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EQUIMEASURABILITY OF FUNCTIONS AND DOUBLY STOCHASTIC OPERATORS

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1. As a continuous version of doubly stochastic matrices, a linear operator T from the real Lebesgue space $\boldsymbol{L}^1(0, 1)$ into itself is called *doubly stochastic* (*d.s.*, in short) if

$$T\mathbf{1}=\mathbf{1}, \tag{1. 1}$$

$$T^*\mathbf{1}=\mathbf{1}, \tag{1. 2}$$

and

$$T\geqq 0, \tag{1. 3}$$

where $\mathbf{1}$ denotes the function whose range is $\{1\}$, and (1. 3) means that $Tf\geqq 0$ whenever $f\geqq 0$. (1. 2) is equivalent to the requirement that $\int_0^1 Tf d\mu=\int_0^1 f d\mu$ for all $f\in\boldsymbol{L}^1$, where μ denotes the Lebesgue measure on (0, 1). As is easily seen, every d.s. operator is a contraction in both $\boldsymbol{L}^1$ and $\boldsymbol{L}^\infty$ norms ($\|T\|_1\leqq 1$, and $\|T\|_\infty\leqq 1$). Furthermore, $Tf\prec f$ holds for all $f\in\boldsymbol{L}^1$, where $\prec$ denotes the continuous version of the preorder of Hardy—Littlewood and Póly [2, 8].

In the sequel, we denote by $\mathfrak{M}$ the set of all Lebesgue measurable sets in $I=(0, 1)$. $\boldsymbol{e}\equiv\boldsymbol{e}'$, $\boldsymbol{e}, \boldsymbol{e}'\in\mathfrak{M}$, means that the measure of the symmetric difference of $\boldsymbol{e}, \boldsymbol{e}'$ is zero, or equivarently, that $\chi_{\boldsymbol{e}}$, the characteristic function of $\boldsymbol{e}$, is identified with $\chi_{\boldsymbol{e}'}$ as an element of $\boldsymbol{L}^1$. Let $\boldsymbol{e}_1, \boldsymbol{e}_2\in\mathfrak{M}$ with $\mu(\boldsymbol{e}_1)=\mu(\boldsymbol{e}_2)$. A mapping σ from $\boldsymbol{e}_1$ (exactly speaking, defined a.e. on $\boldsymbol{e}_1$) into $\boldsymbol{e}_2$ is called a *measure preserving transformation*[1] (*m.p. transformation*, in short) *from* $\boldsymbol{e}_1$ *into* $\boldsymbol{e}_2$, if

$$\sigma^{-1}(\boldsymbol{e})\in\mathfrak{M} \text{ and } \mu(\sigma^{-1}(\boldsymbol{e}))=\mu(\boldsymbol{e}\cap\boldsymbol{e}_2) \quad \text{for all } \boldsymbol{e}\in\mathfrak{M}. \tag{1. 4}$$

If σ^{-1} is a m.p. transformation from $\boldsymbol{e}_2$ into $\boldsymbol{e}_1$ again, σ is called *invertible measure preserving from* $\boldsymbol{e}_1$ *onto* $\boldsymbol{e}_2$. For each m.p. transformation σ from I into itself, the operator T_σ defined by

$$T_\sigma f(t)=f(\sigma t) \qquad (t\in I) \tag{1. 5}$$

is a d.s. operator, and is called a *d.s. operator induced by* σ. In what follows, $\mathcal{D}$ stands for the set of all d.s. operators and $\Sigma(\Sigma_0)$ for the set of all m.p. (resp. invertible m.p.) transformations on I. Then $\mathcal{D}$ is a convex set and each T_σ, $\sigma\in\Sigma$ is, as is easily verified, multiplicative, that is, $T_\sigma(f\cdot g)=T_\sigma f\cdot T_\sigma g$ for all f, $g\in\boldsymbol{L}^\infty$, and is

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1) Two such transformations will be identified if they differ on a set of measure zero.

on extreme point of $\mathscr{D}$ [7]. Also $T_\sigma f \sim f$ holds, where $f \sim g$ means that f and g are equimeasurable.[2] Since every $T \in \mathscr{D}$ acts as a contraction on $\boldsymbol{L}^\infty$, we can consider $\mathscr{D}$ as a subset of the operator space of $\boldsymbol{L}^\infty$. It is known [8] that, according to a general compactness theorem of Kadison [3], $\mathscr{D}$ is compact in the weak*-operator topology.

