ the part of $c^{(+)}(s,\xi,\eta^+)$ in the interior of $\mathcal{D}^{(2)}[a_k,\gamma]$, and by $c_2^{(+)}(s,\xi,\eta^+)$ remaining part. From (2),

$$\int_{c_2^{(+)}(s,\xi,\eta^+)} |\eta-\xi(a_k,x_0)|^{-r}|d\eta| \leqq K'|\xi-\xi(a_k,x_0)|^{-r+1},$$

for some constant K'. On the other hand $c_1^{(+)}(s,\xi,\eta^+)$ can be written as

$$\eta-\xi(a_k,x_0) = |\xi-\xi(a_k,x_0)|e^{i\theta} \qquad (\psi \leqq \theta < \varphi),$$
$$ds = |\xi-\xi(a_k,x_0)|d\theta,$$

then we have

$$\int_{c_1^{(+)}(s,\xi,\eta^+)} |\eta-\xi(a_k,x_0)|^{-r}|d\eta| \leqq |\xi-\xi(a_k,x_0)|^{-r+1}\int_\psi^\varphi d\theta \leqq \pi|\xi-\xi(a_k,x_0)|^{-r+1}.$$

Thus by adding the above two inequalities, we obtained the inequality (4.13).

(4) For ξ in $\mathcal{D}^{(3)}[a_k,\gamma]$, the contributions of the integral from the path in the $\mathcal{D}^{(3)}[a_k,\gamma]$ is obtained by the same method as (2), and the contribution from other part is obtained from (3). Thus proved the desired inequality for this case.

(5) Finally for ξ in $\mathcal{D}_M^{(+)}[\xi,a_k,\gamma]$, the inequality (4.12) can be proved easily by combining the above procedures and we omit them here.

§ 5. Existence theorem.

We prove in this section the main existence theorem. At first the system of integral equation (4.9) will be considered in the region $D_M[x,a_k,\gamma]$ defined in the

§ 4 and let a_k be one of the turning points a_j $(j=1, 2, \cdots, m)$ and fix it throughout this section. Here the matrix $U_0(x)$, which was defined only in the neighborhood of the turning point a_k, is to be considered as defined in the region $D_M[x, a_k, \gamma]$ by an analytic continuation.

The system (4.9) is splitting into two subsystems, that is, a system for $\tilde{U}^{(1)}(x, \varepsilon)$, $\tilde{V}^{(1)}(x, \varepsilon)$, and a system for $\tilde{U}^{(2)}(x, \varepsilon)$, $\tilde{V}^{(2)}(x, \varepsilon)$:

$$
(5.1)\quad \begin{cases} \tilde{U}^{(1)}(x, \varepsilon) = \tilde{U}_0(x)\displaystyle\int^x \Lambda(x, a_k)\Lambda(\tau, a_k)^{-1}\tilde{U}_0(\tau)^{-1}\varepsilon^{-1} \\ \qquad \times\{A_2(\tau, \varepsilon)(\tilde{U}_0(\tau)+\tilde{U}^{(1)}(\tau, \varepsilon))+B_2(\tau, \varepsilon)\tilde{V}^{(1)}(\tau, \varepsilon)\}\Lambda(\tau, a_k)\Lambda(x, a_k)^{-1}d\tau, \\ \tilde{V}^{(1)}(x, \varepsilon) = \displaystyle\int^x V_0(x, \varepsilon)V_0(\tau, \varepsilon)^{-1}\varepsilon^{-1} \\ \qquad \times\{C_2(\tau, \varepsilon)(\tilde{U}_0(\tau)+\tilde{U}^{(1)}(\tau, \varepsilon))+D_2(\tau, \varepsilon)\tilde{V}^{(1)}(\tau, \varepsilon)\}\Lambda(\tau, a_k)\Lambda(x, a_k)^{-1}d\tau, \end{cases}
$$

