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ON A PIECE OF SURFACE IN A FIBRED SPACE

BY MARIKO KONISHI

In 1955 Heinz [1] proved the following

THEOREM A. *Let $z=z(x, y)$ be a 2-dimensional surface in a 3-dimensional Euclidean space defined over the disk $x^2+y^2<R^2$, where $z(x, y)$ is a C^2-class function. Let H and K denote its mean curvature and Gaussian curvature respectively.*

If $|H| \leqq c>0$, *then* $R \leqq \frac{1}{c}$.

If $K \geqq c>0$, *then* $R \leqq \left(\frac{1}{c}\right)^{1/2}$.

If $K \leqq -c<0$, *then* $R \leqq e\left(\frac{3}{c}\right)^{1/2}$.

(c=*constant in all cases.*)

Generalizing this, in 1965 Chern [2] obtained

THEOREM B. *Let M^n be a compact piece of an oriented hypersuface (of dimension n) with smooth boundary ∂M^n, which is immersed in Euclidean space E^{n+1}. Suppose that the mean curvature $H_1 \geqq c$ (c=const.). Let a be a fixed unit vector which makes an angle $\leqq \pi/2$ with all the normals of M^n. Then*

$$nc V_a \leqq L_a$$

where V_a is the volume of the orthogonal projection of M^n and L_a that of ∂M^n on the hyperplane perpendicular to a. When M^n is defined by the equation

$$z=z(x_1, \cdots, x_n), \qquad x_1{}^2+\cdots+x_n{}^2 \leqq R^2,$$

where $x_1, \cdots, x_n, z$ are rectangular coordinates in the space E^{n+1} and $a=(0, \cdots, 0, 1)$, then $cR \leqq 1$.

Katsurada [5] extended this theorem to a compact piece of a hypersurface in a Riemann manifold admitting a conformal killing vector field. The purpose of the present paper is to study this problem in a fibred space with some properties; that is, to prove Theorem 3.

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1. Fibred spaces.[1)]

The set $(\tilde{M}, M, \pi; \tilde{E}, \tilde{G})$ is called a *fibred space* if it satisfies the following five conditions:

1) $\tilde{M}$, M are two differentiable manifolds of dimension $n+1$ and n respectively.
2) π is a differentiable mapping from $\tilde{M}$ onto M and of maximum rank n.
3) The inverse image $\pi^{-1}(p)$ of a point $p \in M$ is a 1-dimensional connected submanifold of $\tilde{M}$. We denote $\pi^{-1}(p)$ by F_p and call F_p the fiber over the point P.
4) $\tilde{G}$ is a positive definite Riemannian metric.
5) $\tilde{E}$ is a unit vector field in $\tilde{M}$ tangent to the fiber everywhere.

Moreover, if $\pounds\tilde{G}=0$ (here and in the sequel $\pounds$ denotes Lie derivation with respect to $\tilde{E}$), we call $\tilde{G}$ an *invariant metric*. Let $\tilde{U}$ be a coordinate neighborhood and $(x^\alpha)=(x^1, \cdots, x^{n+1})$ be local coordinates defined in $\tilde{U}$, where and in the sequel the indices $\alpha, \beta, \cdots$ run over the range $\{1, 2, \cdots, n+1\}$. We denote the components of $\tilde{E}$ and $\tilde{G}$ with respect to these coordinates by E^γ and $G_{\gamma\beta}$ respectively. If $(\xi^i) =(\xi^1, \cdots, \xi^n)$ are local coordinates in $\pi(\tilde{U})$, π has a local expression

$$\xi^i = f^i(x^\alpha), \tag{1.1}$$

f^i $(i=1, \cdots, n)$ being certain functions, where and in the sequel, the indices $i, j, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$. Then the differential of π has the local expression

$$d\xi^i = E_\alpha{}^i\, dx^\alpha,$$

where we have put $E_\alpha{}^i = \partial_\alpha f^i$, ∂_α denoting the differential operator $\partial/\partial x^\alpha$. We see that the n local covector fields $\zeta^i = E_\alpha{}^i\, dx^\alpha$ are linearly independent in $\tilde{U}$. Putting

$$E_\beta = G_{\gamma\beta} E^\gamma, \tag{1.2}$$

we denote by $\tilde{\eta}$ the 1-form whose components are E_β in $\tilde{U}$.

We now find

$$E^\alpha E_\alpha{}^i = 0$$

because the vector field $\tilde{E}$ is tangent to fibers, i.e, $d\pi(\tilde{E})=0$. Consequently, the inverse of the matrix $(E_\alpha{}^i, E_\alpha)$ has the form

$$(E_\alpha{}^i, E_\alpha)^{-1} = \begin{pmatrix} E^\beta{}_h \\ E^\beta \end{pmatrix}$$

and thus for each fixed index h, $E^\beta{}_h$ are components of a local vector field $\tilde{A}_h$ in U.

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1) As to notations and the definitions of fibred spaces we follow [7] and [8].

If we assume that $\tilde{G}$ satisfies the condition

$$\pounds(G_{\gamma\beta}E^{\gamma}{}_{j}E^{\beta}{}_{i})=0, \tag{1. 3}$$

then we can induce a metric g on M whose components are $g_{ji}=G_{\gamma\beta}E^{\gamma}{}_{j}E^{\beta}{}_{i}$ in U. In this sense, when a Riemannian metric $\tilde{G}$ satisfies (1. 3), $\tilde{G}$ is called a *projectable metric* and g is called the *induced metric* in M from $\tilde{G}$. In the sequel, a fibred space $(\tilde{M}, M, \pi; \tilde{E}, \tilde{G})$ is called, for simplicity, a *fibred space with projectable* (*resp. invariant*) *metric* when $\tilde{G}$ is projectable (resp. invariant) metric.

2. A piece of hypersurface in a fibred space.

Let $(\tilde{M}, M, \pi; \tilde{E}, \tilde{G})$ be a fibred space with projectale metric $\tilde{G}$. Consider a compact piece $\tilde{M}^n$ of an orientable hypersurface of dimension n in $\tilde{M}$ and denote by $\partial\tilde{M}^n$ the boundary of the compact piece $\tilde{M}^n$. We suppose that $\tilde{M}^n$ meets at most once each fiber. For simplicity, we say that such a piece $\tilde{M}^n$ of a hypersurface is a simple covering of the projection $M^n=\pi(\tilde{M}^n)$.

We now assume that $\tilde{M}^n$ has a local expression

$$x^{\gamma}=x^{\gamma}(u^{j}), \tag{2. 1}$$

where $(u^{j})=(u^{1}, \cdots, u^{n})$ are local parameters of $\tilde{M}^n$, and that the boundary $\partial\tilde{M}^n$ has a local expression

$$u^{j}=u^{j}(r^{a}),$$

where $(r^{a})=(r^{1}, \cdots, r^{n-1})$ are local parameters of $\partial\tilde{M}^n$. The indices $a, b, c, \cdots$ run over the range $\{1, 2, \cdots, n-1\}$.