Let $\mathfrak{x}$ and $\mathfrak{y}$ be n-vetors $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ respectively. It is clear that

(1. 6) *if $\mathfrak{y}$ is a n-vector whose coordinates y_i are obtained by a permutation of the coordinates of $\mathfrak{x}$, then there exists a n-square permutation matrix $\boldsymbol{P}$ such that $\mathfrak{x}=\mathfrak{y}\boldsymbol{P}$.*

A continuous version of this statement would be the following:

(1. 7) *if $f \sim g$, f, $g \in \boldsymbol{L}^1$, there exists an $\sigma \in \Sigma$ such that $T_\sigma f = g$.*

Unfortunately, however, the statement (1. 7) is not valid in general. It is only known [1, 8] that if $f \sim g$, f, $g \in \boldsymbol{L}^1$, there exists an $T \in \mathscr{D}$ such that $Tf=g$. More precisely, Ryff [8] has shown that such a T can be chosen from d.s. operators of the form $T^*_{\sigma_1} T_{\sigma_2}$, σ_1, $\sigma_2 \in \Sigma$.

In §2, we shall present an alternative proof of this Ryff's theorem in a somewhat different form. Namely we shall show that if $f \sim g$, f, $g \in \boldsymbol{L}^1$ there exists an $T \in \mathscr{D}$ such tha $Tf=g$ which is a w^*-cluster point of a sequence of members of T_σ, $\sigma \in \Sigma_0$.

In §3, some fundamental properties of d.s. operators will be studied. In [6] Mirsky called a d.s. operator T a *permutator* if $f \sim Tf$ holds for all $f \in \boldsymbol{L}^1$. We shall show that each permutator T is nothing but a d.s. operator induced by a m.p. transformation σ, i.e., $T=T_\sigma$ (Theorem 5). Also some characterizations for the d.s. operators induced by m.p. transformations will be given.

Finally, in §4, we shall give a necessary and sufficient condition for $f \sim g$, f, $g \in \boldsymbol{L}^1$, under which we can find an $\sigma \in \Sigma$ such that $T_\sigma f = g$ holds.

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2. We shall give an alternative proof of the Ryff's theorem:

THEOREM 1. *If f and g are equimeasurable on $I=(0, 1)$, then $Tf=g$ holds for a d.s. operator T which is a w^*-cluster point of a sequence of members of T_σ, $\sigma \in \Sigma_0$.*

To prove this theorem we use a lemma due to Lorentz [4, p. 60].

LEMMA 1 (Lorentz). *Let f and g be eqimeasurabe. If C is any set of real numbers for which $f^{-1}(C)$ is measurable, then so is $g^{-1}(C)$ and both sets have the same measure.*

The following lemma is known. For the convenience of readers, we present here a proof based on the preceeding lemma.

2) f and g are called equimeasurable if d_f, the distribution function of f, is equal to d_g.

LEMMA 2. *If* $\mu(\boldsymbol{e}_1)=\mu(\boldsymbol{e}_2)$, $\boldsymbol{e}_1, \boldsymbol{e}_2\in\mathfrak{M}$, *then there exists an* $\sigma\in\Sigma_0$ *such that* $\sigma(\boldsymbol{e}_1)\equiv\boldsymbol{e}_2$.

