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# ON CERTAIN SUBMANIFOLDS OF CODIMENSION 2 OF AN ALMOST TACHIBANA MANIFOLD

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*Dedicated to Professor Kentaro Yano on his sixtieth birthday*

## §0. Introduction.

Blair, Ludden and Yano [1] introduced a structure which is naturally induced on a submanifold of codimension 2 of an almost complex manifold. Yano and Okumura [6] introduced what they call an $(f, g, u, v, \lambda)$-structure and gave characterizations of even-dimensional sphere. In a previous paper [5], Yano and the present author proved that

THEOREM A. *Let $M$ be a complete manifold with normal metric $(f, g, u, v, \lambda)$-structure satisfying*

$$(0.1)\qquad (dv)_{ji}=2cf_{ji},$$

*or, equivalently*

$$(0.2)\qquad \mathcal{L}_u g_{ji}=-2c\lambda g_{ji},$$

*where $c$ is a non-zero constant on $M$. If $\lambda(1-\lambda^2)$ is an almost everywhere non-zero function and* dim $M>2$, *then $M$ is isometric with an even-dimensional sphere.*

In the present paper, using theorem A, we study submanifolds of codimension 2 of an almost Tachibana manifold $\tilde{M}$.

In §1, we recall the properties of $(f, g, u, v, \lambda)$-structure of a submanifold of codimension 2 in $\tilde{M}$ and find differential equations which the induced $(f, g, u, v, \lambda)$-structure satisfies.

We study in §2 totally umbilical submanifolds of codimension 2 of $\tilde{M}$ and in §3 submanifolds of codimension 2 of 6-dimensional sphere $S^6$.

## §1. Submanifolds of codimension 2 of an almost Tachibana manifold.

In this section, we recall some properties of submanifolds of codimension 2 in an almost Tachibana manifold as examples of the manifold with $(f, g, u, v, \lambda)$-structure (cf. [5], [6]). Let $\tilde{M}$ be a $(2n+2)$-dimensional almost Tachibana manifold covered by

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a system of coordinate neighborhoods $\{\tilde{U};\ y^\kappa\}$, where here and in the sequel the indices $\kappa, \lambda, \mu, \nu, \cdots$ run over the range $\{1, 2, \cdots, 2n+2\}$, and let $(F_\lambda{}^\kappa, G_{\mu\lambda})$ be the almost Tachibana structure, that is, $F_\lambda{}^\kappa$ is the almost complex structure;

$$F_\mu{}^\kappa F_\lambda{}^\mu = -\delta_\lambda^\kappa, \tag{1.1}$$

and $G_{\mu\lambda}$ a Riemannian metric such that

$$G_{\gamma\beta} F_\mu{}^\gamma F_\lambda{}^\beta = G_{\mu\lambda}, \tag{1.2}$$

and

$$\nabla_\mu F_\lambda{}^\kappa + \nabla_\lambda F_\mu{}^\kappa = 0, \tag{1.3}$$

where we denote by $\{{}_\mu{}^\kappa{}_\lambda\}$ and $\nabla_\mu$ the Christoffel symbols formed with $G_{\mu\lambda}$ and the operator of covariant differentiation with respect to $\{{}_\mu{}^\kappa{}_\lambda\}$ respectively.

Let $M$ be a $2n$-dimensional differentiable manifold which is covered by a system of coordinate neighborhoods $\{U;\ x^h\}$, where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$ and which is differentiably immersed in $\tilde{M}$ as a submanifold of codimension 2 by the equations

$$y^\kappa = y^\kappa(x^h). \tag{1.4}$$

We put

$$B_i{}^\kappa = \partial_i y^\kappa, \qquad (\partial_i = \partial/\partial x^i),$$

then $B_i{}^\kappa$ is, for each $i$, a local vector field of $\tilde{M}$ tangent to $M$ and the vectors $B_i{}^\kappa$ are linearly independent in each coordinate neighborhood. $B_i{}^\kappa$ is, for each $\kappa$, a local 1-form of $M$.