If we put

$$B_{j}{}^{\gamma}=\frac{\partial x^{\gamma}}{\partial u^{j}},$$

then $\tilde{B}_{j}=(B_{j}{}^{\gamma})$ are vectors tangent to $\tilde{M}^n$. We choose a unit vector $\tilde{C}$ normal to $\tilde{M}^n$ in such a way that the determinant of the matrix $(C^{\gamma}, B_{j}{}^{\gamma})$ is positive, C^{γ} being the components of $\tilde{C}$. We put

$$E^{\beta}=v^{j}B_{j}{}^{\beta}+\alpha C^{\beta} \tag{2. 2}$$

on the compact piece $\tilde{M}^n$. Denoting by $\tilde{g}_{ji}=B_{j}{}^{\gamma}B_{i}{}^{\beta}G_{\gamma\beta}$ the metric tensor on $\tilde{M}^n$ induced from $\tilde{G}$ and setting

$$v_{i}=\tilde{g}_{ji}v^{j},$$

we have

$$v_{i}=B_{i}{}^{\gamma}E_{\gamma}, \tag{2. 3}$$

because of (1. 2) and (2. 2). Hence we have

$$\nabla_{j}v_{i}=\alpha h_{ji}+B_{j}{}^{\gamma}B_{i}{}^{\beta}\nabla_{\gamma}E_{\beta} \tag{2. 4}$$

along $\tilde{M}^n$, where h_{ji} denotes the second fundamental tensor of $\tilde{M}^n$. Transvecting (2. 3) with $\tilde{g}^{ji}$, we get

$$\tilde{g}^{ji}\nabla_j v_i=\alpha(nH_1)+\frac{1}{2}\tilde{g}^{ji}B_j{}^\gamma B_i{}^\beta \pounds G_{\gamma\beta}, \tag{2. 5}$$

where H_1 is the mean curvature of $\tilde{M}^n$, i.e. $H_1=(1/n)\tilde{g}^{ji}h_{ji}$. Integrating both sides of (2. 5) over $\tilde{M}^n$ and applying Stokes' theorem, we have

$$\int_{\partial\tilde{M}^n} v_j D^j\, d\tilde{\sigma}=n\int_{\tilde{M}^n} H_1\alpha\, d\tilde{V}+\frac{1}{2}\int_{\tilde{M}^n}\tilde{g}^{ji}B_j{}^\gamma B_i{}^\beta \pounds G_{\gamma\beta}\, d\tilde{V}, \tag{2. 6}$$

$\tilde{D}=(D^j)$ being the unit vector field normal to $\tilde{C}$ and to the boundary $\partial\tilde{M}^n$. In the integral formula (2. 6) $d\tilde{\sigma}$ and $d\tilde{V}$ denote the volume elements of $\partial\tilde{M}^n$ and $\tilde{M}^n$ respectively, that is,

$$d\tilde{\sigma}=\sqrt{\det\bar{g}}\, dr^1\wedge\cdots\wedge dr^{n-1},$$
$$d\tilde{V}=\sqrt{\det\tilde{g}}\, du^1\wedge\cdots\wedge du^n,$$

where we have put

$$\bar{B}_a=(\bar{B}_a{}^j)=\left(\frac{\partial u^j}{\partial r^a}\right), \qquad \bar{g}_{cb}=\tilde{g}(\bar{B}_c, \bar{B}_b),$$

$\det\bar{g}$ and $\det\tilde{g}$ denoting the determinants formed with $(\bar{g}_{cb})$ and $(\tilde{g}_{ji})$ respectively.

From the definitions of $\tilde{g}$ and $\tilde{C}$, we have

$$\sqrt{\det\tilde{G}}\det(\tilde{C}, \tilde{B}_j)=\sqrt{\det\tilde{g}}. \tag{2. 7}$$

Here and in the sequel $\det(\tilde{C}, \tilde{B}_j)$ denotes the determinant of the matrix $(C^\alpha, B_j{}^\alpha)$.

On the other hand, since the Riemannian metric g induced on the base space M from $\tilde{G}$ has the components

$$g_{ji}=\tilde{G}(\tilde{A}_j, \tilde{A}_i),$$

we have

$$\det\tilde{G}\{\det(\tilde{E}, \tilde{A}_j)\}^2=\det g. \tag{2. 8}$$

Since $\tilde{M}^n$ is nowhere tangent to fibres, we can choose $(\xi^j)=(\xi^1, \cdots, \xi^n)$ as the local parameters of $\tilde{M}^n$. If we substitute the local expression (2. 1) with $u^j=\xi^j$ in (1. 1), we have the identity

$$\xi^j=f^j(x^\alpha(\xi^k)).$$

Then, differentiating the equation above, we have

$$\delta_i{}^j=E_\alpha{}^j B_i{}^\alpha \tag{2. 9}$$

and consequently

$$\tilde{B}_i=\tilde{A}_i+\tilde{\eta}(\tilde{B}_i)\tilde{E}, \qquad \tilde{\eta}(\tilde{B}_i)=v_i \tag{2. 10}$$

in $\tilde{U}$.

Taking account of (2. 2) and (2. 10), we get

$$\text{(2. 11)} \qquad \begin{aligned} &\det(\tilde{E}, \tilde{B}_j) = \det(v^i\tilde{B}_i + \alpha\tilde{C}, \tilde{B}_j) = \alpha \det(\tilde{C}, \tilde{B}_j), \\ &\det(\tilde{E}, \tilde{A}_j) = \det(\tilde{E}, \tilde{B}_j - v_j\tilde{E}) = \det(\tilde{E}, \tilde{B}_j). \end{aligned}$$

Consequently, if we assume $\alpha > 0$, we have from (2. 7), (2. 8) and (2.11)

$$\text{(2. 12)} \qquad |\alpha| \sqrt{\det \bar{g}} = \sqrt{\det g}.$$

The metric $\bar{g}$ of $\partial\tilde{M}^n$ induced from $\tilde{g}$ being defined by

$$\bar{g}_{cb} = \tilde{G}(\widetilde{BB}_c, \widetilde{BB}_b),$$

we have

$$\text{(2. 13)} \qquad (\det \tilde{G})\{\det(\tilde{C}, \widetilde{BB}_c, \widetilde{BB}_b)\}^2 = \det \bar{g},$$

where $(\widetilde{BD})^\gamma = B_j{}^\gamma D^j$ and $(\widetilde{BB}_b)^\gamma = B_j{}^\gamma B_b{}^j$. On the other hand, denoting by $*g$ the metric induced on $\pi(\partial\tilde{M}^n)$ from the induced metric g of M, we have