Proof. Let $k_i(t)=\int_0^t \chi_{e_i}d\mu$, $0<t<1$, $i=1, 2$. The functions k_i, $i=1, 2$, are positive, continuous, and non-decreasing on I. Also denote by f_i the function $k_i\chi_{e_i}$. Then it is easy to see that f_1 and f_2 are equimeasurable, and $k_i^{-1}(\lambda)$ is a single point or a closed interval in I for any $\lambda\in(0, \alpha)$, $\alpha=\mu(\boldsymbol{e}_1)=\mu(\boldsymbol{e}_2)$. We put J_i the set of all $\lambda\in(0, \alpha)$ such that $k_i^{-1}(\lambda)$ is not a set of a single point. Then J_i is a countable set for each i. Putting $\tilde{\boldsymbol{e}}_i=f_i^{-1}\{(0, \alpha)-J_1\cup J_2\}$, $i=1, 2$, we see that $\tilde{\boldsymbol{e}}_i\subset\boldsymbol{e}_i$ and $\tilde{\boldsymbol{e}}_i\equiv\boldsymbol{e}_i$. If we define a mapping σ_1 from $\tilde{\boldsymbol{e}}_1$ onto $\tilde{\boldsymbol{e}}_2$ by

$$\sigma_1(s)=f_2^{-1}\{f_1(s)\},\quad s\in\tilde{\boldsymbol{e}}_1, \tag{2.1}$$

σ_1 is a one to one mapping from $\tilde{\boldsymbol{e}}_1$ onto $\tilde{\boldsymbol{e}}_2$. Furthermore, σ_1 is a m.p. transformation from $\tilde{\boldsymbol{e}}_1$ onto $\tilde{\boldsymbol{e}}_2$. In fact, for every $\boldsymbol{e}\in\mathfrak{M}$ with $\boldsymbol{e}\subset\tilde{\boldsymbol{e}}_2$, $\sigma_1^{-1}(\boldsymbol{e})=f_1^{-1}\{f_2(\boldsymbol{e})\}$ is measurable and $\mu(\sigma_1^{-1}(\boldsymbol{e}))=\mu(\boldsymbol{e})$ by Lemma 1. In the same way we can also verify that σ_1^{-1} is a m.p. transformation from $\tilde{\boldsymbol{e}}_2$ onto $\tilde{\boldsymbol{e}}_1$. Thus σ_1 is an invertible m.p. transformation from $\boldsymbol{e}_1$ onto $\boldsymbol{e}_2$, since $\boldsymbol{e}_i\equiv\tilde{\boldsymbol{e}}_i$, $i=1, 2$. Now in the same way we can find an invertible m.p. transformation σ_2 from $\boldsymbol{e}_1^c$ to $\boldsymbol{e}_2^c$. Consequently, putting $\sigma(s)=\sigma_1(s)$ if $s\in\boldsymbol{e}_1$; $\sigma(s)=\sigma_2(s)$ if $s\in\boldsymbol{e}_1^c$, we see that σ is an invertible m.p. transformation on I for which $\sigma(\boldsymbol{e}_1)\equiv\boldsymbol{e}_2$.

From the proof above, it follows that if $\{\boldsymbol{e}_i\}_{i=1}^n$ and $\{\boldsymbol{e}_i'\}_{i=1}^n$ are two systems of mutually disjoint sets of $\mathfrak{M}$ with $\mu(\boldsymbol{e}_i)=\mu(\boldsymbol{e}_i')$ for all $1\leqq i\leqq n$, there exists an $\sigma\in\Sigma_0$ such that $\sigma(\boldsymbol{e}_i)\equiv\boldsymbol{e}_i'$ for all $1\leqq i\leqq n$. Now let $\mathcal{S}$ denote the set of all simple functions on I. Then we have immediately

LEMMA 3. *If* $f\sim g$, $f, g\in\mathcal{S}$, *then there exists an* $\sigma\in\Sigma_0$ *for which* $T_\sigma f=g$ *holds.*