$$
(5.2)\quad \begin{cases} \tilde{U}^{(2)}(x, \varepsilon) = \tilde{U}_0(x)\displaystyle\int^x \Lambda(x, a_k)\Lambda(\tau, a_k)^{-1}\tilde{U}_0(\tau)^{-1}\varepsilon^{-1} \\ \qquad \times\{A_2(\tau, \varepsilon)\tilde{U}^{(2)}(\tau, \varepsilon)+B_2(\tau, \varepsilon)(E+\tilde{V}^{(2)}(\tau, \varepsilon))\}V_0(\tau, \varepsilon)V_0(x, \varepsilon)^{-1}d\tau, \\ \tilde{V}^{(2)}(x, \varepsilon) = \displaystyle\int^x V_0(x, \varepsilon)V_0(\tau, \varepsilon)^{-1}\varepsilon^{-1} \\ \qquad \times\{D_2(\tau, \varepsilon)(E+\tilde{V}^{(2)}(\tau, \varepsilon))+C_2(\tau, \varepsilon)\tilde{U}^{(2)}(\tau, \varepsilon)\}V_0(\tau, \varepsilon)V_0(x, \varepsilon)^{-1}d\tau. \end{cases}
$$

We prove at first the following lemma.

LEMMA 5.1. *The system of integral equations* (5. *i*) $(i=1, 2)$ *have solutions* $\tilde{U}^{(i)}(x, \varepsilon)$, $\tilde{V}^{(i)}(x, \varepsilon)$ *for which there exists a positive constant* K *such that* (1), *for* $q\geqq 2$, *and for* x *in* $D'_M[x, a_k, \gamma]$, *where* $D'_M[x, a_k, \gamma]=D_M[x, a_k, \gamma]-D[a_k, \gamma]$,

$$
(5.3)\quad \begin{aligned} &\|\tilde{U}^{(i)}(x, \varepsilon)\|\leqq K\varepsilon, \\ &\|\tilde{V}^{(i)}(x, \varepsilon)\|\leqq K\varepsilon, \end{aligned}
$$

for x *in* $D[a_k, \gamma]$,

$$
(5.4)\quad \begin{aligned} &\left\|\begin{bmatrix} (x-a_k)^{-1} & 0 \\ 0 & 1 \end{bmatrix}\tilde{U}^{(i)}(x, \varepsilon)\right\|\leqq K|x-a_k|^{-(q/2+1)}\varepsilon, \\ &\|(x-a_k)\tilde{V}^{(i)}(x, \varepsilon)\|\leqq K|x-a_k|^{-(q/2+1)}\varepsilon, \end{aligned}
$$

(2), *for* $q=1$, *and for* x *in* $D'_M[x, a_k, \gamma]$,

$$
(5.3)'\quad \begin{aligned} &\|\tilde{U}^{(i)}(x, \varepsilon)\|\leqq K\varepsilon, \\ &\|\tilde{V}^{(i)}(x, \varepsilon)\|\leqq K\varepsilon, \end{aligned}
$$

for x *in* $D[a_k, \gamma]$,

$$(5.4)' \qquad \begin{aligned} &\|\tilde{U}^{(i)}(x,\varepsilon)\| \leqq K|x-a_k|^{-3/2}\varepsilon, \\ &\|(x-a_k)\tilde{V}^{(i)}(x,\varepsilon)\| \leqq K|x-a_k|^{-3/2}\varepsilon. \end{aligned}$$

Proof. We treat only the case $i=1$, $q\geqq 2$. And the same method will be able to apply to other cases. If we write down (5.1) for each component, some of them contain exponential function coming from $V_0(x,\varepsilon)V_0(\tau,\varepsilon)^{-1}$. Suppose that one of them carries the function $\exp\xi(x,\tau,\varepsilon)$ or $\exp\{-\xi(x,\tau,\varepsilon)\}$, then we choose the curve $c^{(+)}(s,x,x_0^+)$ or $c^{(-)}(s,x,x_0^-)$ as the integral path of (5.1), and if it does not carry the exponential function, the integral path is one of $c^{(\pm)}(s,x,x_0^\pm)$, where

$$\exp\xi(x,\tau,\varepsilon)=\exp\int_\tau^x \sqrt{p_{30}(x)}\left(1+\frac{\varepsilon p_{31}(x)}{p_{30}(x)}\right)dx.$$

From the Lemma 4.1, the exponential function is bounded in each integral path.

