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TENSOR FIELDS AND CONNECTIONS ON A CROSS-SECTION IN THE TANGENT BUNDLE OF ORDER r

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§ 0. Introduction.

Let M be an n-dimensional differentiable manifold and $T_r(M)$ the tangent bundle of order r over M, $r \geqq 1$ being an integer [1], [3], [4]. The prolongations of tensor fields and connections given in the differentiable manifold M to its tangent bundle of order r have been studied in [1], [2], [3] [4], [7], [8] and [9]. If V is a vector field given in M, V determines a cross-section in $T_r(M)$. For the cases $r=1$ and $r=2$, Yano [7] and Tani [5] have studied, on the cross-section determined by a vector field V, the behavior of the prolongations of tensor fields and connections in M to $T(M)$ (i.e., $T_1(M)$) and $T_2(M)$, respectively. The purpose of this paper is to study, on the cross-section determined by a vector field V, the behavior of the prolongations of these geometric objects in M to $T_r(M)$ ($r \geqq 1$).

In § 1 we summarize the results and properties we need concerning the prolongations of tensor fields and connections in M to $T_r(M)$. Proofs of the statements in § 1 can be found in [1], [2], [3], [4] and [8]. In § 2 we study the cross-section determined in $T_r(M)$ by a given vector field V in M. In § 3 we study the behavior of prolongations of tensor fields on the cross-section. In § 4 we study the prolongations of connections given in M to $T_r(M)$ along the cross-section and some of their properties.

We assume in the squel that the manifolds, functions, tensor fields and connections under consideration are all of differentiability of class C^∞. Several kinds of indices are used as follows: The indices $\lambda, \mu, \nu, \cdots, s, t, u, \cdots$ run through the range $0, 1, 2, \cdots r$; the indices $h, i, j, k, m, \cdots$ run through the range $1, 2, \cdots n$. Double indices like $(\nu)h$ are used, where $0 \leqq \nu \leqq r$, $1 \leqq h \leqq n$. The indices $A, B, C, \cdots$ run through the range $(1)1, (1)2, \cdots, (1)n, (2)1, \cdots, (2)n, \cdots, (r)1, \cdots, (r)n$. For a given function f on M, the notation $f^{(0)}$ is sometimes substituted by f^0 for simplicity. Summation notation $\sum_{i=1}^n$ with respect to $h, i, j, k, m, \cdots$ $(=1, 2, \cdots n)$ is omitted while summation notation with respect to $\lambda, \mu, \nu, \cdots, s, t, u \cdots$, from 0 to r, will be kept. For example,

$$\sum_{s=0}^{r} \sum_{h=1}^{n} \binom{r}{s} \mathcal{L}_V^s \nabla_j x^h B_{(s)h} \quad \text{will be written in} \quad \sum_{s=0}^{r} \binom{r}{s} \mathcal{L}_V^s \nabla_j x^h B_{(s)h}.$$

For differentiable manifold N, we denote by $\mathcal{T}_q^p(N)$ the space of all tensor

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fields of type (p, q), i.e., of contravariant degree p and covariant degree q $(p, q\geqq 0)$ and put

$$\mathcal{T}(N)=\sum_{p,q}\mathcal{T}_p^q(N).$$

§1. Prolongations of tensor fields and connections to $T_r(M)$.

Let R be the real line. $T_r(M)$ is the set of all r-jets $J_P^r(F)$ determined by a mapping $F: R\to M$ such that $F(0)=P$. We denote by $\pi_r: T_r(M)\to M$ the bundle projection, i.e., $\pi_r(J_P^r(F))=P$. We shall denote π_r simply by π if there is no confusion. Let $\{U, x^h\}$ be a coordinate neighborhood of M at P. If we take an r-jet $J_P^r(F)$ belonging to $\pi^{-1}(U)$ and put

$$y^{(\nu)h}=\frac{1}{\nu!}\cdot\frac{d^\nu F^h(0)}{dt^\nu}, \tag{1.1}$$

where F has the local expression $x^h=F^h(t)$, $t\in R$, in U such that $P=F(0)$, then the r-jet $J_P^r(F)$ is expressed in a unique way by the set $(y^{(\nu)h})$ $(\nu=0, 1, \cdots, r;\ h=1, \cdots, n)$, $(y^{(0)h})=(x^h)$ being the coordinates of P in U. Thus a system of coordinates $(y^{(\nu)h})$ is introduced in the open set $\pi^{-1}(U)$ of $T_r(M)$. We now call $(y^{(\nu)h})$ the coordinates induced in $\pi^{-1}(U)$ from $\{U, x^h\}$, or simply the induced coordinates in $\pi^{-1}(U)$. We sometimes denote the *induced coordinates* by (y^A) (see §0). Thus $T_r(M)$ is a differentiable manifold of $(r+1)n$ dimensions.

For $\lambda=0, 1, \cdots, r$, we define the *λ-lift* $f^{(\lambda)}$ of a function f in M to $T_r(M)$ by

$$f^{(\lambda)}(J_P^r(F))=\frac{1}{\lambda!}\left[\frac{d^\lambda(f\circ F)}{dt^\lambda}\right]_0, \tag{1.2}$$

F: $R\to M$ being an arbitrary mapping such that $P=F(0)$. The λ-lift $f^{(\lambda)}$ of f is well defined in $T_r(M)$, i.e., the value $f^{(\lambda)}(J_P^r(F))$ is independent of the choice of F: $R\to M$. Clearly, $f^0=f\circ\pi$ $(f^0=f^{(0)}$, see §0). For the sake of convenience, we define that $f^{(\lambda)}=0$ for any negative integer λ. For the lifts of two functions f and g to $T_r(M)$, we have the following formula:

$$(f\circ g)^{(\lambda)}=\sum_{\mu=0}^{\lambda}f^{(\mu)}g^{(\lambda-\mu)}. \tag{1.3}$$

Let X be a vector field in M with components X^h in a coordinate neighborhood $\{U, x^h\}$. We defined the *λ-lift* of X to $T_r(M)$, denoted by $X^{(\lambda)}$, to be the vector field $\tilde{X}$ which locally has components $\tilde{X}^A$ in the open set $\pi^{-1}(U)$ such that

$$\tilde{X}^{(\nu)h}=(X^h)^{(\nu+\lambda-r)} \tag{1.4}$$

relative to the induced coordinates $(y^A)=(y^{(\nu)}{}_h)$ in $\pi^{-1}(U)$, where the right-hand side of (1.4) denotes the $(\nu+\lambda-r)$-lift of the local function X^h. $\tilde{X}$ or $X^{(\lambda)}$ actually determines globally a vector field in $T_r(M)$ (use (1.10)). For the λ-lifts of vector fields, we have the following formulas:

(1. 5) $$X^{(\lambda)}f^{(\mu)}=(Xf)^{(\lambda+\mu-r)}, \qquad f\in\mathcal{T}_0^0(M), \qquad X\in\mathcal{T}_0^1(M);$$

(1. 6) $$\frac{\partial}{\partial y^{(\lambda)i}}=\left(\frac{\partial}{\partial x^i}\right)^{(r-\lambda)};$$

(1. 7) $$\frac{\partial f^{(\lambda)}}{\partial y^{(\mu)i}}=\left(\frac{\partial f}{\partial x^i}\right)^{(\lambda-\mu)}, \qquad f\in\mathcal{T}_0^0(M);$$

(1. 8) $$(fX)^{(\lambda)}=\sum_{\mu=0}^{\lambda}f^{(\mu)}X^{(\lambda-\mu)}, \qquad f\in\mathcal{T}_0^0(M), \qquad X\in\mathcal{T}_0^1(M);$$

(1. 9) $$[X^{(\lambda)}, Y^{(\mu)}]=[X, Y]^{(\lambda+\mu-r)}, \qquad X,Y\in\mathcal{T}_0^1(M)$$

