KÔDAI MATH. SEM. REP.
24 (1972), 50—61

INVARIANT SUBMANIFOLDS OF CODIMENSION 2 OF A MANIFOLD WITH (F, G, u, v, λ)-STRUCTURE

BY YOSHIKO KUBO*

An almost complex manifold, an almost contact manifold and a manifold with a structure tensor f satisfying $f^3+f=0$, all admit a tensor field of type (1, 1). A submanifold of these manifolds is said to be invariant when the tangent space at each point of the submanifold is left invariant by the endomorphism defined by this tensor field.

It is known that the invariant submanifolds of almost complex and contact manifolds inherit properties of the enveloping manifold. For example, an invariant submanifold of a Kählerian manifold is Kählerian and an invariant submanifold of a normal contact manifold is normal [1, 2, 3].

Yano and Okumura [4] have recently introduced the so-called (F, G, u, v, λ)-structure in an even-dimensional manifold and given a characterization of an even-dimensional sphere in terms of this structure.

The purpose of the present paper is to study invariant submanifolds of codimension 2 of a manifold with (F, G, u, v, λ)-structure.

We recall in §1 the definition and properties of (F, G, u, v, λ)-structure and in §2 the fundamental formulas for submanifolds of codimension 2 of a Riemannian manifold. In §3, we obtain fundamental formulas for submainfolds of codimension 2 of a Riemannian manifold with (F, G, u, v, λ)-structure. In the last §4, we get a theorem stating that invariant submanifolds of codimension 2 of a manifold with (F, G, u, v, λ)-structure are also manifolds with (f, g, u, v, λ)-structure and a corollary stating that invariant submanifolds of codimension 2 of an even-dimensional sphere are also spheres.

§1. (F, G, u, v, λ)-structures.

Let M be an m-dimensional differentiable manifold of class C^∞. If there exist in M a tensor field $F_\lambda{}^\kappa$ of type (1, 1), two contravariant vector fields U^λ, V^λ, two covariant vector fields u_λ, v_λ, and a function λ such that[1]

Received January 14, 1971.

* Formerly Yoshiko Watanabe.

[1] (x^λ) are local coordinates of M and $F_\lambda{}^\kappa$, U^λ, V^λ, u_λ, v_λ and λ are components of F, U, V, u, v and λ with respect to this local coordinate system respectively. The indices $\lambda, \kappa, \mu, \nu, \cdots$ run over the range $\{1, 2, \cdots, m\}$ and the so-called Einstein summation convention is used with respect to this system of indices.

$$F_\lambda{}^\kappa F_\kappa{}^\nu = -\delta_\lambda^\nu + U^\nu u_\lambda + V^\nu v_\lambda, \tag{1. 1}$$

$$F_\lambda{}^\kappa U^\lambda = -\lambda V^\kappa, \qquad F_\lambda{}^\kappa u_\kappa = \lambda v_\lambda, \tag{1. 2}$$

$$F_\lambda{}^\kappa V^\lambda = \lambda U^\kappa, \qquad F_\lambda{}^\kappa V_\kappa = -\lambda u_\lambda, \tag{1. 3}$$

$$U^\lambda u_\lambda = 1-\lambda^2, \qquad V^\lambda u_\lambda = 0, \tag{1. 4}$$

$$V^\lambda v_\lambda = 1-\lambda^2, \qquad U^\lambda v_\lambda = 0, \tag{1. 5}$$

then the manifold M is said to have an (F, U, V, u, v, λ)-structure. Yano and Okumura [4] proved.

THEOREM A. *A differentiable manifold M^m with (F, U, V, u, v, λ)-structure is even-dimensional, i.e. $m=2n$.*

DEFINITION. A (F, U, V, u, v, λ)-structure is said to be *normal* if the Nijenhuis tensor N of F satisfies

$$S_{\lambda\kappa}{}^\nu \overset{\text{def}}{=} N_{\lambda\kappa}{}^\nu + (\partial_\lambda u_\kappa - \partial_\kappa u_\lambda) U^\nu + (\partial_\lambda v_\kappa - \partial_\kappa v_\lambda) V^\nu = 0. \tag{1. 6}$$

We assume that, in a manifold M with (F, U, V, u, v, λ)-structure, there exists a positive definite Riemannian metric G such that

$$G_{\lambda\kappa} U^\lambda = u_\kappa, \qquad G_{\lambda\kappa} V^\lambda = v_\kappa, \tag{1. 7}$$

$$G_{\lambda\kappa} F_\nu{}^\lambda F_\tau{}^\kappa = G_{\nu\tau} - u_\nu u_\tau - v_\nu v_\tau. \tag{1. 8}$$

We call an (F, U, V, u, v, λ)-structure with such a Riemannian metric a metric (F, U, V, u, v, λ)-structure and denote the structure by (F, G, u, v, λ).