$$\text{(2. 14)} \qquad \sqrt{\det \tilde{G}} \det(\widetilde{AN}, \tilde{E}, \widetilde{AB}_b) = \sqrt{\det {*g}},$$

where $N = (N^j)$ denotes the unit normal to $\pi(\partial\tilde{M}^n)$, and $(\widetilde{AN})^\gamma = E^\gamma{}_j N^j$ is such that the determinant of the matrix $(\widetilde{AN}, \tilde{E}, \widetilde{AB}_b)$ is positive. The unit vector $\tilde{C}$ normal to $\tilde{M}^n$ is a linear combination of $\widetilde{AN}$, $\tilde{E}$, and $\widetilde{AB}_d$, i.e.,

$$\tilde{C} = a(\widetilde{AN}) + b^d(\widetilde{AB}_d) + \alpha\tilde{E}$$

a, b^d being certain functions where $|a| \leqq 1$. Thus we have

$$\text{(2. 15)} \qquad \det(\tilde{C}, \tilde{E}, \widetilde{AB}_b) = |a| \det(\widetilde{AN}, \tilde{E}, \widetilde{AB}_b).$$

If we put

$$\tilde{E} = (v_j D^j)\widetilde{BD} + d^a(\widetilde{BB}_a) + \alpha\tilde{C}$$

for certain functions d^a, we have

$$\text{(2. 16)} \qquad \begin{aligned} \det(\tilde{C}, \tilde{E}, \widetilde{AB}_b) &= \det(\tilde{C}, \tilde{E}, \widetilde{BB}_b) \\ &= v_j D^j \det(\tilde{C}, \widetilde{BD}, \widetilde{BB}_b), \end{aligned}$$

by virtue of (2. 8) and (2. 10).

Now we suppose that $\tilde{D}$ is chosen in such a way that $\det(\tilde{C}, \widetilde{BD}, \widetilde{BB}_b) > 0$. Then we have $v_j D^j \geqq 0$ and

$$\text{(2. 17)} \qquad (v_j D^j) \sqrt{\det \bar{g}} = |a| \sqrt{\det {*g}}$$

because of (2. 13)$\sim$(2. 16).

Returning to the integral formula (2. 5) and taking account of (2. 12) and

(2.17), we get

$$\int_{\partial\tilde{M}^n} |a| \sqrt{\det {}^*g}\, dr^1 \wedge \cdots \wedge dr^{n-1} = n \int_{\tilde{M}^n} H_1 \sqrt{\det g}\, du^1 \wedge \cdots du^n + \int_{\tilde{M}^n} G^{*\gamma\beta} \pounds G_{\gamma\beta}\, d\tilde{V}$$

and hence by virtue of $|a| \leqq 1$

$$\int_{\partial\tilde{M}^n} \sqrt{\det {}^*g}\, dr^1 \wedge \cdots \wedge dr^{n-1} = n \int_{\tilde{M}^n} H_1 \sqrt{\det g}\, du^1 \wedge \cdots du^n + \int_{\tilde{M}^n} G^{*\gamma\beta} \pounds G_{\gamma\beta}\, d\tilde{V}, \tag{2.18}$$

where we have put $G^{*\gamma\beta} = \tilde{g}^{ji} B_j{}^\gamma B_i{}^\beta$.

If we assume that $H_1 \geqq c > 0$ (c: constant), $\alpha > 0$ and

$$\int_{\tilde{M}^n} G^{*\gamma\beta} \pounds G_{\gamma\beta}\, d\tilde{V} \geqq 0,$$

then we get

$$\int_{\pi(\partial\tilde{M}^n)} d\sigma \geqq nc \int_{\pi(\tilde{M}^n)} dV,$$

where $d\sigma$ and dV are the volume elements of $\pi(\partial\tilde{M}^n)$ and $\pi(\tilde{M}^n)$ respectively. Therefore we obtain

THEOREM 1. *Let $(\tilde{M}, M, \pi : \tilde{E}, \tilde{G})$ be a fibred space with projectable metric $\tilde{G}$. Let $\tilde{M}^n$ be a compact piece of an oriented hypersurface in $\tilde{M}$ with compact smooth boudary $\partial\tilde{M}^n$, which covers simply the projection $\pi(\tilde{M}^n)$. Suppose that its mean curvatvre H_1 satisfies the condition $H_1 \geqq c > 0$ (c: constant) and that $\tilde{E}$ makes an angle $\leqq \pi/2$ with the normals of $\tilde{M}^n$ at each point. If the condition*

$$\int_{\tilde{M}^n} G^{*\gamma\beta} \pounds G_{\gamma\beta}\, d\tilde{V} \geqq 0 \tag{2.19}$$

holds, then the inequality

$$ncV \leqq L \tag{2.20}$$

holds, where V and L denote the volume of the projection of $\tilde{M}^n$ and $\partial\tilde{M}^n$ respectively.

REMARK 1. When $\tilde{M}^n$ is a compact hypersurface, $\partial\tilde{M}^n$ is empty. Thus taking account of (2.18), we see that there is no compact hypersurface satisfying the conditions mentioned in Theorem 1. In other words we can say that if $\tilde{M}^n$ is a compact hypersurface of constant mean curvature, then $\tilde{M}^n$ must be minimal.

REMARK 2. When $(\tilde{M}, M, \pi : \tilde{E}, \tilde{G})$ is a fibred space with invariant Riemanian metric, the condition (2.19) mentioned in Theorem 1 obviously holds.

For a projectable metric $\tilde{G}$, if we put

$$\pounds E_\gamma = \phi_j E_\gamma{}^j,$$

for certain functions ϕ_j, then we have

$$G_{\gamma\beta}=\phi_j(E_\gamma{}^j E_\beta+E_\gamma E_\beta{}^j)$$

by virtue of $\pounds\tilde{E}=0$. Hence we have

$$G^{*\gamma\beta}\pounds G_{\gamma\beta}=g^{ji}B_j{}^\gamma B_i{}^\beta\phi_k(E_\gamma{}^k E_\beta+E_\gamma E_\beta{}^k)$$
$$=2\tilde{g}^{ji}v_j\phi_i$$

by virtue of (2. 3) and (2. 8). Thus we get

$$G^{*\gamma\beta}\pounds G_{\gamma\beta}=2v^i\phi_i=2v^iB_i{}^\gamma\pounds E_\gamma$$
$$=2(E^\gamma-\alpha C^\gamma)\pounds E_\gamma=-2\alpha C^\gamma\pounds E_\gamma.$$

We note the above obtained results in the following remark.

REMARK 3. The condition (2. 19) is equivalent to the condition

$$\int_{\tilde{M}^n} C^\gamma\pounds E_\gamma\, d\tilde{V}\leqq 0.$$

3. A piece of submanifold of co-dimension 2.

In this section we discuss a compact piece $\tilde{M}^{n-1}$ of $(n-1)$-dimensional orientable submanifold of co-dimension 2 in a fibred space $(\tilde{M}, M, \pi:\tilde{E}, \tilde{G})$. We also suppose that $\tilde{M}^{n-1}$ is a simple covering of the projection $M^{n-1}=\pi(\tilde{M}^{n-1})$ in the above mentioned sense.