Let $\tilde{U}^{(1)}(x,\varepsilon)$ and $\tilde{V}^{(1)}(x,\varepsilon)$ are matrices as in the Lemma 5.1, and let $T\tilde{U}^{(1)}$ and $T\tilde{V}^{(1)}$ be the integral expressions of right hand side of (5.1). T can be considered as a mapping of $\tilde{U}^{(1)}(x,\varepsilon)$ and $\tilde{V}^{(1)}(x,\varepsilon)$. Now we prove that the mapping T has a contraction property from which the Lemma 5.1 can be concluded by the standard methods such as methods of fix point or successive approximations.

Firstly let x be in $D'_M[x,a_k,\gamma]$. Then from (3.7) and the Lemma 4.2 we have

$$(5.5) \qquad \|T\tilde{U}^{(1)}\|,\ \|T\tilde{V}^{(1)}\| \leqq L'\varepsilon+L'K\varepsilon^2,$$

for some constant L', where L' denotes unspecified constant in the followings. For x in $D[a_k,\gamma]$, the following inequality is valid from the Lemma (3.1);

$$\|\tilde{U}_0(\tau)^{-1}\varepsilon^{-1}\{A_2(\tau,\varepsilon)(\tilde{U}_0(\tau)+\tilde{U}^{(1)}(\tau,\varepsilon))+B_2(\tau,\varepsilon)\tilde{V}^{(1)}(\tau,\varepsilon)\|\leqq\{L'\varepsilon+L'K\varepsilon^2\}|\tau-a_k|^{-(q/2+2)}.$$

Since we have

$$\Lambda(x,a_k)\Lambda(\tau,a_k)^{-1}=\begin{bmatrix} \left(\dfrac{x-a_k}{\tau-a_k}\right)^{\lambda_k} & c\,\dfrac{(x-a_k)^{\lambda_k}}{(\tau-a_k)^{\mu_k}}\log\dfrac{x-a_k}{\tau-a_k} \\ 0 & \left(\dfrac{x-a_k}{\tau-a_k}\right)^{\mu_k} \end{bmatrix},$$

$T\tilde{U}^{(1)}(x,\varepsilon)$ consists of linear combinations of

$$(5.6) \qquad \begin{aligned} &\int\varphi(x,\tau,\varepsilon)\left(\frac{x-a_k}{\tau-a_k}\right)^{\pm(\lambda_k-\mu_k)}d\tau, \\ &c\int\varphi(x,\tau,\varepsilon)(\tau-a_k)^{\lambda_k-\mu_k}\log\frac{x-a_k}{\tau-a_k}d\tau, \\ &c\int\varphi(x,\tau,\varepsilon)(x-a_k)^{\lambda_k-\mu_k}\log\frac{x-a_k}{\tau-a_k}d\tau, \\ &c\int\varphi(x,\tau,\varepsilon)\left\{\log\frac{x-a_k}{\tau-a_k}\right\}^2d\tau \end{aligned}$$

where $\varphi(x,\tau,\varepsilon)$ can be written

$$\varphi(x, \tau, \varepsilon)=\tilde{\varphi}(x, \tau, \varepsilon)(\tau-a_k)^{-(q/2+2)}$$

with holomorphic function $\tilde{\varphi}(x, \tau, \varepsilon)$ satisfying

$$|\tilde{\varphi}(x, \tau, \varepsilon)| \leqq L'\varepsilon+L'K\varepsilon^2.$$

We assume here that none of the quantities $-(q/2+2)\pm(\lambda_k-\mu_k)$ equal -1 for simplicity. Then by using the integration by parts and the Lemma 4.2, each term of (5.6) can be estimated by

$$\left\|\int \cdot\, d\tau\right\| \leqq (L'\varepsilon+L'K\varepsilon^2)|x-a_k|^{-(q/2+1)}.$$