In a manifold with (F, G, u, v, λ)-structure, we can easily see that F satisfies

$$F_{\lambda\kappa} = -F_{\kappa\lambda}, \tag{1. 9}$$

where

$$F_{\lambda\kappa} = F_\lambda{}^\nu G_{\nu\kappa}.$$

As examples of manifolds with (F, G, u, v, λ)-structure, we know submanifolds of codimension 2 of an almost Hermitian manifold with non-zero mean curvature vector and hypersurfaces of an almost contact metric manifold. Moreover we can see that there always exists a metric (F, G, u, v, λ)-structure in an even-dimensional sphere.

By assuming that u_λ and v_λ in the manifold with (F, G, u, v, λ)-structure satisfy

$$\nabla_\lambda u_\kappa - \nabla_\kappa u_\lambda = 2\phi F_{\lambda\kappa}, \tag{1. 10}$$

$$\nabla_\lambda v_\kappa - \nabla_\kappa v_\lambda = 2F_{\lambda\kappa}, \tag{1. 11}$$

where ∇_λ is the operator of covariant differentiation with respect to Christoffel symbols $\{{}_\kappa{}^\lambda{}_\nu\}$ formed with $G_{\lambda\kappa}$ and ϕ a scalar function, we have the following theorem [4]

THEOREM B. *If a manifold with normal metric (F, G, u, v, λ)-structure satisfies* (1. 10), *and* (1. 11) *and if $\lambda(1-\lambda^2)$ is an almost everywhere non-zero function, then we have*

$$\nabla_\nu F_{\lambda\kappa} = -G_{\nu\lambda}(\phi u_\kappa + v_\kappa) + G_{\nu\kappa}(\phi u_\lambda + v_\lambda). \tag{1.12}$$

THEOREM C. *Let M be a complete manifold with normal metric (F, G, u, v, λ)-structure satisfying* (1. 10) *and* (1. 11). *If $\lambda(1-\lambda^2)$ is an almost everywhere non-zero function and $m>2$ then M is isometric with an even dimensional sphere.*

We know that the (F, G, u, v, λ)-structure in an even-dimensional sphere satisfies (1. 10) and (1. 11) and that $\lambda(1-\lambda^2)$ is an almost everywhere non-zero function over the sphere.

§ 2. Submanifolds of codimension 2 of a Riemannian manifold.

We consider a submanifold N of codimension 2 of a differentiable manifold M of dimension m with positive definite Riemannian metric $G_{\lambda\kappa}$, and let the parametric representation of the submanifold N be

$$x^\lambda = x^\lambda(y^i),$$

where (y^i) are local coordinates in N, and the indices $i, j, k, l, \cdots$ run over the range $\{1, 2, \cdots, m-2\}$.

Put

$$B_i^\lambda = \partial_i x^\lambda,$$

∂_i denoting the operator $\partial/\partial y^i$, and denote a pair of mutually orthogonal unit vector fields normal to N by C^λ and D^λ, which are locally defined in each coordinate neighborhood of N. Then the Riemannian metric induced on N is given by

$$g_{ij} = G_{\lambda\kappa} B_i^\lambda B_j^\kappa, \tag{2.1}$$

and we have

$$G_{\lambda\kappa} C^\lambda B_i^\kappa = 0, \quad G_{\lambda\kappa} D^\lambda B_i^\kappa = 0, \quad G_{\lambda\kappa} C^\lambda C^\kappa = 1, \quad G_{\lambda\kappa} C^\lambda D^\kappa = 0, \quad G_{\lambda\kappa} D^\lambda D^\kappa = 1. \tag{2.2}$$

If we denote by ∇_i the operator of the so-called van der Waerden-Bortolotti covariant differentiation along N, i.e. if we put

$$\nabla_i B_j^\lambda = \partial_i B_j^\lambda + \left\{ {\lambda \atop \kappa\ \ \nu} \right\} B_i^\kappa B_j^\nu - \left\{ {k \atop i\ \ j} \right\} B_k^\nu, \tag{2.3}$$

$$\nabla_i C^\lambda = \partial_i C^\lambda + \left\{ {\lambda \atop \kappa\ \ \nu} \right\} B_i^\kappa C^\nu, \qquad \nabla_i D^\lambda = \partial_i D^\lambda + \left\{ {\lambda \atop \kappa\ \ \nu} \right\} B_i^\kappa D^\nu, \tag{2.4}$$

$\{{}_j{}^i{}_k\}$ being the Christoffel symbols formed with g_{ij}, then, taking account of (2. 2), we have

$$\nabla_i B_j{}^\lambda = h_{ij} C^\lambda + k_{ij} D^\lambda, \tag{2. 5}$$

$$\nabla_i C^\lambda = -h_i{}^j B_j{}^\lambda + l_i D^\lambda, \qquad \nabla_i D^\lambda = -k_i{}^j B_j{}^\lambda - l_i C^\lambda, \tag{2. 6}$$

where h_{ij} and k_{ij} are the second fundamental tensors with respect to C^λ and D^λ respectively, l_i the third fundamental tensor, and $h_i{}^j = h_{il} g^{lj}$, $k_i{}^j = k_{il} g^{lj}$. As is well-known we have

$$h_{ij} = h_{ji}, \qquad k_{ij} = k_{ji},$$

where (g^{ij}) is the inverse of the matrix (g_{ij}). (2. 5) are equations of Gauss and (2. 6) those of Weingarten.