We assume that we can choose two mutually orthogonal unit vectors $C^\kappa$ and $D^\kappa$ of $\tilde{M}$ normal to $M$ in such a way that $2n+2$ vectors $B_i{}^\kappa, C^\kappa, D^\kappa$ give the positive orientation of $\tilde{M}$. The transforms $F_\lambda{}^\kappa B_i{}^\lambda$ of $B_i{}^\lambda$ by $F_\lambda{}^\kappa$ can be expressed as linear combinations of $B_i{}^\kappa, C^\kappa$ and $D^\kappa$, that is,

$$F_\lambda{}^\kappa B_i{}^\lambda = f_i{}^h B_h{}^\kappa + u_i C^\kappa + v_i D^\kappa, \tag{1.5}$$

where $f_i{}^h$ is a tensor field of type (1, 1) and $u_i, v_i$ are 1-forms on $M$, and, the transform $F_\lambda{}^\kappa C^\lambda$ of $C^\lambda$ by $F_\lambda{}^\kappa$ and the transform $F_\lambda{}^\kappa D^\lambda$ of $D^\lambda$ by $F_\lambda{}^\kappa$ can be written as

$$\begin{aligned} F_\lambda{}^\kappa C^\lambda &= -u^i B_i{}^\kappa + \lambda D^\kappa, \\ F_\lambda{}^\kappa D^\lambda &= -v^i B_i{}^\kappa - \lambda C^\kappa, \end{aligned} \tag{1.6}$$

respectively, where

$$u^i = u_t g^{ti}, \qquad v^i = v_t g^{ti},$$

$g_{ji}$ being the Riemannian metric on $M$ induced from that of $\tilde{M}$, and $\lambda$ is a function on $M$. We can easily verify that $\lambda$ is a function globally defined on $M$.

From (1. 2), (1. 5) and (1. 6), we have

$$
\begin{aligned}
&f_j{}^t f_t{}^h = -\delta_j^h + u_j u^h + v_j v^h,\\
&f_j{}^t f_i{}^s g_{ts} = g_{ji} - u_j u_i - v_j v_i,\\
&f_i{}^t u_t = \lambda v_i \quad \text{or} \quad f_i{}^h u^i = -\lambda v^h,\\
&f_i{}^t v_t = -\lambda u_i \quad \text{or} \quad f_i{}^h v^i = \lambda u^h,\\
&u_i u^i = v_i v^i = 1 - \lambda^2, \qquad u_i v^i = 0.
\end{aligned} \tag{1. 7}
$$

If we put

$$f_{ji} = f_j{}^r g_{ri},$$

then we can easily verify that $f_{ji}$ is skew-symmetric.

We call an *$(f, g, u, v, \lambda)$-structure* of $M$ the set of $f$, $g$, $u$, $v$ and $\lambda$ satisfying (1. 7).

An $(f, g, u, v, \lambda)$-structure is said to be *normal* if the tensor field $S_{ji}{}^h$ of type (1, 2) defined by

$$S_{ji}{}^h = N_{ji}{}^h + (\nabla_j u_i - \nabla_i u_j) u^h + (\nabla_j v_i - \nabla_i v_j) v^h \tag{1. 8}$$

vanishes, where $N_{ji}{}^h$ is the Nijenhuis tensor formed with $f_i{}^h$. We denote by $\left\{ {}_j{}^h{}_i \right\}$ and $\nabla_i$ the Christoffel symbols formed with $g_{ji}$ and the operator of covariant differentiation with respect to $\left\{ {}_j{}^h{}_i \right\}$ respectively.

An $(f, g, u, v, \lambda)$-structure is said to be *quasi-normal* if the condition

$$S_{jih} - (f_j{}^t f_{tih} - f_i{}^t f_{tjh}) = 0, \tag{1. 9}$$

is satisfied, where

$$f_{jih} = \nabla_j f_{ih} + \nabla_i f_{hj} + \nabla_h f_{ji}. \tag{1. 10}$$

Yano and the present author [5] proved

LEMMA 1. 1. *For a manifold with quasi-normal $(f, g, u, v, \lambda)$-structure, if $\lambda(1-\lambda^2)$ is almost everywhere non-zero, then we have*

$$\lambda(1-\lambda^2)(\nabla_j u_i - \nabla_i u_j) = u_i f_j{}^s u^t \mathcal{L}_v g_{st} - \{\lambda u_i v^t + (1-\lambda^2) f_i{}^t\} \mathcal{L}_v g_{jt}, \quad (\mathcal{L}_u g_{ji}) u^j v^i = 0, \tag{1. 11}$$