We now assume that $\tilde{M}^{n-1}$ has a local expression

$$x^\alpha=x^\alpha(u^{\bar{j}}), \tag{3. 1}$$

$(u^{\bar{j}})$[2] $=(u^1, \cdots, u^{n-1})$ being local parameters of $\tilde{M}^{n-1}$, and that the boundary $\partial\tilde{M}^{n-1}$ has a local expression

$$u^{\bar{j}}=u^{\bar{j}}(r^{\bar{a}}),$$

$(r^{\bar{a}})$[3] $=(r^1, \cdots, r^{n-2})$ being local parameters of $\partial\tilde{M}^{n-1}$. If we put

$$B_{\bar{j}}{}^\alpha=\frac{\partial x^\alpha}{\partial u^{\bar{j}}},$$

then we have $n-1$ linearly independent vectors $\tilde{D}_{\bar{j}}=(B_{\bar{j}}{}^\alpha)$ tangent to $\tilde{M}^{n-1}$.

Let $\tilde{C}_1=(C_1{}^\alpha)$, $\tilde{C}_2=(C_2{}^\alpha)$ be mutually orthogonal unit vectors normal to $\tilde{M}^{n-1}$ and $\tilde{h}_1=(h_{(1)\bar{j}\bar{i}})$, $\tilde{h}_2=(h_{(2)\bar{j}\bar{i}})$ be the second fundamental tensors with respect to $\tilde{C}_1$, $\tilde{C}_2$, reespectively. A vector field $\tilde{H}=(H^\alpha)$ defined by

$$H^\alpha=\frac{1}{n-1}(h_{(1)\bar{j}\bar{i}}C_1{}^\alpha+h_{(2)\bar{j}\bar{i}}C_2{}^\alpha)\tilde{g}^{\bar{j}\bar{i}}$$

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2) The indicies $\bar{i}, \bar{j}, \cdots$ run over the range $\{1, \cdots, n-1\}$.

3) The indicies $\bar{a}, \bar{b}, \cdots$ run over the range $\{1, \cdots, n-2\}$.

is independent of the choice of $(\tilde{C}_1, \tilde{C}_2)$, and we call $\tilde{H}$ the mean curvature vector field of $\tilde{M}^{n-1}$. The magnitude H_1 of the mean curvature vector field is called the mean curvature of $\tilde{M}^{n-1}$, i.e.,

$$H_1=\frac{1}{n-1}(\tilde{g}^{ji}h_{(1)ji}+\tilde{g}^{ji}h_{(2)ji}).$$

If H_1 is positive, we can take the first unit normal $\tilde{C}_1$ in the direction of the mean curvature vector field H. In this case we see that

$$\tilde{g}^{ji}h_{(2)ji}=0 \quad \text{and} \quad \frac{1}{n-1}\tilde{g}^{ji}h_{(1)ji}=H_1. \tag{3.2}$$

We put

$$E^{\gamma}=v^jB_j{}^{\gamma}+\alpha C_1{}^{\gamma}+\beta C_2{}^{\gamma} \tag{3.3}$$

on the compact piece $\tilde{M}^{n-1}$. Putting

$$v_i=\tilde{g}_{ji}v^j,$$

we have

$$v_i=B_i{}^{\gamma}E_{\gamma}, \tag{3.4}$$

because of (1. 2) and (3. 3). Hence we have

$$\nabla_jv_i=\alpha h_{ji}+\beta k_{ji}+B_j{}^{\gamma}B_i{}^{\beta}\nabla_{\gamma}E_{\beta} \tag{3.5}$$

along $\tilde{M}^{n-1}$. Transvecting (3. 4) with $\tilde{g}^{ji}$, we get

$$\tilde{g}^{ji}\nabla_jv_i=\alpha(nH_1)+\frac{1}{2}\tilde{g}^{ji}B_j{}^{\gamma}B_i\pounds G_{\gamma\beta} \tag{3.6}$$

by virtue of (3. 2). Integrating both sides of (3. 6) over $\tilde{M}^{n-1}$ and applying Stokes' theorem, we have

$$\int_{\partial\tilde{M}^{n-1}}v_jD^jd\tilde{\sigma}=n\int_{\tilde{M}^{n-1}}H_1\alpha\, d\tilde{V}+\frac{1}{2}\int_{\tilde{M}^{n-1}}\tilde{g}^{ji}B_j{}^{\gamma}B_i{}^{\beta}\pounds G_{\gamma\beta}\, d\tilde{V}, \tag{3.7}$$

where $\tilde{D}=(D^j)$ is the unit vector field normal to $\tilde{C}_1$, $\tilde{C}_2$ and to the boundary $\partial\tilde{M}^{n-1}$, and $d\tilde{\sigma}$, $d\tilde{V}$ denote the volume elements of $\partial\tilde{M}^{n-1}$, $\tilde{M}^{n-1}$ respectively. Next we will compare $d\tilde{V}$ with the volume element of $\pi(\tilde{M}^{n-1})$. Since $\tilde{M}^{n-1}$ is nowhere tangent to fibres, we can choose $(u^j)=(u^1, \cdots, u^{n-1})$ as the local parameters of the projection M^{n-1} that is, M^{n-1} has the local expression

$$\xi^i=\xi^i(u^j),$$

and by virtue of (1. 1), we have the identity

$$\xi^i(u^j)=f^i(x^{\alpha}(u^j)).$$

Then, differentiating the equation above, we have

$$\frac{\partial \xi^i}{\partial u^j}=E_\alpha{}^i B_j{}^\alpha$$

and consequently, if we set

$$B_j=(B_j{}^i)=\left(\frac{\partial \xi^i}{\partial u^j}\right),$$

(3. 8)

$$\widetilde{AB}_j=(E_i{}^\alpha B_j{}^i),$$

then we have

(3. 9)

$$\widetilde{AB}_j=\tilde{B}_j-v_j\tilde{E}$$

in $\tilde{U}$. From the definition of $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{g}$, we have

$$\sqrt{\det \tilde{G}}\det(\tilde{C}_1,\tilde{C}_2,\tilde{B}_j)=\sqrt{\det\tilde{g}}.$$