§ 3. Submanifolds of codimension 2 of a Riemannian manifold with (F, G, u, v, λ)-structure.

We now assume that the enveloping manifold M is a Riemannian manifold of dimension $m=2n$ with (F, G, u, v, λ)-structure, and that there is given in M a submanifold N of codimension 2. Then, for the transforms of $B_i{}^\lambda$, C^λ and D^λ by $F_\lambda{}^\kappa$ we have equations of the form

$$F_\lambda{}^\kappa B_i{}^\lambda = f_i{}^j B_j{}^\kappa + p_i C^\kappa + q_i D^\kappa, \tag{3. 1}$$

$$F_\lambda{}^\kappa C^\lambda = -p^i B_i{}^\kappa + \alpha D^\kappa, \tag{3. 2}$$

$$F_\lambda{}^\kappa D^\lambda = -q^i B_i{}^\kappa - \alpha C^\kappa, \tag{3. 3}$$

where $p^i = p_j g^{ji}$ and $q^i = q_j g^{ji}$. We can see that $f_i{}^j$ defines a global tensor field of type (1, 1) in N independent of the choice of C^λ and D^λ, p^i and q^i are two local vector fields and α is a global scalar field in N independent of the choice of C^λ and D^λ. On the submanifold N, the vector field u^λ and v^λ have the forms

$$u^\lambda = u^i B_i{}^\lambda + \rho C^\lambda + \nu D^\lambda, \tag{3. 4}$$

$$v^\lambda = v^i B_i{}^\lambda + \tau C^\lambda + \varepsilon D^\lambda, \tag{3. 5}$$

where u^i and v^i are vector fields in N and ρ, ν, τ, ε are scalar fields in N.

Considering the transform of (3. 1) by $F_\lambda{}^\kappa$ and taking account of (1. 1), (3. 1) and (3. 2), we have

$$F_\kappa{}^\alpha F_\lambda{}^\kappa B_i{}^\lambda = f_i{}^j F_\kappa{}^\alpha B_j{}^\kappa + p_i F_\kappa{}^\alpha C^\kappa + q_i F_\kappa{}^\alpha D^\kappa,$$

$$(-\delta_\lambda^\alpha + u_\lambda u^\alpha + v_\lambda v^\alpha) B_i{}^\lambda = f_i{}^j (f_j{}^l B_l{}^\alpha + p_j C^\alpha + q_j D^\alpha) + p_i(-p^j B_j{}^\alpha + \alpha D^\alpha) + q_i(-q^j B_j{}^\alpha - \alpha C^\alpha),$$

$$-B_i{}^\alpha + (u^j B_j{}^\alpha + \rho C^\alpha + \nu D^\alpha) u_i + (v^j B_j{}^\alpha + \tau C^\alpha + \varepsilon D^\alpha) v_i$$

$$= (-\delta_i^j + u^j u_i + v^j v_i) B_j{}^\alpha + (\rho u_i + \tau v_i) C^\alpha + (\nu u_i + \varepsilon v_i) D^\alpha$$

$$= (f_i{}^l f_l{}^j - p_i p^j - p_i p^j) B_j{}^\alpha + (f_i{}^j p_j - \alpha q_i) C^\alpha + (f_i{}^j q_j + \alpha p_i) D^\alpha,$$

and consequentry

$$f_i{}^l f_l{}^j = -\delta_i^j + u_i u^j + v_i v^j + p_i p^j + q_i q^j, \tag{3.6}$$

$$f_i{}^j p_j = \rho u_i + \tau v_i + \alpha q_i, \tag{3.7}$$

$$f_i{}^j q_j = \nu u_i + \varepsilon v_i - \alpha p_i, \tag{3.8}$$

where $u_i = g_{ij} u^j$ and $v_i = g_{ij} v^j$.