*and*

$$\lambda(1-\lambda^2)(\nabla_j v_i - \nabla_i v_j) = -v_i f_j{}^s v^t \mathcal{L}_u g_{st} - \{\lambda v_i u^t - (1-\lambda^2) f_i{}^t\} \mathcal{L}_u g_{jt}, \quad (\mathcal{L}_v g_{ji}) u^j v^i = 0, \tag{1. 12}$$

*where $\mathcal{L}_u$ denotes the operator of Lie differentiation with respect to the vector field $u^h$.*

The equations of Gauss of $M$ are

$$
(1.13) \quad \begin{aligned} \nabla_j B_i^{\kappa} &= \partial_j B_i^{\kappa} + \left\{ {\kappa \atop \mu\lambda} \right\} B_j^{\mu} B_i^{\lambda} - B_h^{\kappa} \left\{ {h \atop j\,i} \right\} \\ &= h_{ji} C^{\kappa} + k_{ji} D^{\kappa}, \end{aligned}
$$

where $h_{ji}$ and $k_{ji}$ are the second fundamental tensors of $M$ with respect to the normals $C^{\kappa}$ and $D^{\kappa}$ respectively.

The equations of Weingarten are

$$
(1.14) \quad \begin{aligned} \nabla_j C^{\kappa} &= \partial_j C^{\kappa} + \left\{ {\kappa \atop \mu\lambda} \right\} B_j^{\mu} C^{\lambda} = -h_j^{\,i} B_i^{\kappa} + l_j D^{\kappa}, \\ \nabla_j D^{\kappa} &= \partial_j D^{\kappa} + \left\{ {\kappa \atop \mu\lambda} \right\} B_j^{\mu} D^{\lambda} = -k_j^{\,i} B_i^{\kappa} - l_j C^{\kappa}, \end{aligned}
$$

where

$$
h_j^{\,i} = h_{jt} g^{ti}, \qquad k_j^{\,i} = k_{jt} g^{ti}
$$

and $l_j$ is the so-called third fundamental tensor.

Differentiating (1. 5) covarirntly along $M$ and taking account of (1. 13) and (1. 14), we get

$$
(1.15) \quad \begin{aligned} &(\nabla_\mu F_\lambda^{\kappa}) B_j^{\mu} B_i^{\lambda} - (h_{ji} u^h + k_{ji} v^h) B_h^{\kappa} - \lambda k_{ji} C^{\kappa} + \lambda h_{ji} D^{\kappa} \\ &= (\nabla_j f_i^{\,h} - h_j^{\,h} u_i - k_j^{\,h} v_i) B_h^{\kappa} + (\nabla_j u_i + h_{jt} f_i^{\,t} - l_j v_i) C^{\kappa} + (\nabla_j v_i + k_{jt} f_i^{\,t} + l_j u_i) D^{\kappa}. \end{aligned}
$$

Thus, from (1. 3), we have

$$
(1.16) \quad \nabla_j f_i^{\,h} + \nabla_i f_j^{\,h} = -2h_{ji} u^h + h_j^{\,h} u_i + h_i^{\,h} u_j - 2k_{ji} v^h + k_j^{\,h} v_i + k_i^{\,h} v_j.
$$

$$
(1.17) \quad \nabla_j u_i + \nabla_i u_j = -h_{jt} f_i^{\,t} - h_{it} f_j^{\,t} - 2\lambda k_{ji} + l_j v_i + l_i v_j,
$$

$$
(1.18) \quad \nabla_j v_i + \nabla_i v_j = -k_{jt} f_i^{\,t} - k_{it} f_j^{\,t} + 2\lambda h_{ji} - l_j u_i - l_i u_j.
$$

In particular, if $\tilde{M}$ is a Kählerian manifold, that is, if $\nabla_\mu F_\lambda^{\kappa} = 0$, then we have from (1. 15)

$$
\nabla_j f_i^{\,h} = -h_{ji} u^h + h_j^{\,h} u_i - k_{ji} v^h + k_j^{\,h} v_i.
$$

From this, we have $f_{jih} = 0$. Therefore, from (1. 9), we see that a submanifold of codimension 2 with quasi-normal $(f, g, u, v, \lambda)$-structure of a Kählerian manifold is normal.

## § 2. Totally umbilical submanifolds of conimenson 2 in an almost Tachibana manifold.

In this section, we consider totally umbilical submanifolds of codimension 2 with normal $(f, g, u, v, \lambda)$-structure of an almost Tachibana manifold.