On the other hand, denoting by g the metric induced on $\pi(\tilde{M}^{n-1})$ from the induced metric of M, we have

$$\det\tilde{G}\{\det(\tilde{E},\widetilde{AN}_1,\widetilde{AB}_j)\}^2=\det g$$

where $N_1=(N_1{}^i)$ denotes the unit normal to $\pi(\tilde{M}^{n-1})$, and $(\widetilde{AN}_1)^r=E^r{}_jN_1{}^j$. If we suppose that $\widetilde{AN}=\tilde{C}_2$ or equivalently $\beta=0$ in (3. 3), then we have

$$\det(\tilde{E},\widetilde{AN}_1,\widetilde{AB}_j)=\det(\tilde{E},\tilde{C}_2,\widetilde{AB}_j)=\alpha\det(\tilde{C}_1,\tilde{C}_2,\tilde{B}_j)$$

by virtue of (3. 3) and (3. 8). Thus we obtain the relation

$$|\alpha|\sqrt{\det\tilde{g}}=\sqrt{\det g}.$$

Henceforth we assume that $\alpha>0$, and then we have

$$\alpha\, d\tilde{V}=dV.$$

If we denote by $\bar{g}$ the induced metric on $\partial\tilde{M}^{n-1}$ from $\tilde{g}$, $\bar{g}$ is given by

$$\bar{g}_{\bar{c}\bar{b}}=\tilde{G}(\widetilde{BB}_{\bar{c}}\widetilde{BB}_{\bar{b}}),$$

and $\det\bar{g}$ by

(3. 10)

$$\det\tilde{G}\{\det(\tilde{C}_1,\tilde{C}_2,\widetilde{BD},\widetilde{BB}_{\bar{b}})\}^2=\det\bar{g},$$

where $(\widetilde{BB}_{\bar{b}})^r=B_j{}^rB_{\bar{b}}{}^j$ and $(\widetilde{BD})^r=B_j{}^rD^j$. On the other hand, as for the metric $*g$ on $\pi(\partial\tilde{M}^{n-1})$ induced from the metric g of $\pi(\tilde{M}^{n-1})$, we have

(3. 11)

$$\det\tilde{G}\{\det(\widetilde{AN},\tilde{E},\widetilde{AN}_2,\widetilde{AB}_{\bar{b}})\}^2=\det *g,$$

where $N_2=(N_2{}^j)$ denotes the unit normal to $\pi(\partial\tilde{M}^{n-1})$, $(\widetilde{AN}_2)^r=E^r{}_jB_i{}^jN_2{}^i$ and $(\widetilde{AB}_{\bar{b}})^r=E^r{}_jB_i{}^jB_{\bar{b}}{}^i$.

The unit vector $\tilde{C}_1$ normal to $\tilde{M}^{n-1}$ is a linear combination of $\widetilde{AN}_1$, $\widetilde{AN}_2$, $\tilde{E}$ and $\widetilde{AB}_d$, i.e.

$$\tilde{C}_1 = a(\widetilde{AN_1}) + b(\widetilde{AN_2}) + c^a(\widetilde{AB_{\bar a}}) + \alpha\tilde{E},$$

a, b, c^a being certain functions $|a| \leqq 1$, $|b| \leqq 1$.

Therefore we have

$$\det(\tilde{E}, \tilde{C}_1, \widetilde{AN_1}, \widetilde{AB_{\bar b}}) = |b| \det(\tilde{E}, \widetilde{AN_2}, \widetilde{AN_1}, \widetilde{AB_{\bar b}}). \tag{3.12}$$

Putting

$$\tilde{E} = (v_j D^j)\widetilde{BD} + d^a(\widetilde{BB_{\bar a}}) + \alpha\tilde{C}_1 + \beta\tilde{C}_2$$

for certain functions d^a, and taking account of

$$\widetilde{AB_{\bar b}} = \widetilde{BB_{\bar b}} - (v_j B_{\bar b}{}^j)\tilde{E}$$

obtained from (3. 9), we have

$$\begin{aligned} \det(\tilde{E}, \tilde{C}_1\, \widetilde{AN_1}, \widetilde{AB_{\bar b}}) &= (v_j D^j) \det(\widetilde{BD}, \tilde{C}_1, \tilde{C}_2, \widetilde{BB_{\bar b}}) \\ &= (v_j D^j) \det(\tilde{C}_1, \tilde{C}_2, \widetilde{BD}, \widetilde{BB_{\bar b}}). \end{aligned} \tag{3.13}$$

As a result of (3. 10)$\sim$(3. 13), we get

$$|v_j D^j| \sqrt{\det \bar{g}} = |b| \sqrt{\det {}^*g}\,.$$

We can choose D in such a way that $v_j D^j \geqq 0$, and we finally get

$$v_j D^j \sqrt{\det \bar{g}} = |b| \sqrt{\det {}^*g}\,.$$

Returning to the integral formula (3. 7), we get

$$\int_{\pi(\partial\tilde{M}^{n-1})} \sqrt{\det {}^*g}\, dr^1 \wedge \cdots \wedge dr^{n-1} \geqq n \int_{\tilde{M}^{n-1}} H_1 \alpha \sqrt{\det \bar{g}}\, du^1 \wedge \cdots \wedge du^{n-1} + \int_{\tilde{M}^{n-1}} G^{*\gamma\beta} \pounds G_{\gamma\beta}\, d\tilde{V},$$

where we have put $G^{*\gamma\beta} = \bar{g}^{ji} B_j{}^\gamma E_i{}^\beta$.

If we assume that $H_1 \geqq c > 0$ (c: const) and

$$\int_{\tilde{M}^{n-1}} G^{*\gamma\beta} \pounds G_{\gamma\beta}\, d\tilde{V} \geqq 0,$$

then we get

$$\int_{\pi(\partial\tilde{M}^{n-1})} d\sigma \geqq nc \int_{\pi(\tilde{M}^{n-1})} dV,$$

where $d\sigma$ and dV are the volume elements of $\pi(\partial\tilde{M}^{n-1})$ and $\pi(\tilde{M}^{n-1})$ respectively. Summarizing, we obtain

THEOREM 2. *Let $(\tilde{M}, M, \pi; \tilde{E}, \tilde{G})$ be a fibred space with projectable metric $\tilde{G}$. Let $\tilde{M}^{n-1}$ be a compact piece of an oriented submanifold of co-dimension 2 in $\tilde{M}$ with compact smooth boundary $\partial\tilde{M}^{n-1}$, which covers simply the projection $\pi(\tilde{M}^{n-1})$. Suppose that at each point, the mean curvature vetor $\tilde{H}$ is spanned by $B_{\bar 1}, \cdots, B_{\overline{n-1}}$*

and $\tilde{E}$, and that $\tilde{H}$ makes an angle $<\pi/2$ with $\tilde{E}$. If we assume that the mean curuature satisfies the condition $H_1 \geqq c > 0$, c being a constant, and

$$\int_{\tilde{M}^{n-1}} G^{*\gamma\beta} \pounds G_{\gamma\beta}\, d\tilde{V} \geqq 0,$$

then the inequality

$$ncV \geqq L$$

holds, where V and L denote the volume of the projection of $\tilde{M}^{n-1}$ and $\partial\tilde{M}^{n-1}$, respectively.