Let $\{U, x^h\}$ and $\{U', x^{h'}\}$ be two intersecting coordinate neighborhoods of M and the coordinate transformation in $U\cap U'$ be given by

$$x^{h'}=x^{h'}(x^k).$$

Then, if $(y^A)=(y^{(\nu)h})$ and $(y^{A'})=(y^{(\nu)h'})$ are the induced coordinates in $\pi^{-1}(U)$ and $\pi^{-1}(U')$ respectively, the transformation of induced coordinates in $\pi^{-1}(U\cap U')$ $=\pi^{-1}(U)\cap\pi^{-1}(U')$ has the Jacobian matrix of the form

(1. 10) $$\left(\frac{\partial y^{A'}}{\partial y^A}\right)=\left(\frac{\partial y^{(\nu)h'}}{\partial y^{(\mu)h}}\right)=\left(\left(\frac{\partial x^{h'}}{\partial x^h}\right)^{(\nu-\mu)}\right).$$

Let a 1-form ω have the local expression $\omega=\omega_i dx^i$ in a coordinate neighborhood $\{U, x^h\}$. Then in $\pi^{-1}(U)$ we denote by $\tilde{\omega}_U$ the local 1-form defined by

(1. 11) $$\tilde{\omega}_U=\sum_{\mu=0}^{\lambda}\omega_i^{(\mu)}dy^{(\lambda-\mu)i}$$

relative to the induced coordinates $(y^{(\nu)h})$ in $\pi^{-1}(U)$. This actually determines globally a 1-form in $T_r(M)$, which is called the *λ-lift* of ω and denoted by $\omega^{(\lambda)}$ (use (1. 10)). For the λ-lifts of ω, we have the following formulas:

(1. 12) $$\omega^{(\lambda)}(X^{(\mu)})=(\omega(X))^{(\lambda+\mu-r)}, \qquad \omega\in\mathcal{T}_1^0(M), \qquad X\in\mathcal{T}_0^1(M);$$

(1. 13) $$dy^{(\lambda)i}=(dx^i)^{(\lambda)};$$

(1. 14) $$(f\omega)^{(\lambda)}=\sum_{\mu=0}^{\lambda}f^{(\mu)}\omega^{(\lambda-\mu)}, \qquad f\in\mathcal{T}_0^0(M), \qquad \omega\in\mathcal{T}_1^0(M).$$

The above operations of taking lifts are linear mapping $\mathcal{T}_0^0(M)\to\mathcal{T}_0^0(T_r(M))$, $\mathcal{T}_0^1(M)\to\mathcal{T}_0^1(T_r(M))$ and $\mathcal{T}_1^0(M)\to\mathcal{T}_1^0(T_r(M))$ respectively. They have the properties (1. 3), (1. 8) and (1. 14) respectively. Thus we can now define, for any element K of $\mathcal{T}_q^p(M)$, its λ-lift $K^{(\lambda)}$ ($\lambda=0, 1, \cdots, r$), which belongs to $\mathcal{T}_q^p(T_r(M))$ in such a way that the correspondence $K\to K^{(\lambda)}$ defines a linear mapping $\mathcal{T}_q^p(M)\to\mathcal{T}_q^p(T_r((M))$ which is characterized by the properties

$$(S\otimes T)^{(\lambda)}=\sum_{\mu=0}^{\lambda} S^{(\mu)}\otimes T^{(\lambda-\mu)}$$

for any $S, T\in\mathcal{T}(M)$ and $\lambda=0, 1, \cdots, r$. The tensor field $K^{(\lambda)}$ thus defined is called the *λ-lift* of the tensor field K in M to $T_r(M)$. For the λ-lifts of tensor fields, we have the following formulas:

(1. 15) $\quad K^{(\lambda)}(X_1^{(\mu)}, \cdots, X_q^{(\mu)})=(K(X_1, \cdots, X_q))^{\lambda+q(\mu-r)},\quad K\in\mathcal{T}_q^p(M),\ X_1 \cdots, X_q\in\mathcal{T}_0^1(M);$

(1. 16) $\quad \mathcal{L}_{X^{(\lambda)}}K^{(\mu)}=(\mathcal{L}_X K)^{(\lambda+\mu-r)},\qquad X\in\mathcal{T}_0^1(M),\qquad K\in\mathcal{T}(M);$

(1. 17) $\quad (\omega\wedge\pi)^{(\lambda)}=\sum_{\mu=0}^{\lambda}\omega^{(\mu)}\wedge\pi^{(\lambda-\mu)};$

(1. 18) $\quad d\omega)^{(\lambda)}=d\omega^{(\lambda)},$

ω and π being arbitrary differential forms of arbitrary order in M, where $\mathcal{L}_X$ denotes the Lie derivation with respect to a vector field X.

Next we shall give local expressions of lifts of tensor fields of special type in M to $T_r(M)$ relative to the induced coordinates $(y^A)=(y^{(\nu)h})$. Let X be a vector field with local components X^h in M. Then $X^{(\lambda)}$ in $T_r(M)$ has local components of the form

(1. 19)
$$X^{(\lambda)}:\begin{bmatrix} 0 \\ \vdots \\ 0 \\ (X^h)^0 \\ \vdots \\ (X^h)^{(\lambda-1)} \\ (X^h)^{(\lambda)} \end{bmatrix};$$

the lifts of a 1-form ω with local expression $\omega=\omega_i dx^i$ in M have local components of the form

(1. 20)
$$\omega^{(\lambda)}=(\omega_i^{(\lambda)}, \omega_i^{(\lambda-1)}, \cdots, \omega_i^{(1)}, \omega_i^{(0)}, 0, \cdots, 0);$$

the λ-lift of a tensor field $F\in\mathcal{T}_1^1(M)$ with local components F_i^h in M to $T_r(M)$ has local components of the form

(1. 21)
$$F^{(\lambda)}:\begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ (F_i^h)^{(0)} & 0 & 0 & \cdots & \cdots & 0 \\ (F_i^h)^{(1)} & (F_i^h)^{(0)} & 0 & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \\ (F_i^h)^{(\lambda)} & (F_i^h)^{(\lambda-1)} & (F_i^h)^{(\lambda-2)} & \cdots & (F_i^h)^{(0)} & 0\cdots 0 \end{bmatrix}$$

and the λ-lift of a tensor field $g\in\mathcal{T}_2^0(M)$ with local components g_{ji} in M to $T_r(M)$ has local components of the form

$$
(1.22) \qquad g^{(\lambda)}: \begin{bmatrix}
(g_{ji})^{(\lambda)} & (g_{ji})^{(\lambda-1)} & \cdots & (g_{ji})^{(1)} & (g_{ji})^{(0)} & 0\cdots 0 \\
(g_{ji})^{(\lambda-1)} & (g_{ji})^{(\lambda-2)} & \cdots & (g_{ji})^{(0)} & 0 & 0\cdots 0 \\
\cdots & \cdots & & \cdots & & \cdots \\
(g_{ji})^{(0)} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 \\
\cdots & & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & 0
\end{bmatrix}
$$

relative to the induced coordinates in $\pi^{-1}(U)$.