Let $M$ be a submanifold of codimension 2 of an almost Tachibana manifold. Then the mean curvature vector of $M$ is defined to be

$$H^{\kappa}=\frac{1}{2n}h_i{}^iC^{\kappa}+\frac{1}{2n}k_i{}^iD^{\kappa}, \tag{2. 1}$$

and the mean curvature $H$ of $M$ is defined to be the length of $H^{\kappa}$, that is,

$$H^2=\frac{1}{4n^2}\{(h_i{}^i)^2+(k_i{}^i)^2\}. \tag{2. 2}$$

Differentiating (2. 1) covariantly and making use of (1. 13) and (1. 14), we have

$$\nabla_j H^{\kappa}=-\frac{1}{4n^2}\{(h_i{}^i)^2+(k_i{}^i)^2\}B_j{}^{\kappa}+\frac{1}{2n}(\nabla_j h_i{}^i-l_j k_i{}^i)C^{\kappa}+\frac{1}{2n}(\nabla_j k_i{}^i+l_j h_i{}^i)D^{\kappa}.$$

If the covariant derivative $\nabla_j H^{\kappa}$ of the mean curvature vector field of $M$ is tangent to $M$, then

$$\nabla_j h_i{}^i=l_j k_i{}^i, \qquad \nabla_j k_i{}^i=-l_j h_i{}^i. \tag{2. 3}$$

We now suppose that $M$ is totally umbilical. Then from (1. 17) we have

$$\nabla_j u_i+\nabla_i u_j=-2\left(\frac{1}{2n}k_t{}^t\right)\lambda g_{ji}+l_j v_i+l_i v_j, \tag{2. 4}$$

from which, using the second equation of (1. 11), $l_t u^t=0$. Similarly we see, from (1. 18) and the second equation of (1. 12), that $l_t v^t=0$. Taking the symmetric part of the first equation of (1. 12) in $j$ and $i$ and using (2. 4), $l_t v^t=0$ and $l_t u^t=0$, we find $l_j u_i+l_i u_j=0$, from which, $l_j=0$ and consequently $h_i{}^i=$constant, $k_i{}^i=$constant because of (2. 3). Thus the structure is normal (See [5]).

Taking acount of Theorem A and $l_j=0$, we have

THEOLEM 2. 1.[1] *Let $M$ be a complete totally umbilical submanifold of codimension 2 with normal $(f, g, u, v, \lambda)$-structure of an almost Tachibana manifold $\tilde{M}$. Suppose that the covariant derivative of the mean curvature vector of $M$ is tangent to $M$, the mean curvature of $M$ does not vanish and $\lambda(1-\lambda^2)$ is almost everywhere non-zero $(n>1)$. Then $M$ is isometric with an even-dimensional sphere.*

As a direct consequence of (1. 17), (2. 4), $l_j=0$ and Theorem A, we have

THEOREM 2. 2. *Let $M$ be a complete totally umbilical submanifold $(n>1)$ of codimension 2 of an almost Tachibana manifold. If the $(f, g, u, v, \lambda)$-structure on $M$ is nomal, $h_i{}^i$ or $k_i{}^i$ is non-vanishing constant and $\lambda(1-\lambda^2)$ is an almost everywhere non-zero function, then $M$ is isometric with an even-dimensional sphere.*

1) M. Okumura has proved the theorem in the case $\tilde{M}$ is Kählerian, [3].

## § 3. Submanifolds of codimension 2 of a 6-dimensional sphere.

Let $M$ be an almost Tachibana manifold of constant curvature, that is, 6-dimensional sphere $S^6$, [4]. Its curvature form is given by

$$R_{\nu\mu\lambda\kappa}=k(G_{\nu\kappa}G_{\mu\lambda}-G_{\mu\kappa}G_{\nu\lambda}), \tag{3.1}$$

$k$ being a positive constant.