4. Special cases.

In this section we shall prove theorem 3 which is a generalization of Heinz's theorem. For this purpose we need some lemmas, which will be proved by devices similar to those developed in [1] and [3].

Let M be an n-dimensional Riemannian manifold. Let γ be a geodesic starting at $m \in M$ and parametrized by arc-length t,

$$\gamma(t) = \exp_m \rho(t), \qquad \gamma(0) = m,$$

where $\rho(t)$, is a ray in the tangent space M_m of M at the point m. Now a Jacobi field along a geodesic γ is defined by

DEFINITION. If a vector field Y given along a geodesic γ satisfies the differential equation

$$Y'' + R(Y, \dot{\gamma})\dot{\gamma} = 0,$$

the prime denoting covariant differentiation along γ, Y is called a *Jacobi field* along γ, where R is the curvature tensor, that is,

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}.$$

As is well known, we have (cf. [1] p. 172)

LEMMA 1. *Let A be a constant field along the ray ρ in the tangent space M_m, then*

$$Y(t) = d \exp_m tA$$

is a Jacobi field along γ.

LEMMA 2. *Assume that M is a space of constant curvature k. Let γ be a geodesic in M having no conjugate point of $\gamma(0)$ and $E_1, E_2, \cdots, E_n$ be a parallel orthonormal basis along γ. If hE_i $(i=1, 2, \cdots, n)$ is Jacobi field with the conditions $h(0)=0$, $h(r)=1$,[4] then h satisfies one of the following conditions:*

---

4) See Appendix I.

1) $h(t)=\frac{\sin bt}{\sin br}$, *if* $k=b^2$;

2) $h(t)=\frac{t}{r}$, *if* $k=0$;

3) $h(t)=\frac{\sinh bt}{\sinh br}$ *if* $k=-b^2$.

Proof. If hE_i is a Jacobi field along γ with the conditions $h(0)=0$, $h(r)=1$, then h is a solution of the differential equation

$$\frac{d^2h}{dt^2}+kh=0$$

with the conditions $h(0)=0$, $h(r)=1$. Thus we have Lemma 2.

If X and Y are vector fields along γ and orthogonal to γ, the index form of the pair (X, Y) on $(0, r)$ is given by

$$I(X, Y)=\int_0^r \{\langle X', Y'\rangle-\langle R(\dot\gamma, X)\dot\gamma, Y\rangle\}_t\, dt,$$

where $\langle\ ,\ \rangle$ denotes the Riemannian metric in M. For a Jacobi field Y, $I(X, Y)$ reduces to

$$I(X, Y)=\langle X, Y'\rangle|_0^r.$$

LEMMA 3. *Let γ be a geodesic and have no conjugate point of $m=\gamma(0)$. Let Y be an orthogonal Jacobi field along γ and X be any field orthogonal to γ with $X(0)=Y(0)$, $X(r)=Y(r)$. Then $I(X, X)\geqq I(Y, Y)$ and the equality occurs only when $X=Y$.*

Proof. If $X\neq Y$, then $X-Y\neq 0$. Since $I(X, Y)$ is positive definite,[5]

$$\begin{aligned}0<I(X-Y, X-Y)&=I(X, X)-2I(X, Y)+I(Y, Y)\\&=I(X, X)-2\langle X, Y'\rangle|_0^r+\langle Y, Y'\rangle|_0^r\\&=I(X, X)-\langle Y, Y'\rangle|_0^r=I(X, X)-I(Y, Y),\end{aligned}$$

which proves Lemma 3.

Next we consider the Jacobian determinant of the exponential mapping $\exp_m$ at a point $\rho(t)$ and its relation with Jacobi fields. In the sequel $R_i(X)$ and $K(X, Y)$ denote the Ricci curvature with respect to X and the sectional curvature with respect to X and Y, i.e.,

$$K(X, Y)=\frac{\langle R(X, Y)Y, X\rangle}{\langle X, X\rangle\langle Y, Y\rangle-\langle X, Y\rangle^2},$$

5) See Appendix II.

$$R_i(X)=\frac{1}{n-1}\sum_{i=1}^{n}K(X_i, X),$$

X_i being an orthonormal frame at m.

LEMMA 4. *Let γ be a geodesic starting at m in M and $j(t)$ the Jacobian determinant of* $\exp_m$ *at a point $\rho(t)$. Then $j(t)$ satisfies one of the following conditions*:

1) $$\left\{\frac{\sin bt}{bt}\right\}^{n-1}\geqq j(t)\geqq\left\{\frac{\sin at}{at}\right\}^{n-1}$$

at least out to the first conjugate point of m along γ, if $R_i(X)\geqq a^2>0$, and $0<K(X, Y)\leqq b^2$ for arbitrary X and Y;

2) *$j(t)=1$, if $K(X, Y)=0$ for any X and Y;*

3) $$1\leqq j(t)\leqq\left\{\frac{\sinh at}{at}\right\}^{n-1},$$

if $R_i(X)\geqq -a^2$, and $K(X, Y)\leqq 0$ for any X and Y.

Proof. We first note that $j(t)$ is given by

$$j(t)=\frac{\|d\exp_m s_1\cdots d\exp_m s_{n-1}\|}{\|s_1\cdots s_{n-1}\|}$$

for any linearly independent $(n-1)$-vectors $s_1, \cdots, s_{n-1}$ which are orthogonal to ρ at $\rho(t)$ (cf. [1]), where

$$\|Y_1\cdots Y_{n-1}\|=\det(\langle Y_i, Y_j\rangle).$$

Let A_i be constant fields along ρ, and assume that $d\exp_m(rA_i)=F_i(r)$, where $\{F_1, \cdots, F_{n-1}\}$ is a parallel orthonormal basis along γ. Put $Y_i(t)=d\exp_m(tA_i)$, then by virtue of Lemma 1, $Y_1, \cdots, Y_{n-1}$ are Jacobi fields along γ which are linearly independent. Then we have

$$j(t)=\frac{\|Y_1\cdots Y_n\|}{t^{n-1}A},$$

where $A=\|A_1\cdots A_{n-1}\|$ is constant. Since $Y_1(r), \cdots, Y_{n-1}(r)$ are orthonormal, we have

$$\frac{d}{dt}\|Y_1\cdots Y_{n-1}\|^2(r)=2\sum_{i=1}^{n-1}\langle Y_i(r), Y_i'(r)\rangle,$$

and therefore

$$\frac{j'(r)}{j(r)}=\sum_{i=1}^{n-1}\langle Y_i(r), Y_i'(r)\rangle-\frac{n-1}{r}. \tag{4.1}$$

For the first case 1), using the assumption, we have

$$\begin{aligned}\langle Y_i(r), Y_i'(r)\rangle &= \int_0^r \{\|Y_i'\|^2 - K(\dot\gamma, Y_i)\|Y_i\|^2\}_t\,dt \\ &= \int_0^r \{\|Y_i'\|^2 - b^2\|Y_i\|^2\}_t\,dt.\end{aligned} \tag{4. 2}$$

On the other hand, if we consider a Jacobi field $\bar Y_i = h(t)E_i(t)$ along a geodesic $\bar\gamma$ on the space S of constant curvature b^2 ($\bar\gamma(t) = \overline{\exp_{\bar m}}(t)$, $\overline{\exp_{\bar m}}: S_{\bar m} \to S$) E_i denoting orthonormal vector fields given in Lemma 2, we have

$$\langle h(r)E_i(r), \dot h(r)E_i(r)\rangle = \langle \bar Y_i(r), \bar Y_i'(r)\rangle = \int_0^r \{\|\bar Y_i'\|^2 - b^2\|\bar Y_i\|^2\}_t\,dt \tag{4. 3}$$

by means of Lemma 2.