Similarly computing the transform of (3.2) by $F_\kappa{}^\alpha$, we have

$$\begin{aligned}
&F_\kappa{}^\alpha F_\lambda{}^\kappa C^\lambda = -p^i F_\kappa{}^\alpha B_i{}^\kappa + \alpha F_\kappa{}^\alpha D^\kappa,\\
&(-\delta_\lambda^\alpha + u_\lambda u^\alpha + v_\lambda v^\alpha) C^\lambda = -p^i (f_i{}^j B_j{}^\alpha + p_i C^\alpha + q_i D^\alpha) + \alpha(-q^i B_i{}^\alpha - \alpha C^\alpha),\\
&-C^\alpha + (u^i B_i{}^\alpha + \rho C^\alpha + \nu D^\alpha)\rho + (v^i B_i{}^\alpha + \tau C^\alpha + \varepsilon D^\alpha)\tau\\
&= (\rho u^i + \tau v^i) B_i{}^\alpha + (-1 + \rho^2 + \tau^2) C^\alpha + (\nu\rho + \tau\varepsilon) D^\alpha\\
&= (-f_i{}^j p^i - \alpha q^j) B_j{}^\alpha - (p^i p_i + \alpha^2) C^\alpha - p^i q_i D^\alpha,
\end{aligned}$$

and hence

$$p_i p^i = 1 - \rho^2 - \tau^2 - \alpha^2, \tag{3.9}$$

$$p_i q^i = -\nu\rho - \varepsilon\tau. \tag{3.10}$$

Moreover from (3.3) we have

$$F_\kappa{}^\alpha F_\lambda{}^\kappa D^\lambda = -q^i F_\kappa{}^\alpha B_i{}^\kappa - \alpha F_\kappa{}^\alpha C^\kappa,$$

from which

$$(-\delta_\lambda^\alpha + u_\lambda u^\alpha + v_\lambda v^\alpha) D^\lambda = -q^i (f_i{}^j B_j{}^\alpha + p_i C^\alpha + q_i D^\alpha) - \alpha(-p^i B_i{}^\alpha + \alpha D^\alpha),$$

or equivalently

$$\begin{aligned}
&-D^\alpha + (u^i B_i{}^\alpha + \rho C^\alpha + \nu D^\alpha)\nu + (v^i B_i{}^\alpha + \tau C^\alpha + \varepsilon D^\alpha)\varepsilon\\
&= (\nu u^i + \varepsilon v^i) B_i{}^\alpha + (\rho\nu + \tau\varepsilon) C^\alpha + (-1 + \nu^2 + \varepsilon^2) D^\alpha\\
&= (-f_i{}^j q^i + \alpha p^j) B_j{}^\alpha - q^i p_i C^\alpha - (q^i q_i + \alpha^2) D^\alpha,
\end{aligned}$$

$$q^i q_i = 1 - \nu^2 - \varepsilon^2 - \alpha^2. \tag{3.11}$$

Forming the transform of (3.4) by $F_\lambda{}^\kappa$ and using (1.2), (3.1), (3.2), and (3.3), we find

$$\begin{aligned}
&F_\lambda{}^\kappa u^\lambda = u^i F_\lambda{}^\kappa B_i{}^\lambda + \rho F_\lambda{}^\kappa C^\lambda + \nu F_\lambda{}^\kappa D^\lambda,\\
&-\lambda v^\kappa = u^i (f_i{}^j B_j{}^\kappa + p_i C^\kappa + q_i D^\kappa) + \rho(-p^i B_i{}^\kappa + \alpha D^\kappa) + \nu(-q^i B_i{}^\kappa - \alpha C^\kappa),\\
&-\lambda(v^i B_i{}^\kappa + \tau C^\kappa + \varepsilon D^\kappa) = (f_i{}^j u^i - \rho p^j - \nu q^j) B_j{}^\kappa + (u^i p_i - \alpha\nu) C^\kappa + (u^i q_i + \alpha\rho) D^\kappa,
\end{aligned}$$

$$f_i{}^j u^i = -\lambda v^j + \rho p^j + \nu q^j, \tag{3.12}$$

$$u^i p_i = \alpha\nu - \lambda\tau, \tag{3.13}$$

$$u^i q_i = -\alpha\rho - \lambda\varepsilon. \tag{3.14}$$

Similarly, we have from (3. 5)

$$F_\lambda{}^\kappa v^\lambda = v^i F_\lambda{}^\kappa B_i{}^\lambda + \tau F_\lambda{}^\kappa C^\lambda + \varepsilon F_\lambda{}^\kappa D^\lambda,$$

$$\lambda u^\kappa = v^i(f_i{}^j B_j{}^\kappa + p_i C^\kappa + q_i D^\kappa) + (-p^i B_i{}^\kappa + \alpha D^\kappa) + \varepsilon(-q^i B_i{}^\kappa - \alpha C^\kappa),$$

$$\lambda(u^i B_i{}^\kappa + \rho C^\kappa + \nu D^\kappa) = (f_i{}^j v^i - \tau p^j - \varepsilon q^j) B_j{}^\kappa + (v^i p_i - \alpha\varepsilon) C^\kappa + (v^i q_i + \alpha\tau) D^\kappa,$$