Finally, we consider lifts of affine connections. Let ∇ be an affine conection in M with components Γ_{ji}^{h} in $\{U, x^h\}$. We now introduce in $\pi^{-1}(U)$ affine connection ∇_U^* with components $\tilde{\Gamma}_{CB}^{A}$ relative to the induced coordinates (y^A) such that

$$
(1.23) \qquad \tilde{\Gamma}_{CB}^{A} = (\Gamma_{ji}^{h})^{(\lambda-\mu-\nu)}
$$

for $A=(\lambda)h$, $B=(\mu)i$ and $C=(\nu)j$. According to (1.10) and (1.23), ∇_U^* actually determines globally an affine connection ∇^* in $T_r(M)$ which is called the *lift of the affine connection* ∇ and denoted also by ∇^*. We have the following properties of ∇^*:

$$
(1.24) \qquad \nabla^*_{X^{(\lambda)}} K^{(\mu)} = (\nabla_X K)^{(\lambda+\mu-r)}, \qquad X \in \mathcal{T}_0^1(M), \qquad K \in \mathcal{T}(M);
$$

$$
(1.25) \qquad \mathcal{L}_{X^{(\lambda)}} \nabla^* = (\mathcal{L}_X \nabla)^{(\lambda)}, \qquad X \in \mathcal{T}_0^1(M).
$$

§ 2. Cross-section determined by a vector field.

Suppose V be a vector field in M with components V^i relative to $\{U, x^h\}$. Denote by $F: I \to M$ the orbit of V passing through a point p in M such that $F(0)=p$, where I is an interval $(-\varepsilon, \varepsilon)$, ε being some positive number. We denote the r-jet $J_p^r(F)$ by $\gamma_V(p)$. Then the correspondence $p \to \gamma_V(p)$ defines a mapping γ_V: $M \to T_r(M)$ such that $\pi \circ \gamma_V$ is the identity mapping of M. Thus γ_V: $M \to T_r(M)$ is a cross-section in $T_r(M)$. We call the submanifold $\gamma_V(M)$ imbedded in $T_r(M)$ the *cross-section* determined by the vector field V. If $\{U, x^h\}$ is a coordinate neighborhood of M, the cross-section $\gamma_V(M)$ is expressed locally in $\pi^{-1}(U)$ by equations

$$
(2.1) \qquad \begin{aligned}
y^{(0)} &= x^h = F^h(0), \\
y^{(1)} &= \frac{dF^h(0)}{dt} V^h(x^i), \\
y^{(2)} &= \frac{1}{2!} \frac{d^2F^h(0)}{dt^2} = \frac{1}{2} V^k \partial_k V^h, \\
y^{(3)} &= \frac{1}{3!} \frac{d^3F^h(0)}{dt^3} = \frac{1}{3!} V^k (V^m \partial_k \partial_m V^h + \partial_k V^m \partial_m V^h), \\
&\cdots\cdots\cdots\cdots\cdots\cdots,
\end{aligned}
$$

$$y^{(\nu)}=\frac{1}{\nu!}=\frac{d^{\nu}F^{h}(0)}{dt^{\nu}}.$$

with respect to the induced coordinates.

Let f be a function on M, we have

$$f^{0}(=f^{(0)})=f,$$

$$f^{(1)}=\frac{d}{dt}(f\circ F)=\partial_i f\cdot y^{(1)i}=V^{i}\partial_i f=(\mathcal{L}_V f)^{0}$$

along $\gamma_V(M)$. A simple calculation yields that along the cross-section $\gamma_V(M)$

$$f^{(\lambda)}=\frac{1}{\lambda!}\mathcal{L}_V^{\lambda}f$$

holds, where $\mathcal{L}_V^{\lambda}=\mathcal{L}_V(\mathcal{L}_{\lambda}^{\lambda-1}f)$ for $\lambda>1$.

According to (2. 1), the submanifold $\gamma_V(M)$ is locolly expressed by a system of equations $y^{(\mu)h}=y^{(\nu)h}(x^i)$ such that

(2. 3)
$$\begin{aligned}
&y^{(0)h}(x^i)=x^h,\\
&y^{(1)h}(x^i)=V^h=(V^h)^0,\\
&y^{(2)h}(x^i)=\frac{1}{2}V^k\partial_k V^h=\frac{1}{2}(V^h)^{(1)},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&y^{(r)h}(x^i)=\frac{1}{r}(V^h)^{(r-1)}
\end{aligned}$$

with respect to the induced coordinates $(y^A)=(y^{(\nu)h})$ in $\pi^{-1}(U)$. Let us put

(2. 4)
$$B^{A}_{(0)i}=\partial_i y^A(x^h).$$

Then we have along $\gamma_V(M)$ n local vector fields $B_{(0)1}, B_{(0)2}, \cdots, B_{(0)n}$ which are tangent to the cross-section. Their components with respect to the induced coordinate $(y^{(\nu)h})$ are

(2. 4)
$$B_{(0)j}=\begin{bmatrix}\delta_j^h\\ \partial^j V^h\\ \frac{1}{2}\partial_j(V^h)^{(1)}\\ \vdots\\ \frac{1}{r}\partial_j(V^h)^{(r-1)}\end{bmatrix}.$$

For an element X of $\mathcal{T}_0^1((M)$ with local components X^i, we denote by $B_{(0)}X$ the vector field with components

$$B^{A}_{(0)i}X^i, \qquad \text{i.e.}\quad B_{(0)}X=B^{A}_{(0)i}X^i\frac{\partial}{\partial y^A},$$

which is defined globally along $\gamma_V(M)$ by virtue of (1.10). For any point σ of $\gamma_V(M)$, the mapping $B_{(0)p}: T_p(M)\to T_\sigma(T_r(M))$ $(\sigma=\gamma_V(p))$ defined by $B_{(0)p}(X_p)=(B_{(0)}X)_\sigma$ is nothing but the differential $(\gamma_V)_p$ of the cross-section mapping γ_V: $M\to T_r(M)$. Thus $B_{(0)p}(T_p(M))$ is the tangent space of the cross-section $\gamma_V(M)$ at the point $\sigma=\gamma_V(p)$.

Along the cross-section $\gamma_V(M)$, for each integer ν such that $0\leqq\nu\leqq r-1$, we consider n local vector fields $B_{(\nu)1}, B_{(\nu)2}, \cdots, B_{(\nu)n}$ which have respectively components of the form

$$(B^A_{(\nu)j})=\begin{bmatrix} 0 \\ \vdots \\ \delta^h_j \\ \partial_j V^h \\ \dfrac{1}{2}\partial_j(V^h)^{(1)} \\ \vdots \\ \dfrac{1}{r-\nu}\partial_j(V^h)^{(r-\nu-1)} \end{bmatrix} \tag{2.5}$$

and n local vector fields $B_{(r)1}, B_{(r)2}, \cdots, B_{(r)n}$ which have respectively components of the form

$$(B^A_{(r)j})=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \delta^h_j \end{bmatrix} \tag{2.6}$$

relative to the induced coordinates (y^A). Again we denote by $B_{(\nu)}X$ the vector field with components $B^A_{(\nu)j}X^j$, i.e., $B_{(\nu)}X=B^A_{(\nu)j}X^j\partial/\partial y^A$. These vector fields are defined globally along $\gamma_V(M)$. For any point σ of $\gamma_V(M)$, the mappings $B_{(\nu)p}$: $T_p(M))\to T_\sigma(T_r(M))$ $(\sigma=\gamma_V(p))$ are defined as follows:

$$B_{(\nu)p}(X_p)=(B_{(\nu)}X)_\sigma \qquad X\in\mathcal{T}^1_0(M).$$

The mappings $B_{(\nu)p}$, including $\nu=0$, are isomorphisms of $T_p(M)$ into $T_\sigma(T_r(M))$.

The $(r+1)n$ vector fields $B_{(\nu)j}$ $(0\leqq\nu\leqq r,\ 1\leqq j\leqq n)$ form a local family of frames along $\gamma_V(M)$, which we shall call *adapted frames* of $\gamma_V(M)$. The n vector fields $B_{(0)j}$ span at each point σ of $\gamma_V(M)$ the tangent plane $T_\sigma(\gamma_V(M))$ of the cross-section $\gamma_V(M)$.

For any element X of $\mathcal{T}^1_0(M)$ with local components X^i, we denote by $B_{(\nu)}X$ the vector field with components

$$B^A_{(\nu)i}X^i, \qquad \text{i.e.}\quad B_{(\nu)}X=B^A_{(\nu)i}X^i\frac{\partial}{\partial y^A}.$$

§3. Prolongations of tensor fields in the cross-section.