Proof of THEOREM 1. First we prove in the case that $0\leqq f$, $g\in\boldsymbol{L}^1$, and $f\sim g$. For every $n\in N$ (N stands for the set of all integers) let $F_{n,0}=f^{-1}[n, \infty)$, $G_{n,0}=g^{-1}[n, \infty)$, $F_{n,k}=f^{-1}[2^{-n}(k-1), 2^{-n}k)$, and $G_{n,k}=g^{-1}[2^{-n}(k-1), 2^{-n}k)$, where $k=1, \cdots, 2^n n$. Since $f\sim g$ and both $\{F_{n,k}\}_{k=0}^{2^n n}$ and $\{G_{n,k}\}_{k=0}^{2^n n}$ are systems of mutually disjoint sets, Lemma 3 shows that for every $n\in N$ there exists an $\sigma_n\in\Sigma_0$ such that $T_{\sigma_n}\chi_{F_{n,k}}=\chi_{G_{n,k}}$ for all $k=0, \cdots, 2^n{}_n$. If we put

$$f_n=\sum_{k=1}^{2^n n}2^{-n}(k-1)\chi_{F_{n,k}}+n\chi_{F_{n,0}},\qquad g_n=\sum_{k=1}^{2^n n}2^{-n}(k-1)\chi_{G_{n,k}}+n\chi_{G_{n,0}},$$

$T_{\sigma_n}f_n=g_n$, $n\in N$ holds. Moreover, since each $F_{m,k}(G_{m,k})$, $0\leqq k\leqq 2^m m$ is contained in an $F_{n,k}$ (resp. $G_{n,k}$) if $n\leqq m$, we have

$$T_{\sigma_m}f_n=g_n,\qquad \text{if}\quad n\leqq m. \tag{2.2}$$

We write $\mathcal{F}_i=\{T_{\sigma_i}, T_{\sigma_{i+1}}, \cdots\}^{-w^*}$, the closure of $\{T_{\sigma_i}, T_{\sigma_{i+1}}, \cdots\}$ in the w^*-operator topology, for each i. Since $\mathcal{D}$, considered as a subset of the operator space of $\boldsymbol{L}^\infty$, is w^*-compact, there exists an $T\in\mathcal{D}$ such that $T\in\bigcap_{i=1}^\infty\mathcal{F}_i$. For each fixed $m\in N$, there is a subnet $\{T_\alpha\}\subset\{T_{\sigma_m}, T_{\sigma_{m+1}}, \cdots\}$ such that $T=w^*-\lim_\alpha T_\alpha$. Since $T_\alpha f_m=g_m$ holds for every T_α, by (2.2) and $f_m\in\boldsymbol{L}^\infty$, we have

$$\int_0^1 uTf_m\,d\mu=\lim_\alpha\int_0^1 uT_\alpha f_m\,d\mu=\int_0^1 ug_m\,d\mu,$$

for every $u\in\boldsymbol{L}^1$. Hence $Tf_m=g_m$ holds for every $m\in N$. Finally, for every m,

$$\|g-Tf\|_1\leqq\|g-g_m\|_1+\|g_m-Tf_m\|_1+\|Tf_m-Tf\|_1\leqq\|g-g_m\|_1+\|f_m-f\|_1,$$

which implies $g=Tf$.

For a proof in the general case we have only to recall that if $f\sim g\in\boldsymbol{L}^1$ we have $f^+\sim g^+$, $f^-\sim g^-$, and if we construct $f_n^+, g_n^+, f_n^-, g_n^-\in\mathcal{S}$ in a similar way as above, we have $f_n^+-f_n^-\sim g_n^+-g_n^-\in\mathcal{S}$ and $f_n^+-f_n^-\to f$, $g_n^+-g_n^-\to g$ in $\boldsymbol{L}^1$ norm.