Therefore we have for some constant L

$$\left\|\begin{bmatrix} (x-a_k)^{-1} & 0 \\ 0 & 1 \end{bmatrix} T\tilde{U}^{(1)}(x, \varepsilon)\right\| \leqq (L\varepsilon+LK\varepsilon^2)|x-a_k|^{-(q/2+1)}. \tag{5.7}$$

By the same method we can deduce

$$\|(x-a_k)T\tilde{V}^{(1)}(x, \varepsilon)\| \leqq (L\varepsilon+LK\varepsilon^2)|x-a_k|^{-(q/2+1)}. \tag{5.8}$$

Hence if ε and $|(x-a_k)^{-(q/2+1)}\varepsilon|$ are limited sufficiently small, that is, the constants ε_0 and N^{-1} in (3.5) are taken sufficiently small, and if K is taken large enough, we have

$$L\varepsilon+LK\varepsilon^2 \leqq K\varepsilon,$$

$$(L\varepsilon+LK\varepsilon^2)|x-a_k|^{-(q/2+1)} \leqq K|x-a_k|^{-(q/2+1)}\varepsilon,$$

from which we can conclude that the mappung T has a contraction property, and then the Lemma is proved.

From the Lemma 5.1 and (4.8), the system of integral equations (4.7) and then the system of differential equations (4.4) have a solutions as follow.

THEOREM 5.1. *The system of differential equations* (4.4) *has a system of solutions* $W(x, \varepsilon)$ *in the region* $D_M[x, a_k, \gamma]$ *such that for* x *in* $D_M[x, a_k, \gamma]$,

$$\begin{aligned} &\|U^{(i)}(x, \varepsilon)\| \leqq K\varepsilon, \\ &\|V^{(i)}(x, \varepsilon)V_0(x, \varepsilon)^{-1}\| \leqq K\varepsilon \end{aligned} \qquad (i=1, 2), \tag{5.9}$$

and for x *in* $D[a_k, \gamma]$,

$$\begin{aligned} &\left\|\begin{bmatrix} (x-a_k)^{\chi(q)} & 0 \\ 0 & 1 \end{bmatrix} U^{(1)}(x, \varepsilon)\Lambda(x, a_k)^{-1}\right\| \leqq K|x-a_k|^{-(q/2+1)}\varepsilon, \\ &\|(x-a_k)V^{(1)}(x, \varepsilon)\Lambda(x, a_k)^{-1}\| \leqq K|x-a_k|^{-(q/2+1)}\varepsilon, \\ &\left\|\begin{bmatrix} (x-a_k)^{\chi(q)} & 0 \\ 0 & 1 \end{bmatrix} U^{(2)}(x, \varepsilon)V_0(x, \varepsilon)^{-1}\right\| \leqq K|x-a_k|^{-(q/2+1)}\varepsilon, \\ &\|(x-a_k)V^{(2)}(x, \varepsilon)V_0(x, \varepsilon)^{-1}\| \leqq K|x-a_k|^{-(q/2+1)}\varepsilon, \end{aligned} \tag{5.10}$$

for some positive constant K, and $\chi(q)=-1$ *if* $q\geqq 2$, $\chi(q)=0$ *if* $q=1$.

The fundamental system of solutions of the original system (2.1) can be obtained from Theorem 5.1 by multiplying the transformations 1, 2 and 3 defined in § 2:

$$Y(x,\varepsilon)=\Omega(x)\begin{bmatrix} E+\varepsilon QR & \varepsilon Q \\ R & E \end{bmatrix}\begin{bmatrix} E & 0 \\ 0 & \tilde{T}_3T_3 \end{bmatrix}[W_0(x,\varepsilon)+W(x,\varepsilon)], \tag{5.11}$$

where $W_0(x,\varepsilon)$ is defined in (4.1).