$$f_i{}^j v^i = \lambda u^j + \tau p^j + \varepsilon q^j, \tag{3.15}$$

$$v^i p_i = \lambda\rho + \alpha\varepsilon, \tag{3.16}$$

$$v^i q_i = \lambda\nu - \alpha\tau. \tag{3.17}$$

On the other hand from (1. 4), (1. 5), (3. 4) and (3. 5) it follows

$$u^\lambda u_\lambda = (u^i B_i{}^\lambda + \rho C^\lambda + \nu D^\lambda)(u_j B^j{}_\lambda + \rho C_\lambda + \nu D_\lambda),$$

$$1 - \lambda^2 = u^i u_i + \rho^2 + \nu^2,$$

$$v^\lambda v_\lambda = (v^i B_i{}^\lambda + \tau C^\lambda + \varepsilon D^\lambda)(v_j B^j{}_\lambda + \tau C_\lambda + \varepsilon D_\lambda),$$

$$1 - \lambda^2 = v^i v_i + \tau^2 + \varepsilon^2,$$

$$u^\lambda v_\lambda = (u^i B_i{}^\lambda + \rho C^\lambda + \nu D^\lambda)(v_j B^j{}_\lambda + \tau C_\lambda + \varepsilon D_\lambda),$$

$$0 = u_i v^i + \rho\tau + \nu\varepsilon,$$

$$u^i u_i = 1 - \lambda^2 - \rho^2 - \nu^2, \tag{3.18}$$

$$v^i v_i = 1 - \lambda^2 - \tau^2 - \varepsilon^2, \tag{3.19}$$

$$u_i v^i = -\rho\tau - \nu\varepsilon. \tag{3.20}$$

Now differentiating (3. 1) covariantly along the submanifold N and using (2. 5) and (2. 6) we obtain

$$\begin{aligned}(\nabla_\nu F_\lambda{}^\kappa) B_j{}^\nu B_i{}^\lambda + F_\lambda{}^\kappa(h_{ji} C^\lambda + k_{ji} D^\lambda)\\ = (\nabla_j f_i{}^l) B_l{}^\kappa + f_i{}^l(h_{jl} C^\kappa + k_{jl} D^\kappa) + (\nabla_j p_i) C^\kappa + p_i(-h_j{}^l B_l{}^\kappa + l_j D^\kappa)\\ + (\nabla_j q_i) D^\kappa + q_i(-k_j{}^l B_l{}^\kappa - l_j C^\kappa).\end{aligned} \tag{3.21}$$

If we assume that the enveloping manifold is a manifold with normal metric (F, G, u, v, λ)-structure satisfying (1. 10) and (1. 11) and $\lambda(1-\lambda^2)$ is an almost everywhere non-zero function, then we have, from Theorem B, (3. 21), (3. 1), (3. 2),

(3. 3) and (3. 5)

$$\nabla_j f_i{}^l = -g_{ji}v^l + \delta_j^l v_i + p_i h_j{}^l + q_i k_j{}^l - p^l h_{ji} - q^l k_{ji}, \tag{3. 22}$$

$$\nabla_j p_i = -g_{ji}\tau - \alpha k_{ji} - h_{jl} f_i{}^l + q_i l_j, \tag{3. 23}$$

$$\nabla_j q_i = -g_{ji}\varepsilon + \alpha h_{ji} - k_{jl} f_i{}^l - p_i l_j. \tag{3. 24}$$

Differentiating (3. 4) and (3. 5) covariantly along the submanifold N and taking account of (2. 2), we find

$$\begin{aligned} B_j{}^\kappa \nabla_\kappa u^\lambda = {} & (\nabla_j u^i) B_i{}^\lambda + u^i(h_{ji}C^\lambda + k_{ji}D^\lambda) + (\nabla_j \rho)C^\lambda + \rho(-h_j{}^i B_i + l_j D^\lambda) \\ & + (\nabla_j \nu) D^\lambda + \nu(-k_j{}^i B_i{}^\lambda - l_j C^\lambda), \\ B_j{}^\kappa \nabla_\kappa v^\lambda = {} & (\nabla_j v^i) B_i{}^\lambda + v^i(h_{ji}C^\lambda + k_{ji}D^\lambda) + (\nabla_j \tau) C^\lambda \\ & + \tau(-h_j{}^i B_i{}^\lambda + l_j D^\lambda) + (\nabla_j \tau) D^\lambda + \varepsilon(-k_j{}^i B_i{}^\lambda - l_j C^\lambda). \end{aligned}$$

So, we have

$$\nabla_j u^i = \rho h_j{}^i + \nu k_j{}^i + B_j{}^\kappa B_\lambda{}^i \nabla_\kappa u^\lambda, \tag{3. 25}$$

$$\nabla_j v^i = \tau h_j{}^i + \varepsilon k_j{}^i + B_j{}^\kappa B_\lambda{}^i \nabla_\kappa v^\lambda. \tag{3. 26}$$

§ 4. Invariant submanifold of codimension 2 of a manifold with (*F, G, u, v, λ*)-structure.