Since $\bar Y_i$ are Jacobi fields, we have, from Lemma 3,

$$\int_0^r \{\|Y_i'\|^2 - b^2\|Y_i\|^2\}_t\,dt \geqq \int_0^r \{\|\bar Y_i'\|^2 - b^2\|\bar Y_i\|^2\}_t\,dt. \tag{4. 4}$$

Combining (3. 2), (3. 3) and (3. 4), we have

$$\langle Y_i(r), Y_i'(r)\rangle \geqq \cot br \tag{4. 5}$$

by virtue of Lemma 2.

Next, taking account of Lemma 3, we have

$$\langle Y_i(r), Y_i'(r)\rangle = I(Y_i, Y_i) = I(hF_i, hF_i) = \int_0^r \{h'^2 - K(\dot\gamma, F_i)h^2\}\,dt. \tag{4. 6}$$

Taking sum with respect to i and taking account of the inequality $R_i(\dot\gamma) \geqq a^2 > 0$ and (4. 1), we find

$$\frac{j'(r)}{j(r)} \leqq (n-1)\int_0^r \{(h')^2 - a^2h^2\}\,dt - \frac{n-1}{r}, \tag{4. 7}$$

which implies together with (4. 2)

$$(n-1)\left(\cot ar - \frac{1}{r}\right) \geqq \frac{j'(r)}{j(r)} \geqq (n-1)\left(\cot br - \frac{1}{r}\right).$$

Integrating each side of this inequality from s to t, ($s \in (0, t)$), we get

$$\left(\frac{\sin at}{at}\right)^{n-1}\left(\frac{as}{\sin as}\right)^{n-1} \geqq \frac{j(t)}{j(s)} \geqq \left(\frac{\sin bt}{bt}\right)^{n-1}\left(\frac{bs}{\sin bs}\right)^{n-1}.$$

Taking the limit as $s \to 0$, we have

$$\left(\frac{\sin at}{at}\right)^{n-1} \geqq j(t) \geqq \left(\frac{\sin bt}{bt}\right)^{n-1}$$

by virtue of $j(0) = 1$.

For the second case 2), $j'(r)/j(r)$ being zero, we have $j(t)=1$.

For the last case 3), (4. 2) reduces to

$$\langle Y_i(r), Y_i'(r)\rangle \geqq \int_0^r \|Y_i'\|^2\,dt \geqq \frac{1}{r}$$

by means of Lemmas 2 and 3. Moreover (4. 7) reduces to

$$0 \leqq \frac{j'(r)}{j(r)} \leqq (n-1)\int_0^r \{(h')^2+a^2h^2\}\,dt - \frac{n-1}{r},$$

where $h(t)=\sinh at/\sinh ar$. Thus we get

$$1 \leqq j(t) \leqq \left(\frac{\sinh at}{at}\right)^{n-1}.$$

Consequently, Lemma 4 has been proved completely.

We are now going to prove

THEOEM 3. *Let $(\tilde{M}, M, \pi; \tilde{E}, \hat{G})$ be a fibred space with projectable metric and $\tilde{M}^n$ a compact piece of an oriented hypersurface in $\tilde{M}$ with properties stated in Theorem 2. Assume that the projection M^n of $\tilde{M}^n$ to M is a Riemannian sphere with radius R lying in a normal coordinate neighborhood. Then R satisfies one of the following inequalities*:

1) $nc\left(\frac{a}{b}\right)^{n-1}\int_0^R\left(\frac{\sin bt}{\sin aR}\right)^{n-1}dt \leqq 1$, *if* $R_i(X) \geqq a^2 > 0$, *and* $0 < K(X, Y) \leqq b^2$ *on* M^n;

2) $cR \leqq 1$, *if* $K(X, Y)=0$;

3) $cR \leqq \left(\frac{\sinh aR}{a}\right)^{n-1}$, *if* $R_i(X) \geqq -a^2$ *and* $K(X, Y)<0$,

where c is the constant appearing in Theorem 2.

Proof. If m is the origin of the Riemannian sphere M^n, M^n is the image of R-ball $B(R)$ in the tangent space M^n_m under the exponential mapping and its boundary ∂M^n is the image of $(n-1)$-dimensional sphere $S^{n-1}(R)$ with radius R. Let γ be a geodesic in M^n orthogonal to ∂M^n and $j(t)$ be the Jacobian determinant of $\exp_m$ at $\rho(t)$. If dB and dS are the volume elements of $B(R)$ and the unit sphere $S^{n-1}(1)$ respectively, the volume element dV of M^n is given by

$$dV = j(t)\,dB = j(t)t^{n-1}\,dt\,dS.$$

Thus we get

$$\text{volume } M^n \geqq \int_{B(R)} j(t)\,dB = \int_0^R\int_{S^{n-1}} j(t)t^{n-1}\,dt\,dB.$$

Taking account of Lemma 4, we have for the case 1)

$$\text{volume } M^n \geqq (\text{volume } S^{n-1}) \int_0^R \left(\frac{\sin bt}{b}\right)^{n-1} dt$$

$$\geqq (\text{volume } \partial M^n) \left(\frac{a}{b}\right)^{n-1} \int_0^R \left(\frac{\sin bt}{\sin aR}\right)^{n-1} dt.$$

On the other hand, we have already in Theorem 2 the inequality

$$nc \text{ volume } M^n \leqq \text{volume } \partial M^n.$$

Thus, summing up, we obtain the following required inequality

$$nc \left(\frac{a}{b}\right)^{n-1} \int_0^R \left(\frac{\sin bt}{\sin aR}\right)^{n-1} dt \leqq 1.$$

For the cases 2) and 3) we reach the corresponding inequalities in the same way.