In this section, we consider a submanifold of codimension 2 of $S^6$. Substituting (3.1) into the Gauss, Codazzi, Ricci-equations;

$$R_{\nu\mu\lambda\kappa}B_k{}^{\nu}B_j{}^{\mu}B_i{}^{\lambda}B_h{}^{\kappa}=R_{kjih}-h_{kh}h_{ji}+h_{jh}h_{ki}-k_{kh}k_{ji}+k_{jh}k_{ki},$$

$$\begin{cases} R_{\nu\mu\lambda\kappa}B_k{}^{\nu}B_j{}^{\mu}B_i{}^{\lambda}C^{\kappa}=\nabla_k h_{ji}-\nabla_j h_{ki}-l_k k_{ji}+l_j k_{ki}, \\ R_{\nu\mu\lambda\kappa}B_k{}^{\nu}B_j{}^{\mu}B_i{}^{\lambda}D^{\kappa}=\nabla_k k_{ji}-\nabla_j k_{ki}+l_k h_{ji}-l_j h_{ki}, \end{cases}$$

$$R_{\nu\mu\lambda\kappa}B_k{}^{\nu}B_j{}^{\mu}C^{\lambda}D^{\kappa}=\nabla_k l_j-\nabla_j l_k+h_{kt}k_j{}^t-h_{jt}k_k{}^t,$$

we have respectively

$$R_{kjih}-h_{kh}h_{ji}+h_{jh}h_{ki}-k_{kh}k_{ji}+k_{jh}k_{ki}=k(g_{kh}g_{ji}-g_{jh}g_{ki}), \tag{3.2}$$

and

$$\begin{cases} \nabla_k h_{ji}-\nabla_j h_{ki}-l_k k_{ji}+l_j k_{ki}=0, \\ \nabla_k k_{ji}-\nabla_j k_{ki}+l_k h_{ji}-l_j h_{ki}=0, \end{cases} \tag{3.3}$$

and

$$\nabla_k l_j-\nabla_j l_k+h_{kt}k_j{}^t-h_{jt}k_k{}^t=0. \tag{3.4}$$

Now, we consider a submanifold $M$ of codimension 2 of an almost Tachibana manifold satisfying the fallowing conditions;

$$f_j{}^t h_t{}^h=h_j{}^t f_t{}^h, \tag{3.5}$$

$$f_j{}^t k_t{}^h=k_j{}^t f_t{}^h. \tag{3.6}$$

We see that (3.5) and (3.6) are global conditions over the submanifold $M$.

LEMMA 3.1. *For $(f, g, u, v, \lambda)$-structure of $M$ with* (3.5) *and* (3.6), *if $\lambda$ does not vanssh almost everywhere, we have*

$$h_{ji}u^i=\alpha u_j, \qquad h_{ji}v^i=\alpha v_j, \tag{3.7}$$

*and*

$$k_{ji}u^i=\bar{\alpha}u_j, \qquad k_{ji}v^i=\bar{\alpha}v_j, \tag{3.8}$$

*where $\alpha$ and $\bar{\alpha}$ are scalars of $M$* [2].

LEMMA 3.2. *Let $M$ be a submanifold of codimension* 2 *of an almost Tachibana* *manifold. If the* $(f, g, u, v, \lambda)$*-structure on* $M$ *is quais-normal and satisfies* (3. 5) *and* (3. 6), *and assume that* $\lambda(1-\lambda^2)$ *is almost everywhere non-zero, then we have*

$$\nabla_j u_i = -h_{jt}f_i{}^t - \lambda k_{ji}, \tag{3. 9}$$

$$\nabla_j v_i = -k_{jt}f_i{}^t + \lambda h_{ji}, \tag{3. 10}$$

*Proof.* By assumptions, (1. 17) and (1.18) can be respectively written as

$$\nabla_j u_i + \nabla_i u_j = -2\lambda k_{ji} + l_j v_i + l_i v_i, \qquad \nabla_j v_i + \nabla_i v_j = 2\lambda h_{ji} - l_j u_i - l_i u_j.$$

Substituting the equations above into the first equations of (1. 11), (1. 12) respectively, we have

$$\nabla_j u_i - \nabla_i u_j = -2h_{jt}f_i{}^t + \frac{1}{\lambda}(u_j f_i{}^t l_t - u_i f_j{}^t l_t) + l_j v_i, \tag{3. 11}$$

$$\nabla_j v_i - \nabla_i v_j = -2k_{jt}f_i{}^t + \frac{1}{\lambda}(v_j f_i{}^t l_t - v_i f_j{}^t l_t) - l_j u_i \tag{3. 12}$$

by virtue of Lemma 3. 1, from which, taking the symmetric part in $j$ and $i$ and using the second equations of (1. 11) and (1. 12), $l_j u_i + l_i u_j = 0$ and $l_i v_j + l_j v_i = 0$ and consequently $l_j = 0$. Thus (3. 9) and (3. 10) proved.