Suppose X is a given vector field in M. We consider along $\gamma_V(M)$ the λ-lift $X^{(\lambda)}$ of X. We shall describe $X^{(\lambda)}$ with respect to the adapted frames $B_{(0)j}$ of

$\gamma_V(M)$. The result is as follows:

PROPOSITION 3.1. *Along $\gamma_V(M)$ the λ-lift $X^{(\lambda)}$ of X is written in*

$$\begin{aligned} X^{(\lambda)} &= \sum_{\nu=0}^{\lambda} \frac{1}{\nu!} B_{(r-\lambda+r)} \mathcal{L}_V^{\nu} X \\ &= B_{(r-\lambda)} X + B_{(r-\lambda+1)} \mathcal{L}_V X + \frac{1}{2!} B_{(r-\lambda+2)} \mathcal{L}_V^2 X + \cdots \\ &\quad + \frac{1}{(\lambda-1)!} B_{(r-1)} \mathcal{L}_V^{\lambda-1} X + \frac{1}{\lambda!} B_{(r)} \mathcal{L}_V^{\lambda} X. \end{aligned} \tag{3.1}$$

Proof. By (1.19), $X^{(\lambda)}$ has the form

$$X^{(\lambda)} = \sum_{\nu=0}^{\lambda} (X^h)^{(\nu)} \frac{\partial}{\partial y^{(r-\lambda+\nu)h}}$$

with respect to the natural frame $\{\partial/\partial y^A\}$.

We first calculate $(X^h)^{(\nu)}$ along $\gamma_V(M)$ as follows:

$$(X^h)^{(0)} = X^h;$$

$$(X^h)^{(1)} = V^j \partial_j X^h = X^i \partial_i V^h + \mathcal{L}_V X^h;$$

$$\begin{aligned} (X^h)^{(2)} &= \frac{1}{2} V^j \partial_j ((X^h)^{(1)}) \\ &= \frac{1}{2} V^k \partial_k (\mathcal{L}_V X^h + X^j \partial_j V^h) \\ &= \frac{1}{2} (\mathcal{L}_V^2 X^h + \partial_j V^h \mathcal{L}_V X^j + V^k \partial_k X^j \partial_j V^h + V^k X^j \partial_k \partial_j V^h) \\ &= \frac{1}{2} [\mathcal{L}_V^2 X^h + \partial_j V^h \mathcal{L}_V X^j + \partial_j V^h (\mathcal{L}_V X^j + X^k \partial_k V^j) + V^k X^j \partial_k \partial_j V^h] \\ &= \frac{1}{2} \mathcal{L}_V^2 X^h + \partial_j V^h \mathcal{L}_V X^j + \frac{1}{2} X^k (\partial_j V^h \partial_k V^j + V^j \partial_j \partial_k V^h) \\ &= \frac{1}{2} X^j \partial_j (V^h)^{(1)} + \partial_j V^h \mathcal{L}_V X^j + \frac{1}{2} \mathcal{L}_V^2 X^h. \end{aligned}$$

By induction, we have the following formulas:

$$\begin{aligned} (X^h) &= \frac{1}{\nu} X^j \partial_j (V^h)^{(\nu-1)} + \frac{1}{\nu-1} (\mathcal{L}_V X^j) \partial_j (V^h)^{(\nu-2)} \\ &\quad + \frac{1}{2!(\nu-2)} (\mathcal{L}_V^2 X^j) \partial_j (V^h)^{(\nu-3)} + \cdots \\ &\quad + \frac{1}{\mu!(\nu-\mu)} (\mathcal{L}_V^{\mu} X^j) \partial_j (V^h)^{(\nu-\mu-1)} + \cdots \end{aligned} \tag{3.2}$$

$$+\frac{1}{(\nu-1)!}(\mathcal{L}_V^{\nu-1}X^j)\partial_j V^h+\frac{1}{\nu!}\mathcal{L}_V^{\nu}X^h.$$

Thus (3. 1) follows from (1. 19), (2. 5) and (3. 2).

Let ω be an element of $\mathcal{T}_1^0(M)$ with local expression $\omega=\omega_i dx^i$. Then, by (1. 20), $\omega^{(\lambda)}$ has components of the form

$$\omega^{(\lambda)}=(\omega_i^{(\lambda)}, \omega_i^{(\lambda-1)}, \cdots, \omega_i^{(1)}, \omega_i^0, 0, \cdots, 0)$$

with respect to the natural coframe $\{dy^A\}$. Along the cross-section $\gamma_V(M)$, let the coframes dual to the adapted frames $\{B_{(\nu)j}\}$ be $\{B^{(\nu)j}\}$. We denote by $B^{(\nu)}\omega$ the 1-form with components $B_A^{(\nu)j}\omega_j$ with respect to the coframes $\{dy^A\}$. Then we have

PROPOSITION 3. 2. *Along $\gamma_V(M)$ the λ-lifts $\omega^{(\lambda)}$ of ω are written in*

$$\omega^{(\lambda)}=\frac{1}{\lambda!}B^{(0)}\mathcal{L}_V^{\lambda}\omega+\frac{1}{(\lambda-1)!}B^{(1)}\mathcal{L}_V^{\lambda-1}\omega+\cdots \tag{3. 3}$$
$$+\frac{1}{2!}B^{(\lambda-2)}\mathcal{L}_V^2\omega+B^{(\lambda-1)}\mathcal{L}_V\omega+B^{(\lambda)}\omega.$$

Proof. By (1. 12) we have

$$\omega^{(\lambda)}(X^{(\nu)})=(\omega(X))^{(\lambda+\nu-r)}$$

and by (2. 2)

$$(\omega(X))^{(\lambda+\nu-r)}=(\omega_i X^i)^{(\lambda+\nu-r)}=\frac{1}{(\lambda+\nu-r)!}\mathcal{L}_V^{\lambda+\nu-r}(\omega_i X^i)$$
$$=\frac{1}{(\lambda+\nu-r)!}\sum_{\mu=0}^{\lambda+\nu-r}\binom{\lambda+\nu-r}{\mu}(\mathcal{L}_V^{\lambda+\nu-r-\mu}\omega_i)(\mathcal{L}_V^{\mu}X^i)$$
$$=\sum_{\mu=0}^{\lambda+\nu-r}\frac{1}{(\lambda+\nu-r-\mu)!\,\mu!}(\mathcal{L}_V^{\lambda+\nu-r-\mu}\omega_i)(\mathcal{L}_V^{\mu}X^i),$$

where $\binom{\lambda+\nu-r}{\mu}$ denotes the binomial coefficient.

On the other hand, with respect to the coframes $\{B^{(\nu)j}\}$, we consider a 1-form $\bar{\omega}^{(\lambda)}$ defined by

$$\bar{\omega}^{(\lambda)}=\sum_{\mu=0}^{\lambda}\frac{1}{(\lambda-\mu)!}B^{(\mu)i}\mathcal{L}_V^{\lambda-\mu}\omega_i.$$

Then by (3. 2) we have

$$\bar{\omega}^{(\lambda)}X^{(\nu)}=\sum_{\mu=0}^{\lambda-r+\nu}\frac{1}{(\lambda-r+\nu-\mu)!\,\mu!}(\mathcal{L}_V^{\mu}X^i)(\mathcal{L}_V^{\lambda+\nu-r-\mu}\omega_i).$$

Since X is arbitrary in the above formulas, the formula (3. 3) follows from $\omega^{(\lambda)}X^{(\nu)}=\bar{\omega}^{(\lambda)}X^{(\nu)}$.