As a special case, we consider a fibred space $(S^{n+1}, CP(l), \pi; \tilde{E}, \tilde{G})$, where S^{n+1} is a unit sphere with natural metric $\tilde{G}$ induced from E^{n+2} and $CP(l)$ is the complex projective space of complex dimension l $(2l=n)$. We shall prove

THEOREM 4. *Let $\tilde{M}^n$ be a compact piece of an oriented hypersurface of S^{n+1} with properties stated in Theorem 2. Assume that the projection M^n of $\tilde{M}^n$ to $CP(l)$ is a Riemannian sphere with radius $R<\pi/2$. Then R satisfies the following inequality*

$$2c \tan \frac{R}{2} \leqq 1,$$

where c is the constant appearing in Theorem 2.

Proof. Let m be the origin of the Riemannian sphere. A holomorphic sectional curvature on $CP(l)$ being constant $(=1)$, the curvature tensor is given by

$$\begin{aligned} \langle R(X, Y)Z, W\rangle = \frac{1}{4}\{&\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle \\ &+\langle JX, W\rangle\langle JY, Z\rangle - \langle JY, W\rangle\langle JX, Z\rangle - 2\langle JX, Y\rangle\langle JZ, W\rangle\}, \end{aligned} \tag{4. 8}$$

where J is the complex structure in $CP(l)$. Let γ be a geodesic starting at m and orthogonal to ∂M^n. We choose a parallel orthonormal basis along $\dot{\gamma}$, $E_1, E_{1^*}, \cdots, E_l, E_{l^*}$ in such a way that

$$E_1 = \dot{\gamma}, \qquad E_{\alpha^*} = JE_\alpha \qquad (\alpha = 1, \cdots, l).$$

If $h_i E_i$ (i: not summed $i=1, 1^*, \cdots, l, l^*$) is a Jacobi field along γ with the conditions $h(0)=0$ and $h(r)=1$, then $h_i(t)$ satisfies

$$h_{1^*}(t) = \frac{\sin t}{\sin r}, \qquad h_i(t) = \frac{\sin (t/2)}{\sin (r/2)} \qquad (i = 2, 2^*, \cdots, l, l^*). \tag{4. 9}$$

In fact, h_{1*} and h_i satisfy the differential equations

$$\frac{d^2h_{1*}}{dt^2}+h_{1*}=0, \qquad h_{1*}(0)=0, \qquad h_{1*}(r)=1;$$

$$\frac{d^2h_i}{dt^2}+\frac{1}{4}h_i=0, \qquad h_i(0)=0, \qquad h_i(r)=1.$$

Next we have an estimation of the Jacobian determinant $j(t)$ of $\exp_m$, that is,

$$j(t)=\frac{1}{t^{n-1}}\left(2\sin\frac{t}{2}\right)^{n-1}\cos\frac{t}{2} \tag{4.10}$$

at least out to the first conjugate point of m along γ. In fact, taking $n-1$ Jacobi fields $Y_1, \cdots, Y_{n-1}$ in the same way as in proof of Theorem 3, we have again

$$\frac{j'(r)}{j(r)}=\sum_{i=1}^{n-1}\langle Y_i(r), Y_i'(r)\rangle-\frac{n-1}{r}.$$

If h_iE_i (i: not summed) is a Jacobi field along γ such that $h_i(0)=0$, $h_i(r)=1$, then we have

$$\langle Y_i(r), Y_i'(r)\rangle=h_i(r)\dot{h}_i(r)=\begin{cases}\cot r, & i=1^*,\\ \dfrac{1}{2}\cot\dfrac{r}{2}, & i\neq 1^*.\end{cases}$$

Therefore we obtain

$$\frac{j'(r)}{j(r)}=\frac{n-2}{2}\cot\frac{r}{2}+\cot r-\frac{n-1}{r}.$$

Integrating this from s to t ($s\in(0, t)$), we get

$$\frac{j(t)}{j(s)}=\frac{s^{n-1}(2\sin(t/2))^{n-2}\sin t}{t^{n-1}(2\sin(s/2))^{n-2}\sin s}.$$

Now taking the limit as $s\to 0$, we have

$$j(t)=\frac{1}{t^{n-1}}\left(2\sin\frac{t}{2}\right)^{n-2}\sin t$$

by virtue of $j(0)=1$.

Denoting by dS the volume element of the unit sphere S^{n-1}, we have

$$\begin{aligned}\text{volume } M^n&=\int_0^R\int_{S^{n-1}} j(t)\, t^{n-1}\, dt\, dS\\ &=(\text{volume } S^{n-1})\int_0^R\left(2\sin\frac{t}{2}\right)^{n-1}\cos\frac{t}{2}\, dt\\ &=2^n(\text{volume } S^{n-1})\int_0^{\sin(R/2)} u^{n-1}\, du \quad \left(u=\sin\frac{t}{2}\right)\end{aligned}$$

$$=\frac{1}{n}\left(2\sin\frac{R}{2}\right)^n(\text{volume } S^{n-1})$$

$$=\frac{2}{n}\tan\frac{R}{2}j(R)R^{n-1}\ (\text{volume } S^{n-1})$$

$$=\frac{2}{n}\tan\frac{R}{2}\ (\text{volume } \partial M^n)$$

by virtue of (4. 3). Since $(S^{n+1}, CP(l), \pi : \tilde{E}, \tilde{G})$ is a fibred space with invariant metric, the inequality

$$nc\,(\text{volume } M^n) \leqq \text{volume } \partial M^n$$

has been established. Thus we obtain the required inequality

$$2c\tan\frac{R}{2} \leqq 1.$$

5. **Appendix** (cf. [1] or [6]).

We give here the definition of conjugate points and properties which our argument requires.

Let $\gamma : [0, l] \to M$ be a geodesic starting at m and parametrized by arc length t;

$$\gamma(t) = \exp_m \rho(t), \qquad \gamma(0) = m.$$

We call t_0 a conjugate point to 0 along γ if $d\exp_m$ is singular at $\rho(t_0)$ and call $\gamma(t_0)$ a *conjugate point* to $\gamma(0)=m$ along γ.

I) *The uniqueness of Jacobi field.*

Let r be a non-conjugate point to 0 along γ and $v \in M_m$ and $w \in M_{\gamma(r)}$. Then there exists exactly one Jacobi field Y along γ such that $Y(0)=v$ and $Y(r)=w$.

II) *The relation to the index form.*

The following two propositions are equivalent:

1) γ has no conjugate point.

2) $I(X, X) > 0$ for any $X \neq 0$ such that $X(0)=X(l)=0$.

III) *Theorem of Morse-Schoenberg*:

1) If $K(X, Y) \leqq k$ and $l < \pi/\sqrt{k}$, then γ has no conjugate point,

1)′ if $K(X, Y) \leqq 0$, then γ has no conjugate point,

2) if $0 \leqq k < K(X, Y)$, there exists at least one conjugate point along γ at distance at most $\pi/\sqrt{k}$,

where k is a constant.

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DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.