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ON THE MINIMUM MODULUS OF A MEROMORPHIC ALGEBROID FUNCTION OF LOWER ORDER LESS THAN ONE HALF

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1. Ostrovskii [4] has proved the following:

Let $f(z)$ be a meromorphic function of lower order λ.
If $\lambda<1/2$, then

$$\limsup_{r\to\infty}\frac{\overset{+}{\log}\,\mu(r,f)}{T(r,f)}\geqq\frac{\pi\lambda}{\sin\pi\lambda}[\cos\pi\lambda-1+\delta(\infty)],$$

where $\mu(r,f)=\inf\{|f(z)|;\ |z|=r\}$ and $\delta(a)$ is the Nevanlinna deficiency of $f(z)$ at a.

In this note we shall extend the above theorem to an n-valued meromorphic algebroid function of lower order less than one half.

It is well known that for algebroid functions even if a function $y(z)$ is entire and of order zero Wiman's theorem does not always hold on the covering Riemann surface defined by $y(z)$. If, however, we use the minimum modulus of the maximum of the determinations of $y(z)$, then Wiman's theorem for it holds. Recently Ozawa [5] has extended Wiman's theorem of $\cos\pi\lambda$-type ([2]) to an n-valued entire algebroid function of lower order less than one.

2. Let $y(z)$ be an n-valued meromorphic algebroid and non-algebraic function of lower order λ defined by an irreducible equation

$$F(z,y)\equiv y^n+A_1y^{n-1}+\cdots+A_{n-1}y+A_n=0, \tag{2.1}$$

where each A_i $(i=1,2,\cdots,n)$ is meromorphic in $|z|<+\infty$ and n is an integer greater than one. Following Ozawa [5] we define the minimum modulus $\mu(r,y)$ of $y(z)$ by $\mu(r,y)=\inf\{\max_{1\leqq j\leqq n}|y_j(z)|;\ |z|=r\}$, where y_j is the j-th determination of $y(z)$.

Then we shall prove the following

THEOREM. *If $\lambda<1/2$, then*

$$\limsup_{r\to\infty}\frac{\overset{+}{\log}\,\mu(r,y)}{T(r,y)}\geqq\frac{\pi\lambda}{\sin\pi\lambda}\left[\frac{1}{k}\cos\pi\lambda-1+\delta(\infty)\right],$$

where k is the number of coefficients A_j transcendental in the defining equation (2.1).

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3. According to Selberg [6] we have the following relation between the coefficients A_j in (2. 1) and the determinations y_j of $y(z)$:

$$\log|A_j| \leqq \sum_{i=1}^{n} \overset{+}{\log}|y_i| + {}_nC_{[n/2]} \qquad (j=1, 2, \cdots, n).$$

Therefore we get

$$\log \max_{1\leqq j\leqq n} |A_j| \leqq n \overset{+}{\log} \max_{1\leqq i\leqq n} |y_i| + O(1),$$

which implies

$$\log \mu(r, A) \leqq n \overset{+}{\log} \mu(r, y) + O(1), \tag{3. 1}$$

where $A = \max_{1\leqq j\leqq n} |A_j|$.

Moreover for all transcendental coefficients A_j we obtain the following inequalities:

$$\frac{1}{n} N(r, \infty, A_j) - O(\log r) \leqq N(r, \infty, y) \leqq \frac{1}{n} \Sigma N(r, \infty, A_i) + O(\log r) \tag{3. 2}$$

and

$$\frac{1}{n} T(r, A_j) - O(\log r) \leqq T(r, y) \leqq \frac{1}{n} \Sigma T(r, A_i) + O(\log r). \tag{3. 3}$$

Here in (3. 2), (3. 3) and in the sequal each summation Σ is taken over all i such that the A_i in (2. 1) are transcendental. From this last inequality we see that if $y(z)$ is of lower order λ, then every A_j is of lower order at most λ. The converse is also true.

Denoting the number of transcendental coefficients A_j in (2. 1) by k we derive from (3. 1)

$$\frac{n \overset{+}{\log} \mu(r, y)}{T(r, y)} \geqq \frac{\log \mu(r, A) + O(1)}{T(r, y)} \geqq \frac{\Sigma \log \mu(r, A_j) + O(1)}{kT(r, y)}. \tag{3. 4}$$

4. A lemma. Let $f(z)$ be a meromorphic function of lower order λ, $\lambda<1$, with $f(0)=1$. Following Ostrovskii and Goldberg we can construct for $f(z)$

$$H_f(r) = \sum_{|a_i|<R} \log\left(1 + \frac{r}{|a_i|}\right) + \sum_{|b_i|<R} \log\left(1 + \frac{r}{|b_i|}\right),$$

where $r<R$ and $\{a_i\}$ and $\{b_i\}$ are zeros and poles of $f(z)$, respectively, and we have $H_f(r) \leqq \text{const.}\, T(2R, f)$ for $r \leqq R$.

Then we have the following

LEMMA ([1], [4]). *For* $0<\xi<\eta<R$ *and* $0<\sigma<1$

$$\int_\xi^\eta \left\{ \overset{+}{\log} \mu(r, f) + \frac{\pi\sigma}{\sin \pi\sigma} [N(r, \infty, f) - \cos \pi\sigma N(r, 0, 1/f)] \right\} \frac{dr}{r^{1+\sigma}}$$

$$\geqq C' H_f(\xi) \xi^{-\sigma} - C''(1-\sigma)^{-1} R^{-\sigma} T(2R, f),$$

where C' and C'' are two positive constants.