Now we shall write down the λ-lifts of tensor fields of special type in M with respect to the adapted frame. For an element h of $\mathcal{T}_2^0(M)$ with local components h_{ij}, we have

$$(3.4)\qquad h^{(\lambda)}: \begin{bmatrix} \frac{1}{\lambda!}\mathcal{L}_V^{\lambda} h_{ji} & \frac{1}{(\lambda-1)!}\mathcal{L}_V^{\lambda-1} h_{ji}\cdots & \frac{1}{2!}\mathcal{L}_V^2 h_{ji} & \mathcal{L}_V h_{ji} & h_{ij} & 0\cdots 0 \\ \frac{1}{(\lambda-1)!}\mathcal{L}_V^{\lambda-1} h_{ji} & \frac{1}{(\lambda-1)!}\mathcal{L}_V^{\lambda-2} h_{ji}\cdots & \mathcal{L}_V h_{ji} & h_{ij} & 0 & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & & \\ \frac{1}{2!}\mathcal{L}_V^2 h_{ji} & \mathcal{L}_V h_{ji} & \cdots & \cdots & & 0 \\ \mathcal{L}_V h_{ji} & h_{ij} & \cdots & \cdots & & 0 \\ h_{ij} & 0 & \cdots & \cdots & & 0 \\ 0 & \cdots & \cdots & \cdots & & 0 \\ \cdots & \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & \cdots & & 0 \end{bmatrix}$$

and, for an element F of $\mathcal{T}_1^1(M)$ with local components F_i^h,

$$(3.5)\qquad F^{(\lambda)}: \begin{bmatrix} 0 & 0 & \cdots & \cdots & & 0 \\ \cdots & \cdots & \cdots & \cdots & & \\ 0 & 0 & \cdots & \cdots & & 0 \\ F_i^h & 0 & \cdots & \cdots & & 0 \\ \mathcal{L}_V F_i^h & F_i^h & \cdots & \cdots & & 0 \\ \frac{1}{2!}\mathcal{L}_V^2 F_i^h & \mathcal{L}_V F_i^h & \cdots & \cdots & & 0 \\ \cdots & \cdots & \cdots & \cdots & & \\ \frac{1}{(\lambda-1)!}\mathcal{L}_V^{\lambda-1} F_i^h \cdots & \frac{1}{(\lambda-2)!}\mathcal{L}_V^{\lambda-2} F_i^h \cdots & F_i^h & 0 & & 0\cdots 0 \\ \frac{1}{\lambda!}\mathcal{L}_V^{\lambda} F_i^h \cdots & \frac{1}{(\lambda-1)!}\mathcal{L}_V^{\lambda-1} F_i^h \cdots & \mathcal{L}_V F_i^h & F_i^h & & 0\cdots 0 \end{bmatrix}$$

For an element S of $\mathcal{T}_2^1(M)$ with local component S_{ji}^h, we have

$$(3.6)\qquad \begin{aligned} (S^{(\lambda)})_{\nu(j)\omega(k)}{}^{\mu(i)} &= \frac{1}{(\lambda+\mu-r-\nu-\omega)!}\mathcal{L}_V^{\lambda+\mu-r-\nu-\omega} S_{jk}^{i}, \\ &= 0 \qquad \text{if } \lambda+\mu<r+\nu+\omega. \end{aligned}$$

In §2 we have shown that for the mapping γ_V: $M\to\gamma_V(M)$, $\gamma'_V(X)=B_{(0)}X$ for any X in $\mathcal{T}_0^1(M)$. $B_{(0)}$: $T(M)\to T((\gamma'_V(M))$ is a linear isomorphism. Let $B_{(0)}^{-1}$ be the inverse of this linear isomorphism $B_{(0)}$. Then $B_{(0)}^{-1}$: $T(\gamma_V(M))\to T(M)$. The dual map $(B_{(0)}^{-1})^*$ of $B_{(0)}^{-1}$ sends $\mathcal{T}_1^0(M)$ to $\mathcal{T}_1^0(\gamma_V(M))$. $(B_{(0)}^{-1})^*$ is nothing but $B^{(0)}$. We now denote $B^{(0)}$ also by γ'_V, i.e., $\gamma'_V(\omega)=B^{(0)}\omega$ for $\omega\in\mathcal{T}_1^0(M)$. Then we can extend the mapping γ'_V to a linear mapping γ'_V: $\mathcal{T}(M)\to\mathcal{T}(\gamma_V(M))$ by setting

$$\gamma'_V(P\otimes Q)=\gamma'_V(P)\otimes\gamma'_V(Q)$$

for arbitrary tensor fields P, Q in M.

Now we shall define an operation, denoted by $\#$, in $\mathcal{T}(T_r(M))$ as follows:

If $\tilde{X}\in\mathcal{T}_0^1(T_r(M))$, $\tilde{X}=\sum_{\nu=0}^{r}\tilde{X}^{(\nu)i}B_{(\nu)i}$, then $\tilde{X}^{\#}=\tilde{X}^{(0)i}B_{(0)i}\in\mathcal{T}_0^1(\gamma_V(M))$;

If $\tilde{\omega}$ is a tensor field of type (0, 1) in $T_r(M)$ defined along $\gamma_V(M)$, then

$$\tilde{\omega}^{\#}(B_{(0)}X)=\tilde{\omega}(B_{(0)}X);$$

If $\tilde{h}$ is a tensor field of type (0, 2) in $T_r(M)$ defined along $\gamma_V(M)$, then

$$\tilde{h}^{\#}(B_{(0)}X, B_{(0)}Y)=\tilde{h}(B_{(0)}X, B_{(0)}Y);$$

If $\tilde{F}$ is a tensor field of type (1, 1) in $T_r(M)$ such that, for any vector field $\tilde{A}$ tangent to $\gamma_V(M)$ $\tilde{F}\tilde{A}$ is also tangent to $\gamma_V(M)$, then $F^{\#}(B_{(0)}X)=\tilde{F}(B_{(0)}X)$;

If $\tilde{S}$ is a tensor field of type (1, 2) in $T_r(M)$ such that, for any vector fields $\tilde{A}$, $\tilde{B}$ tangent to $\gamma_V(M)$, $\tilde{B}(\tilde{A}, \tilde{B})$ is also tangent to $\gamma_V(M)$, then

$$\tilde{S}^{\#}(B_{(0)}X, B_{(0)}Y)=\tilde{S}(B_{(0)}X, B_{(0)}Y).$$

In the above definitions the relations are supposed to hold for arbitrary elements X and Y in $\mathcal{T}_0^1(M)$. We sometimes call $\tilde{h}^{\#}$, $\tilde{F}^{\#}$ and $\tilde{S}^{\#}$ respectively the *tensor fields induced in* $\gamma_V(M)$ from h, F and S.

For the operation $\#$, we have the following propositions by (3. 1), (3. 3) and (3. 4):

PROPOSITION 3. 3. (a) *For any X in $\mathcal{T}_0^1(M)$, $(X^{(\lambda)})^{\#}=0$ if $\lambda=0, 1, \cdots, r-1$. $X^{(r)}$ is tangent to $\gamma_V(M)$, if and only if $\mathcal{L}_VX=0$, and in this case $X^{(r)}=\gamma'_VX$.*

(b) *For any ω in $\mathcal{T}_1^0(M)$, $(\omega^{(\lambda)})^{\#}=\dfrac{1}{\lambda!}\gamma'_V(\mathcal{L}_V^{\lambda}\omega)$, $\lambda=0, 1, \cdots, r$.*

(c) *For any h in $\mathcal{T}_2^0(M)$, $(h^{(\lambda)})^{\#}=\dfrac{1}{\lambda!}\gamma'_V(\mathcal{L}_V^{\lambda}h)$, $\lambda=0, 1, \cdots, r$.*

$$(h^0)^{\#}(B_{(0)}X, B_{(0)}Y)=(h(X, Y))^0.$$

COROLLARY. *Let g be a Riemannian metric in M. Then $(g^0)^{\#}$ is a Riemannian metric in $\gamma_V(M)$ and γ_V is an isometry with respect to g in M and $g^{(0)\#}$ in $\gamma_V(M)$.*

Let $\tilde{F}$ be a (1, 1) tensor field defined along $\gamma_V(M)$. If $T_\sigma(\gamma_V(M))$, $\sigma\in\gamma_V(M)$, is invariant by the action of the tensor F, the cross-section $\gamma_V(M)$ is said to be *invariant* by F.