3. In the sequel, we denote by R the set of all real numbers. For each $f\in\boldsymbol{L}^1$ and each $\lambda\in R$, we denote by $\boldsymbol{e}(f;\lambda)$ the λ-spectral set, that is, the set $\{t:f(t)>\lambda\}\subset I$; and we denote by $\mathfrak{M}_f$ the σ-algebra generated by these sets. $f^{(\alpha)}$ is the α-truncation of f:

$$f^{(\alpha)}(t)=\alpha(t)\ \text{ if } f(t)>\alpha,\qquad f^{(\alpha)}(t)=f(t)\ \text{ if } f(t)\leqq\alpha. \tag{3. 1}$$

Each function $f\in\boldsymbol{L}^1$ will be called smooth if $\mu\{t:f(t)=\lambda\}=0$ for all $\lambda\in R$.

LEMMA 4. *Let $Tf=g$, $T\in\mathcal{D}$, and $f, g\in\boldsymbol{L}^1$. Then the following statements are equivalent.*

(1) $f\sim g$;

(2) $T(f^{(\alpha)})=g^{(\alpha)}$ *for all* $\alpha\in R$;

(3) $T\chi_{\boldsymbol{e}(f;\lambda)}=\chi_{\boldsymbol{e}(g;\lambda)}$ *for all* $\lambda\in R$.

Proof. (1) implies (2): Since $g=Tf\geqq T(f^{(\alpha)})$ and $\alpha 1\geqq T(f^{(\alpha)})$, we have $g^{(\alpha)}\geqq T(f^{(\alpha)})$. Moreover (1) implies

$$\int_0^1 g^{(\alpha)}\,d\mu=\int_0^1 f^{(\alpha)}\,d\mu=\int_0^1 T(f^{(\alpha)})\,d\mu.$$

Hence we obain (2).

(2) implies (3): For each $f\in\boldsymbol{L}^1$ and each $\lambda\in R$, let denote by $\tilde{e}(f;\lambda)$ the set $\{t:f(t)\geqq\lambda\}$. Then we have

$$\mu\{t:\eta\leqq f(t)<\xi\}=\mu\{t:f^{(\xi)}(t)-f^{(\eta)}(t)\neq(\xi-\eta)\chi_{\tilde{e}(f;\xi)}\}$$

for each $f\in\boldsymbol{L}^1$ and each pair $\xi,\eta\in R$ with $\eta<\xi$. Hence we have

$$\chi_{\tilde{e}(f;\xi)}=\lim_{\eta\uparrow\xi}\frac{f^{(\xi)}-f^{(\eta)}}{\xi-\eta}\qquad(\text{in } \boldsymbol{L}^1 \text{ norm}) \tag{3. 2}$$

for each $f\in\boldsymbol{L}^1$ and each $\xi\in R$. Therefore we get

$$T\chi_{\tilde{e}(f;\xi)}=\chi_{\tilde{e}(g;\xi)},\qquad \xi\in R, \tag{3. 3}$$

on account of (2). Then we can easily obtain (3) by the equality $\tilde{e}(f;\lambda)$

$=\cup_1^\infty \tilde{e}(f;\lambda+1/n)$.

Finally, the implication (3)$\Rightarrow$(1) is clear.

THEOREM 2. *Let* $Tf=g$, $T\in\mathcal{D}$ *and* $f, g\in \boldsymbol{L}^1$. *Then* $f\sim g$ *if and only if* $T^*g=f$.[3)]

Proof. Since $Tf=g$, $T\in\mathcal{D}$ implies $g\prec f$, it is easy to see that $Tf=g$ and $T^*g=f$ imply $f\sim g$. On the other hand, if $Tf=g$ and $f\sim g$, applying the statement (3) in Lemma 4, we have

$$\int_0^1 T^*\chi_{e(g;\lambda)}\,d\mu=\int_0^1 \chi_{e(g;\lambda)}\,d\mu=\int_0^1 \chi_{e(g;\lambda)}T\chi_{e(f;\lambda)}\,d\mu$$
$$=\int_0^1 T^*\chi_{e(g;\lambda)}\cdot\chi_{e(f;\lambda)}\,d\mu.$$