LEMMA 3. 3. *Let* $M$ *be a submanifold of codimension* 2 *of* $S^6$. *If the* $(f, g, u, v, \lambda)$*-structure on* $M$ *is quasi-normal and satisfies* (3. 5) *and* (3. 6), *and assume that* $\lambda(1-\lambda^2)$ *is almost everywhere non-zero on* $M$, *then we have*

$$h_i{}^t h_t{}^h = \alpha h_i{}^h, \qquad k_i{}^t k_t{}^h = \bar{\alpha} k_i{}^h, \tag{3. 13}$$

*and* $\alpha$ *and* $\bar{\alpha}$ *are both constants.*

*Proof.* Differentiating the first equation of (3. 7) covariantly, we obtain

$$(\nabla_k h_{ji})u^i + h_{ji}(\nabla_k u^i) = (\nabla_k \alpha)u_j + \alpha \nabla_k u_j,$$

or, using (3. 9),

$$(\nabla_k h_{ji})u^i + h_{ji}(h_k{}^t f_t{}^i - \lambda k_k{}^i) = (\nabla_k \alpha)u_j + \alpha(-h_{kt}f_j{}^t - \lambda k_{kj}),$$

and consequently, taking the skew-symmetric part, we have

$$h_{ji}h_k{}^t f_t{}^i - h_{ki}h_j{}^t f_t{}^i = (\nabla_k \alpha)u_j - (\nabla_j \alpha)u_k - \alpha h_{kt}f_j{}^t + \alpha h_{jt}f_k{}^t$$

because of $\nabla_k h_{ji} - \nabla_j h_{ki} = 0$, $h_{jt}k_k{}^t = h_{kt}k_j{}^t$, or, using (3. 5),

$$2h_{ji}h_k{}^t f_t{}^i = (\nabla_k \alpha)u_j - (\nabla_j \alpha)u_k - 2\alpha h_{kt}f_j{}^t, \tag{3. 14}$$

from which, transvecting with $u^k$,

$$-2\alpha\lambda h_{jt}v^t = (u^k \nabla_k \alpha)u_j - (\nabla_j \alpha)(1-\lambda^2) - 2\alpha^2 \lambda v_j,$$

that is,

$$(1-\lambda^2)\nabla_j\alpha=(u^k\nabla_k\alpha)u_j. \tag{3.15}$$

Thus, $\nabla_j\alpha$ being proportional to $u_j$, we find from (3. 14)

$$h_{ji}f_k{}^t h_t{}^i=\alpha h_{jt}f_k{}^t$$

since $h_k{}^t$ and $f_t{}^i$ commute.

Transvecting this equation with $f_h{}^k$, we find

$$(h_{ji}h_t{}^i)(-\delta_h^t+u_hu^t+v_hv^t)=\alpha h_{jt}(-\delta_h^t+u_hu^t+v_hv^t),$$

or, using (3. 7),

$$h_i{}^t h_t{}^h=\alpha h_i{}^h.$$

Differentiating the second equation of (3. 7) covariantly and taking account of (3. 10), we find

$$(\nabla_j h_i{}^h)v_h-h_i{}^h(k_{jt}f_h{}^t-\lambda h_{jh})=(\nabla_j\alpha)v_i-\alpha(k_{ji}f_i{}^t-\lambda h_{ji}),$$

from which, taking the skew-symmetric part

$$h_j{}^h k_{it}f_h{}^t-h_i{}^h k_{jt}f_h{}^t=(\nabla_j\alpha)v_i-(\nabla_i\alpha)v_j+\alpha(k_{it}f_j{}^t-k_{jt}f_i{}^t),$$

because of the equation of Conazzi (3. 3) with $l_j=0$.

Transvecting the above equation with $v^j$ and making use of (1. 7), (3. 7) and (3. 8), we obtain

$$(1-\lambda^2)\nabla_j\alpha=(v^k\nabla_k\alpha)v_j. \tag{3.16}$$

From (3. 15) and (3. 16), we see that $\alpha$ is constant.