From (3. 5), we have

$$F^{(\nu)}(B_{(0)}X)=B_{(r-\nu)}(FX)+B_{(r-\nu+1)}((\mathcal{L}_VF)X)+\frac{1}{2!}B_{(r-\nu+2)}((\mathcal{L}_V^2F)X)$$
$$+\frac{1}{\mu!}B_{(r-\nu+\mu)}((\mathcal{L}_V^{\mu}F)X)+\cdots+\frac{1}{\nu!}B_{(r)}((\mathcal{L}_V^{\nu}F)X)$$

for any vector field X in M. Thus we have

PROPOSITION 3. 4. *For* $F\in\mathcal{T}_1^1(M)$, *the cross-section* $\gamma_V(M)$ *is invariant by* $F^{(r)}$ *if and only if* $\mathcal{L}_V F=0$. *In this case* $(F^{(r)})^{\#}=\gamma'_V F$ *holds. The lifts* $F^{(\lambda)}$ $(\lambda=0,1,\cdots,r-1)$ *do not leave* $\gamma_V(M)$ *invariant unless* $F=0$.

PROPOSITION 3.5 *If* F *is an almost complex structure in* M *such that* $\mathcal{L}_V F=0$, *then* $(F^{(r)})^{\#}$ *is an almost complex structur in* $\gamma_V(M)$.

If (g, F) is an almost Hermitian structure in M and $\mathcal{L}_V F=0$ holds, then

$$\begin{aligned}(g^0)^{\#}((F^{(r)})^{\#}B_{(0)}X,\ (F^{(r)})^{\#}B_{(0)}Y)&=(\gamma'_V g)((\gamma'_V F)B_{(0)}X,\ (\gamma'_V F)B_{(0)}Y)\\&=(g(FX, FY))^0.\end{aligned}$$

Thus we have

PROPOSITION 3. 6. *Suppose that there is given an almost Hermitian structure* (g, F) *in* M. *If* $\mathcal{L}_V F=0$, *then* $((g^{(0)})^{\#}, (F^{(r)})^{\#})$ *is an almost Hermitian structure in* $\gamma_V(M)$.

By (3. 6), we have for any $S\in\mathcal{T}_2^1(M)$

$$\begin{aligned}S^{(\lambda)}(B_{(0)}X, B_{(0)}Y)&=B_{(r-\lambda)}(S(X,Y))+B_{(r-\lambda+1)}((\mathcal{L}_V S)(X,Y))\\&\quad+\frac{1}{2!}B_{(r-\lambda+2)}((\mathcal{L}_V^2 S)(X,Y))+\cdots+\frac{1}{\lambda!}B_{(r)}((\mathcal{L}_V^\lambda S)(X,Y)).\end{aligned}$$

Thus we have

PROPOSITION 3. 7. *If* $S\in\mathcal{T}_2^1(M)$, *the vector field* $S^{(r)}(B_{(0)}X, B_{(0)}Y)$ *is tangent to* $\gamma_V(M)$ *for arbitrary* X, Y *of* $\mathcal{T}_2^1(M)$, *if and only if* $\mathcal{L}_V S=0$, *and in this case* $(S^{(r)})^{\#}=\gamma'_V S$. *The vector fields* $S^{(\lambda)}(B_{(0)}X, B_{(0)}Y)$ $(0<\lambda<r)$ *are not tangent to* $\gamma_V(M)$ *unless* $S=0$.

Let F be a tensor of type (1, 1) in M and N_F its Nijenhuis tensor. Then it is easy to check that $\mathcal{L}_V F_i^j=0$ implies $\mathcal{L}_V(N_F)_{ij}^h=0$. Thus we have

COROLLARY 1. *Let* F *be an element of* $\mathcal{T}_1^1(M)$ *such that* $\mathcal{L}_V F=0$, *then* $(N_F)^{(r)}(B_{(0)}X, B_{(0)}Y)$ *is tangent to* $\gamma_V(M)$ *for arbitaary elements* X *and* Y *of* $\mathcal{T}_0^1(M)$. *In this case* $((N_F)^{(r)})^{\#}=\gamma'_V N_F$.

COROLLARY 2. *If a complex structure* F *in* M *satisfies the condition* $\mathcal{L}_V F=0$, *then* $(F^{(r)})^{\#}$ *is a complex structure in* $\gamma_V(M)$.

§4. Prolongations of affine connections in the cross-section.

Suppose an affine connection ∇ with coefficients Γ_{ji}^h is given in M. For a vector field X with components X^i and a tensor field P of type (1, 2) with component P_{ji}^h, we have the following formulas [6]:

$$\mathcal{L}_V(\nabla_j X^h)-\nabla_j(\mathcal{L}_V X^h)=(\mathcal{L}_V\Gamma_{ji}^h)X^i,$$

$$\nabla_k(\mathcal{L}_V\Gamma_{ji}^h)-(\nabla_j(\mathcal{L}^A\Gamma_{ki}^h)=\mathcal{L}_V R_{kji}^h,$$

$$\mathcal{L}_V(\nabla_k P_{ij}^h)-\nabla_k(\mathcal{L}_V P_{ij}^h)=(\mathcal{L}_V\Gamma_{km}^h)P_{ij}^m-(\mathcal{L}_V\Gamma_{kj}^m)P_{mi}^h-(\mathcal{L}_V\Gamma_{kj}^m)P_{im}^h,$$

where $R_{kji}{}^h$ are components of the curvature tensor of ∇.

Using the third formula for any tensor field P with local componente P_{ji}^h, we have easily

$$\mathcal{L}_V^q(\nabla_j X^h)-\nabla_j(\mathcal{L}_V^q X^h)=\sum_{s=0}^{q-1}\binom{n}{s}(\mathcal{L}_V^{q-s}\Gamma_{jk}^h)(\mathcal{L}_V^s X^k), \tag{4.1}$$

$$\nabla_k(\mathcal{L}_V^q\Gamma_{ji}^h)-\nabla_j(\mathcal{L}_V^q\Gamma_{ki}^h)+\sum_{s=1}^{q-1}\binom{q}{s}[(\mathcal{L}_V^{q-s}\Gamma_{ji}^m)(\mathcal{L}_V^s\Gamma_{km}^h)-(\mathcal{L}_V^{q-s}\Gamma_{ki}^m)(\mathcal{L}_V^s\Gamma_{jm}^h)]=\mathcal{L}_V^q R_{kji}{}^h \tag{4.2}$$

$$\partial_k(\mathcal{L}_V^q\Gamma_{ji}^h)-\partial_j(\mathcal{L}_V^q\Gamma_{ki}^h)+\sum_{s=0}^{q-1}\binom{q}{s}[(\mathcal{L}_V^{q-s}\Gamma_{ji}^m)(\mathcal{L}_V^s\Gamma_{km}^h)-(\mathcal{L}_V^{q-s}\Gamma_{ki}^m)(\mathcal{L}_V^s\Gamma_{jm}^h)]=\mathcal{L}_V^q R_{kji}{}^h \tag{4.3}$$

for any positive integer q.