We now assume that the tangent space of the submanifold N of codimension 2 in a manifold with (F, G, u, v, λ)-structure is invariant under the action of $F_\lambda{}^\kappa$ at every point, and we call such a submanifold an invariant submanifold.

For an invariant submanifold, we have

$$F_\lambda{}^\kappa B_i{}^\lambda = f_i{}^j B_j{}^\kappa, \tag{4. 1}$$

that is,

$$p_i = 0 \quad \text{and} \quad q_i = 0,$$

in (3. 1). Thus we have

$$F_\lambda{}^\kappa C^\lambda = \alpha D^\kappa, \qquad F_\lambda{}^\kappa D^\lambda = -\alpha C^\kappa, \tag{4. 2}$$

from (3. 2) and (3. 3) respectively,

$$f_i{}^j f_j{}^l = -\delta_i^l + u^l u_i + v^l v_i, \tag{4. 3}$$

$$\rho u_i + \tau v_i = 0, \tag{4. 4}$$

$$\nu u_i + \varepsilon v_i = 0, \tag{4. 5}$$

from (3. 6), (3. 7) and (3. 8) respectively,

$$\alpha^2=1-\rho^2-\tau^2, \tag{4. 6}$$

$$\nu\rho+\varepsilon\tau=0, \tag{4. 7}$$

$$\alpha^2=1-\nu^2-\varepsilon^2, \tag{4. 8}$$

from (3. 9), (3. 10) and (3. 11) respectively,

$$f_i{}^j u^i=-\lambda v^j, \tag{4. 9}$$

$$\alpha\nu=\lambda\tau, \tag{4. 10}$$

$$\alpha\rho=-\lambda\varepsilon, \tag{4. 11}$$

from (3. 12), (3. 13) and (3. 14) respectively and finally

$$f_i{}^j v^i=\lambda u^j, \tag{4. 12}$$

$$\alpha\varepsilon=-\lambda\rho, \tag{4. 13}$$

$$\alpha\tau=\lambda\nu, \tag{4. 14}$$

from (3. 15), (3. 16) and (3. 17) respectively.

Now, first of all, we prepare the following Lemma.

LEMMA 1. *In an invariant submanifold N of a manifold with (F, G, u, v, λ)-structure we have*

$$\rho^2=\varepsilon^2, \tag{4. 15}$$

$$\nu^2=\tau^2. \tag{4. 16}$$

Proof. Suppose first that P is a point of N where $\lambda(\mathrm{P})\neq 0$. Since the simultaneous equations (4. 11) and (4. 13) with unknowns $\lambda(\mathrm{P})$ and $\alpha(\mathrm{P})$ have non-trivial solutions, we have (4. 15). Similarly we can prove (4. 16).

In the next place we suppose that $\lambda(\mathrm{P})=0$. Then we have

$$\rho(\mathrm{P})\alpha(\mathrm{P})=0, \tag{4. 17}$$

and

$$\varepsilon(\mathrm{P})\alpha(\mathrm{P})=0, \tag{4. 18}$$

from (4. 11) and (4. 13) respectively. In this case, we distinguish two cases, that is, $\alpha(\mathrm{P})=0$ and $\alpha(\mathrm{P})\neq 0$. If $\lambda(\mathrm{P})=0$ and $\alpha(\mathrm{P})\neq 0$, then, by virture of (4. 10) and (4. 14), we get

$$\nu(\mathrm{P})=\tau(\mathrm{P})=0.$$

Thus it follows that

$$\rho^2(\mathrm{P})=\varepsilon^2(\mathrm{P})=\nu^2(\mathrm{P})=\tau^2(\mathrm{P})=0,$$

because of (4. 17) and (4. 18).

Suppose that $\lambda(\mathrm{P})=0$ and $\alpha(\mathrm{P})=0$. Then from (4. 6), (4. 8), (3. 18) and (3. 19) we have

$$1-\rho^2-\tau^2=0, \tag{4.19}$$

$$1-\nu^2-\varepsilon^2=0, \tag{4.20}$$

$$u^iu_i=1-\rho^2-\nu^2, \tag{4.21}$$

$$v^iv_i=1-\tau^2-\varepsilon^2. \tag{4.22}$$

Substituting (4. 19) and (4. 20) into (4. 21) and (4. 22) respectively, we have

$$u^iu_i=\tau^2-\nu^2, \tag{4.23}$$

and

$$v^iv_i=\nu^2-\tau^2. \tag{4.24}$$

These imply that

$$u_iu^i=-v_iv^i=\tau^2-\nu^2,$$

from which

$$u_iu^i=v_iv^i=0,$$

and consequently

$$\tau^2=\nu^2 \tag{4.25}$$

at P. Substituting (4. 25) into (4. 19) and (4. 20), we get

$$\rho^2(\mathrm{P})=\varepsilon^2(\mathrm{P}).$$

This completes the proof of the Lemma.