Similarly we can prove

$$k_i{}^t k_t{}^h=\bar{\alpha}k_i{}^h,\qquad \bar{\alpha}=\text{constant}.$$

LEMMA 3. 4. *Under the same assumptions as those in Lemma* 3. 3, *the mean curvature of $M$ is constant.*

*Proof.* Let $\alpha'$ be an eigenvalue of $h_i{}^h$ at a point of $M$ and $p^i$ the eigenvector corresponding to $\alpha'$ at the point. Then we have

$$h_i{}^h p^i=\alpha' p^h.$$

Applying this $h_h{}^j$ and taking account of (3. 13), we find

$$\alpha\alpha' p^j=\alpha'^2 p^j,$$

from which

$$\alpha'=\alpha\qquad\text{or}\qquad\alpha'=0.$$

Thus the only eigenvalue of $h_i{}^h$ is $\alpha$ or 0 and consequently the eigenvalues of $h_i{}^h$ are constant.

Similarly we can prove that $k_i{}^h$ has only two constant eigenvalues $\bar{\alpha}$ and 0.

Let $r$ and $s$ be multiplicities of the eigenvalues $\alpha$ of $h_i{}^h$ and $\bar{\alpha}$ of $k_i{}^h$ respectively. Then $\alpha$ and $\bar{\alpha}$ being constant, $r$ and $s$ are also constant. So we have

$$h_i{}^i = r\alpha, \qquad k_i{}^i = s\bar{\alpha}.$$

Substituting these into the equation giving the mean curvature of $M$;

$$H^2 = \frac{1}{16}\{(h_i{}^i)^2 + (k_i{}^i)^2\}, \tag{3. 17}$$

we have $H$=const. This completes the proof of the lemma.

We now assume that the mean curvature vector does not vanish everywhere on $M$ and choose the second unit normal $D^\kappa$ in such a way that $B_i{}^\kappa$, $C^\kappa$, $D^\kappa$ form the positive orientation of $S^6$. Then from the equation giving the mean curvature vector of $M$:

$$H^\kappa = \frac{1}{4}\{(h_i{}^i C^\kappa + k_i{}^i D^\kappa)\},$$

we have

$$k_i{}^i = 0, \tag{3. 18}$$

which implies that

$$k_{ji} k^{ji} = 0,$$

because of (3. 13). This shows that $k_{ji}=0$ and consequently

$$\nabla_j v_i = \lambda h_{ji}, \tag{3. 19}$$

by virtue of (3. 10).

Differentiating

$$v_i v^i = 1 - \lambda^2$$

covariantly and taking account of (3. 19), we obtain

$$\nabla_j \lambda = -\alpha v_j. \tag{3. 20}$$

Substituting (3. 19) into the Ricci-identity:

$$\nabla_k \nabla_j v^h - \nabla_j \nabla_k v^h = R_{kji}{}^h v^i,$$

we have

$$R_{kji}{}^h v^i = (\nabla_k \lambda) h_j{}^h - (\nabla_j \lambda) h_k{}^h + \lambda(\nabla_k h_j{}^h - \nabla_j h_k{}^h),$$

or, using (3. 3) with $l_j=0$ and (3. 20),

$$R_{kjih} v^i - \alpha v_j h_{kh} + \alpha v_k h_{jh} = 0. \tag{3. 21}$$

On the other hand, transvecting (3. 2) with $v^i$ and using $k_{ji}=0$, we have

$$R_{kjih}v^i - \alpha v_j h_{kh} + \alpha v_k h_{jh} = k(g_{kh}v_j - g_{jh}v_k).$$

From these and (3. 21), we have $k=0$. This contradicts (3. 1). Thus, the mean curvature vector vanishes identically on $M$ and consequently

$$h_i{}^i=0, \qquad k_i{}^i=0.$$

So, using (3. 13), we have

$$h_{ji}h^{ji}=0, \qquad k_{ji}k^{ji}=0$$

which implies that

$$h_{ji}=0, \qquad k_{ji}=0. \tag{3. 22}$$

Thus we have

THEOREM 3. 5. *Let $M$ be a submanifold of codimension 2 of $S^6$. If the $(f, g, u, v, \lambda)$-structure on $M$ is quasi-normal, $\lambda(1-\lambda^2)$ is almost everywhere non-zero on $M$, and the linear transformation $h_j{}^i$ and $k_j{}^i$ which are defined by the second fundrmental tensors commute with $f_j{}^i$, then $M$ is totally geodesic.*

*If the submanifold $M$ is complete, then it is a great sphere.*

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