Let ∇^* be the lift of the affine connection ∇. Then ∇^* is an affine connection in $T_r(M)$. We shall now prove

PROPOSITION 4.1. $\displaystyle \nabla^*_{B_{(0)j}}B_{(s)k}=\sum_{u=0}^{r-s}\frac{1}{u!}(\mathcal{L}_V^u\Gamma_{jk}^h)^0 B_{(s+u)h}.$

Proof. By (1.24) and (3.1), we have

$$\nabla^*_{Y^{(r)}}X^{(r)}=(\nabla_Y X)^{(r)}=\sum_{s=0}^{r}\frac{1}{s!}(\mathcal{L}_V^s\nabla_Y X^h)^0 B_{(s)h}. \tag{4.4}$$

On the other hand, we have

$$\begin{aligned}\nabla^*_{Y^{(r)}}X^{(r)}&=\nabla^*_{Y^{(r)}}\left(\sum_{s=0}^{r}\frac{1}{s!}(\mathcal{L}_V^s X^h)^0 B_{(s)h}\right)\\ &=\sum_{s=0}^{r}\frac{1}{s!}(\mathcal{L}_V^s X^h)^0\nabla^*_{Y^{(s)}h}+\sum_{s=0}^{r}\frac{1}{s!}Y^i\partial_i(\mathcal{L}_V^s X^h)^0 B_{(s)h}.\end{aligned} \tag{4.4$'$}$$

For any $\sigma\in\gamma_V(M)$ there is a vector field Y in M with initial condition $Y=Y^j\partial/\partial x^j$, $\mathcal{L}_V Y=0, \cdots, \mathcal{L}_V^r Y=0$ at $p=\pi(\sigma)$. Then $Y^{(r)}=Y^j B_{(0)j}$ at σ. Taking the coefficients of Y^j in right-hand sides of (4.4) and (4.4)$'$, we have

$$\sum_{s=0}^{r}\frac{1}{s!}(\mathcal{L}_V^s\nabla_j X^h)^0 B_{(s)h}=\sum_{s=0}^{r}(\mathcal{L}_V^s X^h)^0\nabla^*_{B_{(0)j}}B_{(s)h}+\sum_{s=0}^{r}\frac{1}{s!}(\partial_j\mathcal{L}_V^s X^h)^0 B_{(s)h}.$$

Hence we have

$$\begin{aligned}\sum_{s=0}^{r}\frac{1}{s!}(\mathcal{L}_V^s X^h)^0\nabla^*_{B_{(0)j}}B_{(s)h}&=\sum_{s=0}^{r}\frac{1}{s!}(\mathcal{L}_V^s\nabla_j X^h-\partial_j\mathcal{L}_V^s X^h)^0 B_{(s)h}\\ &=\sum_{s=0}^{r}\frac{1}{s!}(\mathcal{L}_V^s\nabla_j X^h-\nabla_j\mathcal{L}_V^s X^h+\Gamma_{ji}^h\mathcal{L}_V^s X^i)^0 B_{(s)h}\\ &=\sum_{s=0}^{r}\frac{1}{s!}\left(\sum_{u=0}^{s}\binom{s}{u}(\mathcal{L}_V^{s-u}\Gamma_{jk}^h)(\mathcal{L}_V^u X^k)\right)^0 B_{(s)h},\end{aligned}$$

where we have used (4. 1). Since X is arbitrary, we may compare the coefficients of $(X^k)^{(0)}$, $(\mathcal{L}_V X^k)^{(0)}$, $(\mathcal{L}_V^2 X^k)^{(0)}$, $\cdots$ in the equation above and have

$$\nabla^*_{B_{(0)j}} B_{(s)k} = \sum_{u=0}^{r-s} \frac{1}{u!} (\mathcal{L}_V^u \Gamma^h_{jk})^0 B_{(s+u)h} \tag{4. 5}$$

which is to be proved.

Putting

$$'\nabla^*_j B_{(s)i} = \nabla^*_{B_{(0)j}} B_{(s)i} - (\Gamma^h_{ji})^0 B_{(s)h}, \tag{4. 6}$$

then we have

$$\begin{aligned} &'\nabla^*_j B_{(s)i} = \sum_{u=1}^{r-s} \frac{1}{u!} (\mathcal{L}_V^u \Gamma^h_{ji})^0 B_{(s+u)h}, \qquad s=0, 1, \cdots, r-1; \\ &'\nabla^*_j B_{(r)i} = 0. \end{aligned} \tag{4. 7}$$

Thus we have now

PROPOSITION 4. 2. *The cross-section $\gamma_V(M)$ is totally geodesic in $T_r(M)$ with respect to the connection ∇^* if and only if the vector field V is infinitesimal affine transformation in M with respect to V, i.e., $\mathcal{L}_V \Gamma^h_{ji} = 0$.*

For any X of $\mathcal{T}^1_0(M)$ we get, from (4. 6),

$$(X^k)^{0\prime} \nabla^*_j B_{(0)i} = (X^i)^0 \nabla^*_{B_{(0)j}} B_{(0)i} - (X^i)^0 (\Gamma^h_{ji})^0 B_{(0)h}$$

and then

$$\nabla^*_{B_{(0)j}} (B_{(0)} X) = (\nabla_j X^h)^0 B_{(0)h} + (X^h)^{0\prime} \nabla^*_j B_{(0)h}.$$

So, for any $Y \in \mathcal{T}^1_0(M)$, we get

$$\nabla^*_{B_{(0)}Y} (B_{(0)} X) = B_{(0)} (\nabla_Y X) + (X^h)^0 (Y^j)^{0\prime} \nabla^*_j B_{(0)h}. \tag{4. 8}$$

Thus $B_{(0)}(\nabla_Y X)$ is the tangent component to $\gamma_V(M)$ of $\nabla^*_{B_{(0)}Y}(B_{(0)}X)$, according to (4. 7). We can now define an affine connection $\nabla^\sharp$ in $\gamma_V(M)$ by the equation

$$\nabla^\sharp_{B_{(0)}Y} B_{(0)} X = B_{(0)} (\nabla_Y X). \tag{4. 9}$$

We then have some propositions concerning $\nabla^\sharp$.

PROPOSITION 4. 3. *For an element h of $\mathcal{T}^0_2(M)$ and an element Z of $\mathcal{T}^1_0(M)$, we have*

$$\nabla^\sharp_{B_{(0)}Z} h^{(0)\sharp} = (\nabla_Z h)^{(0)\sharp}. \tag{4. 10}$$

Especially, let g be a Riemannian metric in M and ∇ the Riemannian connection determined by g in M, then the connection $\nabla^\sharp$ induced in $\gamma_V(M)$ from ∇ is the Riemannian connection determined by the induced metric $g^{(0)\sharp}$ of $\gamma_V(M)$.

Proof. First we have

$$\begin{aligned}(B_{(0)}Z)(h^{(0)\sharp}(B_{(0)}X, B_{(0)}Y)) &= \nabla^{\sharp}_{B_{(0)}Z}(h^{(0)\sharp}(B_{(0)}X, B_{(0)}Y)) \\ &= (\nabla^{\sharp}_{B_{(0)}Z}h^{(0)\sharp})(B_{(0)}X, B_{(0)}Y) \\ &\quad + h^{(0)\sharp}(\nabla^{\sharp}_{B_{(0)}Z}B_{(0)}X, B_{(0)}Y) \\ &\quad + h^{(0)\sharp}(B_{(0)}X, \nabla^{\sharp}_{B_{(0)}Z}B_{(0)}Y)\end{aligned}$$

By Proposition 3. 3 (c) and (4. 9), we get

$$h_{(0)}{}^{\sharp}(B_{(0)}X, B_{(0)}Y) = (h(X, Y))^0,\ h^{(0)\sharp}(\nabla^{\sharp}_{B_{(0)}Z}B_{(0)}X, B_{(0)}Y) = (h(\nabla_Z X, Y))^0,$$
$$h^{(0)\sharp}(B_{(0)}X, \nabla^{\sharp}_{B_{(0)}Z}B_{(0)}Y) = (h(X, \nabla_Z Y))^0.$$

On the other hand, we have

$$(Zh(X, Y))^0 = ((\nabla_Z h)(XY))^0 + (h(\nabla_Z X, Y))^0 + (h(X, \nabla_Z Y))^0.$$

Thus, we have

$$\begin{aligned}\nabla^{\sharp}_{B_{(0)}Z}h^{(0)\sharp}(B_{(0)}X, B_{(0)}Y) &= (Zh(X, Y))^0 - (h(\nabla_Z X, Y))^0 - (h(X, \nabla_Z Y))_0 \\ &= ((\nabla_Z h)(X, Y))^0 = (\nabla_Z h)^{(0)\sharp}(B_{(0)}X, B_{(0)}Y),\end{aligned}$$

which implies (4. 10) because X and Y are arbitrary.