Let us put

$$Y_0(x,\varepsilon)=\Omega(x)\begin{bmatrix} E & 0 \\ R_0(x) & E \end{bmatrix}\begin{bmatrix} E & 0 \\ 0 & \tilde{T}_3 \end{bmatrix}W_0(x,\varepsilon), \tag{5.12}$$

$$\widetilde{W}(x,\varepsilon)=\Omega(x)^{-1}[Y(x,\varepsilon)-Y_0(x,\varepsilon)]\begin{bmatrix} \Lambda(x,a_k) & 0 \\ 0 & V_0(x,\varepsilon) \end{bmatrix}^{-1}. \tag{5.13}$$

Here we can estimate the above $\widetilde{W}(x,\varepsilon)$ from the Theorem 5.1 and the order estimate of transformations at the turning point a_k.

THEOREM 5.2. *There exists a positive constant K such that*

$$\|\widetilde{W}(x,\varepsilon)\|\leqq K\varepsilon \qquad \textit{for } x \textit{ in } D'_M[x,a_k,\gamma],$$

$$\left\|\begin{bmatrix} E & 0 \\ 0 & (x-a_k)E \end{bmatrix}\widetilde{W}(x,\varepsilon)\right\|\leqq K|x-a_k|^{-(q/2+1)}\varepsilon \qquad \textit{for } x \textit{ in } D[a_k,\gamma].$$

Proof. The first inequality is trivial and so we consider the second one. From (4.1), (4.8), (5.11) and (5.12), we have

$$\widetilde{W}(x,\varepsilon)=\begin{bmatrix} \varepsilon QR & \varepsilon Q\tilde{T}_3T_3 \\ \varepsilon R_1 & \tilde{T}_3(T_3-E) \end{bmatrix}\begin{bmatrix} \tilde{U}_0(x) & 0 \\ 0 & E \end{bmatrix}+\begin{bmatrix} E+\varepsilon QR & \varepsilon Q\tilde{T}_3T_3 \\ R & \tilde{T}_3T_3 \end{bmatrix}\begin{bmatrix} \tilde{U}^{(1)} & \tilde{U}^{(2)} \\ \tilde{V}^{(1)} & \tilde{V}^{(2)} \end{bmatrix}. \tag{5.14}$$

By considering the order estimation made in § 3, especially for $R(x,\varepsilon)$, we have

$$\|\varepsilon Q(x)R(x,\varepsilon)\tilde{U}_0(x)\|\leqq K'\{|x-a_k|^{-(q/2+1)}\varepsilon\}^2,$$

$$\|(x-a_k)\varepsilon R_1(x)\tilde{U}_0(x)\|\leqq K'|x-a_k|^{-(q/2+1)}\varepsilon.$$

And also from the Lemma 5.1,

$$\begin{aligned} \|\varepsilon Q(x)R(x,\varepsilon)\tilde{U}^{(i)}(x,\varepsilon)\| &\leqq K'\{|x-a_k|^{-(q/2+1)}\varepsilon\}^2, \\ \|\varepsilon Q(x)\tilde{T}_3T_3(x)\tilde{V}^{(i)}(x,\varepsilon)\| &\leqq K'|x-a_k|^{-(q/2+1)}\varepsilon, \\ \|(x-a_k)R(x,\varepsilon)\tilde{U}^{(i)}(x,\varepsilon)\| &\leqq K'|x-a_k|^{-(q/2+1)}\varepsilon, \\ \|(x-a_k)\tilde{T}_3T_3(x)\tilde{V}^{(i)}(x,\varepsilon)\| &\leqq K'|x-a_k|^{-(q/2+1)}\varepsilon \end{aligned} \qquad (i=1,2)$$

for some constant K'. From these inequalities and (5.14) the Theorem is straightforward.

As a conclusion, we would like to say that our theory gives the rigorous mathematical foundations on the W-K-B approximation used frequently in treating the Orr-Sommerfeld equations, and makes a small progress in the sense that the domain of existence is extended in the neighborhood of a turning point by introducing the domain of influence. There remain problems such as the connection problem, error analysis and application of our theory, and these will be treated in future.

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