We also have

$$T^*\chi_{e(g;\lambda)}\geqq T^*\chi_{e(g;\lambda)}\cdot\chi_{e(f;\lambda)}.$$

Therefore

$$T^*\chi_{e(g;\lambda)}=T^*\chi_{e(g;\lambda)}\cdot\chi_{e(f;\lambda)} \tag{3. 4}$$

holds. (3. 4) means $T^*\chi_{e(g;\lambda)}\leqq\chi_{e(f;\lambda)}$. Hence we obtain

$$T^*\chi_{e(g;\lambda)}=\chi_{e(f;\lambda)}.$$

From this we can show easily that $T^*g=f$ holds.

Ryff [8] proved the following:

THEOREM 3 (Ryff). *To each* $f\in\boldsymbol{L}^1$ *there corresponds a* $\sigma\in\Sigma$ *such that* $T_\sigma f^*=f$.[4)]

Now we prove the following theorem, which plays an essential role in the rest of the present paper.

THEOREM 4. *For every smooth functon* $f\in\boldsymbol{L}^1$, *there corresponds one and only one d.s. operator* T *such that* $Tf^*=f$. *This operator* T *is induced by some* $\sigma\in\Sigma$. *Moreover,* $f^*=Sf$, $S\in\mathcal{D}$ *implies* $S=T^*$.

Proof. By virtue of Lemma 4, if $Tf^*=f$, and $T\in\mathcal{D}$, then we have $T\chi_{e(f^*;\lambda)}=\chi_{e(f;\lambda)}$, $\lambda\in R$. And our assumption that f be smooth implies $\mathfrak{M}_{f^*}=\mathfrak{M}$. Thus T coincides with T_σ, where $\sigma\in\Sigma$ is obtained by Theorem 3.

Next, suppose $f^*=Sf$. Then we have $S^*f^*=f$ by Theorem 2, we must have $S^*=T$, that is, $S=T^*$.

In Theorem 5 below, we shall give some simple characterizations of d. s. operators induced by m. p. transformations. Also, some of the statements are

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3) $T\in\mathcal{D}$ implies $T^*\in\mathcal{D}$, where T^* is a unique extension of the adjoint of T to an operator acting on $\boldsymbol{L}^1$.

4) f^* is the decreasing rearrangement of f.

nearly clear if we use the result due to v. Neumann [6, p. 582, Satz 1]. For completeness and because the special case is much simpler than the general case, we intend to prove our Theorem 5 by mere use of the preceeding arguments.[5)]

THEOREM 5. *Let T be an d.s. operator. Then the following statements are equivalent.*

(1) *T is a permutator, that is, $f\sim Tf$ for all $f\in \boldsymbol{L}^1$;*

(2) *T is truncation invariant, that is, $Tf^{(\alpha)}=(Tf)^{(\alpha)}$ for all $\alpha\in R$ and all $f\in \boldsymbol{L}^1$;*

(3) *T is multiplicative, that is, $T(f\cdot g)=Tf\cdot Tg$ for all $f, g\in \boldsymbol{L}^\infty$;*

(4) *T is an isometry in $\boldsymbol{L}^1$;*

(5) *$T^*T=I$;*

(6) *T is induced by a $\sigma\in\Sigma$.*

In particular, a d.s. operator T is induced by a $\sigma\in\Sigma_0$ if and only if $TT^*=T^*T=I$.

Proof. First, the equivalence (1)$\Longleftrightarrow$(2) follows from Lemma 4.