5. Proof of Theorem. From our assumption there exist k transcendental coefficients A_j in the defining equation (2.1) of $y(z)$. For each A_j of such transcendental functions we can choose a value a_j satisfying

$$\lim_{r\to\infty}\frac{N(r,0,A_j-a_j)}{T(r,A_j)}=1 \tag{5.1}$$

(for example take a_j not contained in a set $\{a\}$ of inner capacity zero [3]).

Then we define $\{B_j(z)\}$ as follows. For each A_j of k transcendental functions we put

$$B_j(z)=\frac{A_j(z)-a_j}{c_0}\qquad \text{if}\quad A_j(z)-a_j=c_0+c_mz^m+\cdots,$$

where $c_m\neq 0$ and m is a non-zero integer, or

$$B_j(z)=\frac{A_j(z)-a_j}{c'_{m'}z^{m'}}\qquad \text{if}\quad A_j(z)-a_j=c'_{m'}z^{m'}+\cdots,$$

where $c'_{m'}\neq 0$, $c_0\neq 0$ and m' is a non-zero integer.

From this definition of B_j and (5.1) we obtain for each B_j

$$\begin{aligned} &N(r,\infty,B_j)=N(r,\infty,A_j)+O(\log r)\\ &N(r,0,B_j)=(1+o(1))T(r,A_j),\\ &\log\mu(r,B_j)\leqq\log\mu(r,A_j)+O(\log r)\qquad \text{as}\quad r\to\infty.\end{aligned}\tag{5.2}$$

Further we have each $B_j(0)=1$. Consequently as Ostrovskii [4] did we can construct the function $H_j(r)$ in §4 for each $B_j(z)$. We can apply the above Lemma to such functions $B_j(z)$. Hence with arbitrarily fixed ξ, η, R and σ we obtain for $0<\xi<\eta<R$, $0<\sigma<1$ and for each transcendental B_j

$$\begin{aligned}&\int_\xi^\eta\left\{\overset{+}{\log}\,\mu(r,B_j)+\frac{\pi\sigma}{\sin\pi\sigma}[N(r,\infty,B_j)-\cos\pi\sigma\cdot N(r,0,B_j)]\right\}\frac{dr}{r^{1+\sigma}}\\ &\geqq C'_jH_j(\xi)\xi^{-\sigma}-C''_j(1-\sigma)^{-1}R^{-\sigma}T(2R,B_j),\end{aligned}$$

where C'_j and C''_j are two positive constants. Summing up these inequalities we have for $0<\xi<\eta<R$ and $0<\sigma<1$

$$\begin{aligned}&\int_\xi^\eta\left\{\Sigma\,\overset{+}{\log}\,\mu(r,B_j)+\frac{\pi\sigma}{\sin\pi\sigma}[\Sigma N(r,\infty,B_j)-\cos\pi\sigma\Sigma N(r,0,B_j)]\right\}\frac{dr}{r^{1+\sigma}}\\ &\geqq\Sigma C'_jH_j(\xi)\xi^{-\sigma}-\Sigma C''_j(1-\sigma)^{-1}R^{-\sigma}T(2R,B_j),\end{aligned}\tag{5.3}$$

where each Σ is taken over all j such that the A_j in (2.1) are transcendental.

Now we choose σ so that $\lambda<\sigma<1$. No matter how large ξ is we can choose the quantity $R=2\eta$ such that the right side of (5.3) will be positive since each B_j

is of lower order at most λ from (5. 2) and (3. 3). It follows that

$$\limsup_{r\to\infty}\left\{\Sigma \overset{+}{\log}\,\mu(r, B_j)+\frac{\pi\sigma}{\sin \pi\sigma}[\Sigma N(r, \infty, B_j)-\cos \pi\sigma\Sigma N(r, 0, B_j)]\right\}\geqq 0.$$

Thus for an arbitrarily given $\varepsilon>0$ there exists a sequence $\{r_n\}$, $r_n\to\infty$ as $n\to\infty$, such that

$$\Sigma \overset{+}{\log}\,\mu(r_n, B_j)\geqq \frac{\pi\sigma}{\sin \pi\sigma}[\cos \pi\sigma\Sigma N(r_n, 0, B_j)-\Sigma N(r_n, \infty, B_j)]-\varepsilon.$$

From this inequality and (5. 2) we deduce

$$\Sigma \overset{+}{\log}\,\mu(r_n, A_j)+O(\log r_n)\geqq \frac{\pi\sigma}{\sin \pi\sigma}[\cos \pi\sigma\Sigma(1+o(1))T(r_n, A_j)$$
$$-\Sigma N(r_n, \infty, A_j)+O(\log r_n)]-\varepsilon.$$

By dividing both sides of the above inequality by $kT(r_n, y)$ and letting r_n tend to infinity in due consideration of (3. 2) and (3. 3) we have with the arbitrariness of ε

$$\limsup_{r\to\infty}\frac{\Sigma \overset{+}{\log}\,\mu(r, A_j)}{kT(r, y)}\geqq \frac{\pi\sigma}{\sin \pi\sigma}\left[\frac{n}{k}\cos \pi\sigma-n+n\delta(\infty)\right].$$

Further we let σ tend to λ. Thus the combination of these with (3. 4) yields our theorem.

The proof of our theorem is completed.

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