For the case $h=g$, $\nabla_Z g=0$ implies $\nabla^{\sharp}_{B_{(0)}Z}g^{(0)\sharp}=0$. It is also clear by (4. 9) that if ∇ is without torsion, so is $\nabla^{\sharp}$. Hence Proposition 4. 3 is proved.

Let an element F of $\mathcal{T}^1_1(M)$ satisfy $\mathcal{L}_V F=0$. Then, for any vector field $\tilde{A}$ tangent to $\gamma_V(M)$, by Proposition 3. 4, $F^{(r)}\tilde{A}$ is also tangent to $\gamma_V(M)$. We can then define an element $F^{(r)\sharp}$ of $\mathcal{T}^1_1(\gamma_V(M))$ by

(4. 11) $$F^{(r)\sharp}(B_{(0)}X) = F^{(r)}(B_{(0)}X), \qquad X \in \mathcal{T}^1_0(M).$$

PROPOSITION 4. 4. *Let F be an element of $\mathcal{T}^1_1(M)$ satisfying $\mathcal{L}_V F=0$, then*

(a) $(\nabla^{\sharp}_{B_{(0)}Z}(F^{(r)})^{\sharp})(B_0 X) = B_{(0)}((\nabla_Z F)X), \qquad X, Z \in \mathcal{T}^1_0(M);$

(b) *If $\nabla F=0$ in M, then $\nabla^{\sharp}F^{(r)\sharp}=0$ in $\gamma_V(M)$;*

(c) *If (g, F) is a Kählerian structure in M, so is $(g^{(0)}, F^{(r)\sharp})$ in $\gamma_V(M)$.*

Proof. We have only to prove (a). By use of (4. 9), we have

(4. 12) $$\nabla^{\sharp}_{B_{(0)}Z}(F^{(r)\sharp}(B_{(0)}X)) = (\nabla^{\sharp}_{B_{(0)}Z}F^{(r)\sharp})(B_{(0)}X) + F^{(r)\sharp}(B_{(0)}\nabla_Z X).$$

On the other hand, since $F^{(r)}(B_{(0)}X)$ is tangent to $\gamma_V(M)$, we get $F^{(r)}(B_{(0)}X) = B_{(0)}F^{(r)}(B_{(0)}X)$. Using (4. 9), (3. 5) and the fact $\mathcal{L}_V F=0$, we have

(4. 13) $$\begin{aligned}\nabla^{\sharp}_{B_{(0)}}(F^{(r)\sharp}(B_{(0)}X)) &= \nabla^{\sharp}_{B_{(0)}Z}(F^{(r)}(B_{(0)}X)) \\ &= \nabla^{\sharp}_{B_{(0)}Z}(B_{(0)}FX) \\ &= B_{(0)}\nabla_Z(FX).\end{aligned}$$

Noticing that $F^{(r)\sharp}(B_{(0)}\nabla_Z X) = F^{(r)}(B_{(0)}\nabla_Z X)$, from (4. 12) and (4. 13), we have

$$\begin{aligned}\nabla^{\sharp}_{B_{(0)}Z}(F^{(r)\sharp})(B_{(0)}X)&=B_{(0)}\nabla_Z(FX)-F^{(r)}(B_{(0)}\nabla_Z X)\\&=B_{(0)}((\nabla_Z F)X)+B_{(0)}(F\nabla_Z X)-F^{(r)}(B_{(0)}\nabla_Z X)\\&=B_{(0)}((\nabla_Z F)X),\end{aligned}$$

since $B_{(0)}(F\nabla_Z X)=F^{(r)}(B_{(0)}\nabla_Z X)$.

Finally, we shall calculate the curvature tensor of ∇^* along the cross-section $\gamma_V(M)$. By (4. 5), we have

$$\begin{aligned}\nabla^*_{B_{(0)k}}\nabla^*_{B_{(0)j}}B_{(0)i}&=\nabla^*_{B_{(0)k}}\left(\sum_{s=0}^{r}\frac{1}{s!}(\mathcal{L}_V^s\Gamma_{ji}^h)^0B_{(s)h}\right)\\&=\sum_{s=0}^{r}\frac{1}{s!}\left[\partial_k(\mathcal{L}_V^s\Gamma_{ji}^h)^0B_{(s)h}+(\mathcal{L}_V^s\Gamma_{ji}^h)^0\sum_{u=0}^{r-s}\frac{1}{u!}(\mathcal{L}_V^u\Gamma_{kh}^m)^0B_{(s+u)m}\right]\\&=\sum_{s=0}^{r}\frac{1}{s!}\left[\partial_k(\mathcal{L}_V^s\Gamma_{ji}^m)^0+\sum_{u=0}^{s}\binom{s}{u}(\mathcal{L}_V^u\Gamma_{ji}^h)^0(\mathcal{L}_V^{s-u}\Gamma_{kh}^m)^0\right]B_{(s)m}\end{aligned}$$

and hence

$$\begin{aligned}&\nabla^*_{B_{(0)k}}\nabla^*_{B_{(0)j}}B_{(0)}-\nabla^*_{B_{(0)j}}\nabla^*_{B_{(0)k}}B_{(0)i}\\&=\sum_{s=0}^{r}\frac{1}{s!}\Big[\partial_k(\mathcal{L}_V^s\Gamma_{ji}^m)^0-\partial_j(\mathcal{L}_V^s\Gamma_{ki}^m)^0\\&\qquad+\sum_{u=0}^{s}\binom{s}{u}\{(\mathcal{L}_V^u\Gamma_{ji}^h)^0(\mathcal{L}_V^{s-u}\Gamma_{kh}^m)^0-(\mathcal{L}_V^u\Gamma_{ki}^h)^0(\mathcal{L}_V^{s-u}\Gamma_{jh}^m)^0\}\Big]B_{(s)m}.\end{aligned}$$

Now, by (4. 3), have

$$\nabla^*_{B_{(0)k}}\nabla^*_{B_{(0)j}}B_{(0)i}-\nabla^*_{B_{(0)j}}\nabla^*_{B_{(0)k}}B_{(0)i}=\sum_{s=0}^{r}\frac{1}{s!}(\mathcal{L}_V^sR_{xji}{}^h)^0B_{(s)h}.$$

Thus we have, for the curvature tensor R^* of ∇^*,

$$R^*(B_{(0)k},B_{(0)j})B_{(s)i}=\sum_{s=0}^{r}\frac{1}{s!}(\mathcal{L}_V^sR_{kji}{}^h)^0B_{(s)h}. \tag{4. 14}$$

As a direct consequence of (4. 14), we have

PROPOSITION 4. 5. *For arbitrary elements X and Y of $\mathcal{T}_0^1(M)$, the curvature transformation $R^*(B_{(0)}X, B_{(0)}Y)$ leaves the tangent space of $\gamma_V(M)$ invariant at each point if and only if $\mathcal{L}_VR_{kji}{}^h=0$. In this case $R^{*\sharp}=\gamma_V'R$.*

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