Next, we have the implication (1)$\Rightarrow$(6)$\Rightarrow$(3)$\Rightarrow$(5)$\Rightarrow$(1) as follows: Let T be a permutator. Then, in particular, for smooth $f\in \boldsymbol{L}^1$ we have $f^*\sim Tf^*$. Hence follows (6) from Theorem 4. The implication (6)$\Rightarrow$(3) is obvious. If T is multiplicative, $T\chi_E=(T\chi_E)^2$. So, $\chi_E\sim T\chi_E$ holds for each $E\in\mathfrak{M}$. Therefore by virtue of Theorem 2, $T^*T\chi_E=\chi_E$ for all $E\in\mathfrak{M}$, that is, $T^*T=I$. Finally, let $T^*T=I$. If there exists a function $f\in \boldsymbol{L}^1$ such that $f\sim Tf$ does not hold, then we can find a numbers $s\in I$ for which

$$\int_0^s (Tf)^*\,d\mu<\int_0^s f^*\,d\mu \tag{3.5}$$

holds on account of $Tf\prec f$. Thus we must have

$$\int_0^s f^*\,d\mu=\int_0^s (T^*Tf)^*\,d\mu\leqq\int_0^s (Tf)^*\,d\mu<\int_0^s f^*\,d\mu$$

by (3. 5), which is a contradiction.

The proof of the implication (6)$\Rightarrow$(4)$\Rightarrow$(5) is also given as follows: The implication (6)$\Rightarrow$(4) is obvious. To prove (4)$\Rightarrow$(5), we recall an elementary formula that

$$|a+b|+|a-b|=2(|a|+|b|)\quad a, b\in R \quad\text{if and only if } a\cdot b=0. \tag{3.6}$$

Now let T be an isometry. Then, for each $E\in\mathfrak{M}$,

$$||T\chi_E+T\chi_{E^c}||_1+||T\chi_E-T\chi_{E^c}||_1=2(||T\chi_E||_1+||T\chi_{E^c}||_1) \tag{3.7}$$

5) Also, Satz 2 of v. Neumann [7, p. 584] is easily proved by use of Lemma 1, in (0, 1) case.

follows from

$$\|\chi_E+\chi_{E^c}\|_1+\|\chi_E-\chi_{E^c}\|_1=2(\|\chi_E\|_1+\|\chi_{E^c}\|_1).$$

Therefore we have $T\chi_E T\chi_{E^c}=0$ by (3. 6) and (3. 7). Moreover, $T\in\mathcal{D}$ implies $T\chi_E+T\chi_{E^c}=1$. It follows that $\chi_E\sim T\chi_E$ for all $E\in\mathfrak{M}$. Finally by the same argument used for the proof of (3)⇒(5), we obtain the implication (4)⇒(5).

4. Two pairs of functions (f, f') and (g, g') on I are called *simultaneously equimeasurable*, if for each pair of α, $\beta\in R$, we have

$$\mu\{e(f;\alpha)\cap e(f';\beta)\}=\mu\{e(g;\alpha)\cap e(g';\beta)\} \text{ [4, p. 61].}$$

We write

$$(f, f')\sim(g, g') \tag{4. 1}$$

if (f, f') and (g, g') are simultaneously equimeasurable.

Now we shall call an f to be *strongly equimeasurable* with g, and write $f\rightsquigarrow g$, if for each $f'\in L^1$, there corresponds some g' which satisfies $(f, f')\sim(g, g')$. It is clear that $f\rightsquigarrow g$ implies $f\sim g$.