We remark that the result of Lemma 1 is independent of the choice of normal unit vector C^λ and D^λ, that is, the property is intrinsic.

Now let

$$N_1=\{\mathrm{P}|\rho(\mathrm{P})=0 \quad \text{and} \quad \nu(\mathrm{P})=0\}$$

and

$$N_2=N-N_1,$$

then N_2 is open in N and $N_1\cup N_2=N$.

In N_1 the vector fields u^λ and v^λ have the forms

$$u^\lambda = u^i B_i{}^\lambda, \qquad v^\lambda = v^i B_i{}^\lambda,$$

i.e. u^λ and v^λ are tangent to N_1. So, we get

$$u^i u_i = 1-\lambda^2, \qquad u^i v_i = 0, \qquad v^i v_i = 1-\lambda^2, \tag{4.26}$$

from (3.18), (3.19) and (3.20). Combining these equations with (4.3), (4.9) and (4.12), we see that the invariant submanifold N, of a Riemannian manifold with (F, G, u, v, λ)-structure has also (f, g, u, v, λ)-structure and is even-dimensional because of Theorem A.

LEMMA 2. *In* N_2 *the vector fields* u^λ *and* v^λ *have the forms*

$$u^\lambda = \rho C^\lambda + \nu D^\lambda, \qquad v^\lambda = \nu C^\lambda - \rho D^\lambda, \tag{4.27}$$

that is, u^λ *and* v^λ *are normal to* N_2.

Proof. At the point P where $\lambda(\mathrm{P}) \neq 0$ and $\alpha(\mathrm{P}) = 0$, we get $\rho(\mathrm{P}) = \nu(\mathrm{P}) = \tau(\mathrm{P}) = \varepsilon(\mathrm{P}) = 0$ from (4.10), (4.11), (4.13) and (4.14). This contradicts with (4.6) and (4.8). From Lemma 1, we see that at the point P where $\lambda(\mathrm{P}) = 0$ and $\alpha(\mathrm{P}) \neq 0$, $\rho(\mathrm{P}) = \nu(\mathrm{P}) = \tau(\mathrm{P}) = \varepsilon(\mathrm{P}) = 0$. Since at P of N_2 $\rho(\mathrm{P}) \neq 0$ or $\nu(\mathrm{P}) \neq 0$, at P of N_2 (i) $\lambda(\mathrm{P}) \neq 0$ and $\alpha(\mathrm{P}) \neq 0$ or (ii) $\lambda(\mathrm{P}) = 0$ and $\alpha(\mathrm{P}) = 0$. At the point of (i), multiplying (4.11) by α and (4.13) by λ and adding those, we get $\lambda^2 = \alpha^2$. Moreover substituting $\lambda^2 = \alpha^2$ and (4.6) into (3.18) and (3.19), we get $u^i = v^i = 0$ at the point P. This shows that at P u and v are normal to N_2. Also, at the point of (ii) from Lemma 1 we get $u^i = v^i = 0$. So, we see that u and v have forms (4.27) over N_2 because of (4.7).

These show that the submanifold N_1 is a manifold with (f, g, u, v, λ)-structure and the submanifold N_2 is an almost complex manifold.

Now, we assume that (1.10) is valid in M. Since N_2 is open, taking any vectors X and Y tangent to N_2, we get

$$\begin{aligned} u_\kappa [X, Y]^\kappa &= u_\kappa (X^\lambda \nabla_\lambda Y^\kappa - Y^\lambda \nabla_\lambda X^\kappa) \\ &= -X^\lambda Y^\kappa \nabla_\lambda u_\kappa + Y^\lambda X^\kappa \nabla_\lambda u_\kappa \\ &= -X^\lambda Y^\kappa (\nabla_\lambda u_\kappa - \nabla_\kappa u_\lambda) = -2\phi X^\lambda Y^\kappa F_{\lambda\kappa} \\ &= -2\phi f_{ij} B^i{}_\lambda B^j{}_\kappa x^l B_l{}^\lambda y^a B_a{}^\kappa \\ &= -2\phi f_{ij} x^i y^j \neq 0, \end{aligned}$$

where x^i and y^i are tangent components of X and Y. This means that the vector $[X, Y]$ are not tangent to N. Therefore, there is no invariant submanifold such as N_2.