THEOREM 6. *f is strongly equimeasurable with g if and only if $T_\sigma f=g$ holds for some m.p. transformation σ.*

Proof. If f is strongly equimeasurable with g, by the definition, there is a function $u\in L^1$ which satisfies both $x\sim u$ and

$$(f, x)\sim(g, u).^{6)} \tag{4. 2}$$

Then, by Theorem 4, there is a unique $\sigma\in\Sigma$ so that equality $T_\sigma x=u$, and for every $\alpha\in R$,

$$\int_{(\beta,\beta']}\chi_{e(f;\alpha)}\,d\mu=\int_{\sigma^{-1}(\beta,\beta']}\chi_{e(g;\alpha)}\,d\mu \qquad (\beta, \beta'\in I) \tag{4. 3}$$

holds. It is easy to see that (4. 3) implies

$$\int_E\chi_{e(f;\alpha)}\,d\mu=\int_{\sigma^{-1}(E)}\chi_{e(f;\alpha)}\,d\mu=\int_0^1\chi_{e(g;\alpha)}T_\sigma\chi_E\,d\mu, \tag{4. 4}$$

for each $E\in\mathfrak{M}$.

Substituting $E=e(f;\alpha)$ in (4. 4), we have, on account of $f\sim g$,

$$\int_0^1\chi_{e(g;\alpha)}\,d\mu=\int_0^1\chi_{e(f;\alpha)}\,d\mu=\int_0^1\chi_{e(g;\alpha)}T_\sigma\chi_{e(f;\alpha)}\,d\mu. \tag{4. 5}$$

(4. 5) means $\chi_{e(g;\alpha)}\leqq T_\sigma\chi_{e(f;\alpha)}$. Thus we have $\chi_{e(g;\alpha)}=T_\sigma\chi_{e(f;\alpha)}$, that is, $T_\sigma f=g$, since

6) x denote the function $x(t)=t$.

$\alpha \in R$ is arbitrary.

The converse implication is clear; we have only to set $g' = T_\sigma f'$ for each $f' \in \boldsymbol{L}^1$.

THEOREM 7. *For each $f \in \boldsymbol{L}^1$, the following conditions are quivarent.*

(1) $\mathfrak{M}_f = \mathfrak{M}$;

(2) *for each g with $f \rightsquigarrow g$, there corresponds a unique $T \in \mathcal{D}$ such that $Tf = g$;*

(3) *for each g with $f \rightsquigarrow g$, there corresponds a unique $u \in \boldsymbol{L}^1$ such that $(f; x) \sim (g; u)$.*

Proof. The implication (1)$\Rightarrow$(2) is an immediate consequence of the statement (1) in Lemma 4.

(2) implies (1): In general, if $f \sim g \in \boldsymbol{L}^1$ is not smooth, we can easily construct two d.s. operators T_i, $i=1, 2$ such that $T_1 \neq T_2$ and $T_i f = g$, $i=1.2$, since the Lebesgue measure on I is non-atomic. Thus, under the condition (2), f is smooth.

Then, by the condition $f \rightsquigarrow g$, and by use of Theorem 6, we have

$$T_{\sigma_1} f = g, \qquad \text{for a unique } \sigma_1 \in \Sigma. \tag{4.6}$$

On the other hand, we have

(4.7) $T_\sigma f^* = f$, for a unique $\sigma_2 \in \Sigma$, by Theorem 4.

Then (4.6) and (4.7) imply

$$T_{\sigma_1} T_{\sigma_2} T^*_{\sigma_2} f = g. \tag{4.8}$$

Therefore we must have

$$T_{\sigma_1} = T_{\sigma_1} T_{\sigma_2} T^*_{\sigma_2}. \tag{4.9}$$

And (4.9) holds if and only if $T_{\sigma_2} T^*_{\sigma_2} = I$, by (5) of Theorem 5; this is equivalent to $\sigma_2 \in \Sigma_0$, by Theorem 5 again.

If $\sigma_2 \in \Sigma_0$, then there exists, for each $E \in \mathfrak{M}$, an $F \in \mathfrak{M}$ such that $\chi_E = T_{\sigma_2} \chi_F$. Since f is smooth, F must belongs to $\mathfrak{M}_{f^*}$. Consequently, (4.7) implies $\mathfrak{M}_f = \mathfrak{M}$.

Finally, the implication (3)$\Leftrightarrow$(2) is implicit in the proof of Theorem 6.

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