So, we see that the invariant submanifold N of the manifold M with (F, G, u, v, λ)-structure satisfying (1.10) exists only when u^λ and v^λ are tangent to N and $\alpha \neq 0$ over N from (4.6) and the invariant submanifold N has induced (f, g, u, v, λ)-structure. Moreover if we assume that $\lambda(1-\lambda^2)$ is almost everywhere non-zero and (1. 10) and (1. 11) are valid, we have in the submanifold N, from (3. 23) and (3. 24),

$$\alpha k_{ji}=-h_{jl}f_i{}^l, \qquad \alpha h_{ji}=k_{jl}f_i{}^l.$$

Since h_{jl}, k_{jl} are symmetric and f_{ji} is skew-symmetric, transvecting these equations by g^{ji}, we have $k_i{}^i=0$ and $h_i{}^i=0$. Since α is non-zero on the submanifold N, we have $h_i{}^i=k_i{}^i=0$, i.e., the invariant submanifold N is minimal.

In the submanifold N, by computing the Nijenhuis tensor of $f_i{}^j$ and exterior derivative of u_i and v_i, we get

$$\begin{aligned}S_{ij}{}^k&=N_{ij}{}^k+(\nabla_i u_j-\nabla_j u_i)u^k+(\nabla_i v_j-\nabla_j v_i)v^k\\&=[N_{\lambda\kappa}{}^\nu+(\nabla_\lambda u_\kappa-\nabla_\kappa u_\lambda)u^\nu+(\nabla_\lambda v_\kappa-\nabla_\kappa v_\lambda)v^\nu]B_i{}^\lambda B_j{}^\kappa B_\nu{}^k.\end{aligned}$$

This shows that if the Riemannian manifold with (F, G, u, v, λ)-structure is normal, then an invariant submanifold N with induced (f, g, u, v, λ)-structure of codimension 2 of M is also normal.

Next, if the (F, G, u, v, λ)-structure satisfies (1. 10), and (1. 11) then we have, on the invariant submanifold N,

$$\nabla_i u_j-\nabla_j u_i=(\nabla_\lambda u_\kappa-\nabla_\kappa u_\lambda)B_i{}^\lambda B_j{}^\kappa=2\phi F_{\lambda\kappa}B_i{}^\lambda B_j{}^\kappa=2\phi f_{ij},$$

$$\nabla_i v_j-\nabla_j v_i=(\nabla_\lambda v_\kappa-\nabla_\kappa v_\lambda)B_i{}^\lambda B_j{}^\kappa=2F_{\lambda\kappa}B_i{}^\lambda B_j{}^\kappa=2f_{ij}$$

So, the induced (f, g, u, v, λ)-structure satisfies $\nabla_j u_i-\nabla_i u_j=2\phi f_{ji}$ and $\nabla_j v_i-\nabla_i v_j=2f_{ji}$ and then, if we assume that $\lambda(1-\lambda^2)$ is almost everywhere non-zero over N an invariant submanifold N is also isometric with a sphere if the manifold M is isometric with a sphere because of Theorem C.

Summarizing the above we have

PROPOSITION 1. *Let M be a Riemannian manifold with (F, G, u, v, λ)-structure. If N is an F-invariant submanifold of codimension* 2 *of M, i.e., $F_\lambda{}^\kappa B_i{}^\lambda=f_i{}^j B_j{}^\kappa$, then N is a sum of a manifold with (f, g, u, v, λ)-structure and an almost complex manifold.*

PROPOSITION 2. *Let M be a Riemannian manifold with (F, G, u, v, λ)-structure and satisfy* (1. 10). *If N is an F-invariant submanifold of codimension* 2 *of M, then u^λ and v^λ are tangent to N_1 i.e., $u^\lambda=u^i B_i{}^\lambda$ and $v^\lambda=v^i B_i{}^\lambda$.*

PROPOSITION 3. *Let M be a Riemannian manifold with (F, G, u, v, λ)-structure and satisfy* (1. 10). *If N is an F-invariant submanifold of codimension* 2 *of M, then N has also (f, g, u, v, λ)-structure induced from (F, G, u, v, λ)-structure.*

PROPOSITION 4. *If M is a Riemannian manifold with normal (F, G, u, v, λ)-structure and satisfy* (1. 10), *then an F-invariant submanifold N of codimension* 2 *of M is also a Riemannian manifold with normal (f, g, u, v, λ)-structure.*

PROPOSITION 5. *In an even dimensional sphere M, there always exists*

(F, G, u, v, λ)-structure. If N is an F-invariant submanifold of codimension 2 of M and $\lambda(1-\lambda^2)$ is almost everywhere non-zero over N, then N is also an even dimensional sphere.

BIBLIOGRAPHY

[1] OKUMURA, M., On contact metric immersion. Kōdai Math. Sem. Rep. **20** (1968), 389-409.

[2] SCHOUTEN, S., AND K. YANO, On invariant subspaces in the almost complex X_{2n}. Indag. Math. **17** (1955), 261-269.

[3] YANO, K., AND S. ISHIHARA, Invariant submanifold of almost contact manifold. Kōdai Math. Sem. Rep. **21** (1969), 350-364.

[4] YANO, K., AND M. OKUMURA, On (f, g, u, v, λ)-structures. Kōdai Math. Sem. Rep. **22** (1970), 401-423.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.