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AFFINE ALMOST CONTACT MANIFOLDS AND f-MANIFOLDS WITH AFFINE KILLING STRUCTURE TENSORS

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1. Introduction.

In a recent paper [2], one of the present authors has proved

THEOREM A. *Suppose M^{2n+1} has an almost contact metric structure (ϕ, ξ, η, g) such that ϕ and η are Killing. Then if this structure is normal, it is cosymplectic.*

THEOREM B. *Let M^{2n+2} be a nearly Kaehler manifold and M^{2n+1} a C^∞ orientable hypersurface. Let (ϕ, ξ, η, g) denote the induced almost contact metric structure on M^{2n+1}. Then ϕ is Killing if and only if the second fundamental form h is proportional to $\eta\otimes\eta$.*

THEOREM C. *Let M^{2n+2} be a nearly Kaehler manifold and M^{2n+1} a C^∞ orientable hypersurface. Let η denote the induced almost contact form and suppose the second fundamental form h is proportional to $\eta\otimes\eta$. Then η is Killing.*

The purpose of the present paper is to generalize these theorems to the case of affine almost contact manifold, and to prove a theorem similar to Theorem A for an affine f-manifold with complemented frames with affine Killing structure tensor.

2. Affine Killing tensors.

Let M be a differentiable manifold with an affine connection ∇ without torsion. A curve $x(t)$ of M is called a *path* if its tangent vector $X=dx/dt$ satisfies

$$\nabla_X X=0 \tag{2.1}$$

and t an *affine parameter.*

Let ϕ be a tensor field of type (1, 1) in M and $x(t)$ a path in M, t being an affine parameter. Then we have a vector fieled ϕX along the path. If this vector field ϕX is parallel along the path $x(t)$, then we have $\nabla_X(\phi X)=0$, or

$$(\nabla_X\phi)X=0. \tag{2.2}$$

If this is the case for any path, we have

$$(\nabla_X\phi)Y+(\nabla_Y\phi)X=0 \tag{2.3}$$

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for any vector fields X and Y. We call a tensor field ϕ of type (1, 1) satisfying (2.3) an *affine Killing tensor field.* [9].

Let η be a 1-form in M and $x(t)$ a path in M, t being an affine parameter. Then we have a function $\eta(X)$ along the path. If this function is constant along the path, then we have $\nabla_X(\eta(X))=0$, or

$$(\nabla_X\eta)(X)=0. \tag{2.4}$$

If this is the case for any path, we have

$$(\nabla_X\eta)(Y)+(\nabla_Y\eta)(X)=0 \tag{2.5}$$

for any vector fields X and Y. We call a 1-form η satisfying (2.5) an *affine Killing 1-form.*

3. Affine f-manifolds with complemented frames with f and η^u affine Killing.

A $(2n+s)$-dimensional differentiable manifold M^{2n+s} is said to have an *affine f-structure with complemented frames* if there exist a tensor field f of type (1, 1) and of constant rank $2n$, s vector fields ξ_v and s 1-forms η^u satisfying

$$\begin{gathered} f^2X=-X+\eta^u(X)\xi_u,\ f\xi_v=0, \\ \eta^u(fX)=0, \qquad \eta^u(\xi_v)=\delta_v^u \end{gathered} \tag{3.1}$$

for any vector field X, where the indices u, v, w, x run over the range $\{1, 2, \cdots, s\}$, [4]. In the special case $s=1$, this structure is said to be an *affine almost contact structure* [5] and we use the customary notation $\phi=f$, $\xi=\xi_1$ $\eta=\eta^1$.

If

$$[f, f]+d\eta^u\otimes\xi_u=0, \tag{3.2}$$

$[f, f]$ being the Nijenhuis tensor of f, or

$$(\nabla_{fX}f)Y-(\nabla_{fY}f)X-f\{(\nabla_Xf)Y-(\nabla_Yf)X\}+\{\nabla_X\eta^u)(Y)-(\nabla_Y\eta^u)(X)\}\xi_u=0 \tag{3.3}$$

for any vector fields X and Y, ∇ being an affine connection without torsion, the affine f-structure with complemented frames is said to be *normal.*

When the f-structure with complemented frames is normal, putting $X=\xi_v$, $Y=\xi_w$, we obtain

$$\begin{gathered} -f\{(\nabla_{\xi_v}f)\xi_w-(\nabla_{\xi_w}f)\xi_v\}+\{(\nabla_{\xi_v}\eta^u)(\xi_w)-(\nabla_{\xi_w}\eta^u)(\xi_v)\}\xi_u=0, \\ f^2\{\nabla_{\xi_v}\xi_w-\nabla_{\xi_w}\xi_v\}-\eta^u(\nabla_{\xi_v}\xi_w-\nabla_{\xi_w}\xi_v)\xi_u=0, \end{gathered}$$

from which

$$[\xi_v, \xi_w]=0. \tag{3.4}$$

THEOREM 3.1. *Suppose M^{2n+s} has an affine f-structure with complemented frames (f, ξ_u, η^u) such that it is normal and f and the η^u's are affine Killing. Then* *the structure is cosymplectic, that is,*

$$\nabla f=0, \qquad \nabla \xi_u=0, \qquad \nabla \eta^u=0.$$

Proof. The assumptions are

$$(\nabla_X f)Y+(\nabla_Y f)(X)=0 \tag{3.5}$$

and

$$(\nabla_X \eta^u)(Y)+(\nabla_Y \eta^u)(X)=0 \tag{3.6}$$

for any vector fields X and Y.

Thus, equation (3. 3) can be written as

$$-(\nabla_Y f)fX+(\nabla_X f)fY-2f(\nabla_X f)Y+2(\nabla_X \eta^u)(Y)\xi_u=0,$$

$$-(\nabla_Y f^2)X+f(\nabla_Y f)X+(\nabla_X f^2)Y-f(\nabla_X f)Y-2f(\nabla_X f)Y+2(\nabla_X \eta^u)(Y)\xi_u=0,$$

$$-(\nabla_Y \eta^u)(X)\xi_u-\eta^u(X)\nabla_Y \xi_u+(\nabla_X \eta^u)(Y)\xi_u+\eta^u(Y)\nabla_X \xi_u$$
$$-4f(\nabla_X f)Y+2(\nabla_X \eta^u)(Y)\xi_u=0$$

or

$$-\eta^u(X)\nabla_Y \xi_u+\eta^u(Y)\nabla_X \xi_u-4f(\nabla_X f)Y+4(\nabla_X \eta^u)(Y)\xi_u=0. \tag{3.7}$$

Consequently

$$\eta^v\{-\eta^u(X)\nabla_Y \xi_u+\eta^u(Y)\nabla_X \xi_u-4f(\nabla_X f)Y+4(\nabla_X \eta^u)(Y)\xi_u\}=0,$$

from which

$$-\eta^u(X)\eta^v(\nabla_Y \xi_u)+\eta^u(Y)\eta^v(\nabla_X \xi_u)+4(\nabla_X \eta^v)(Y)=0,$$

or

$$\eta^u(X)(\nabla_Y \eta^v)(\xi_u)-\eta^u(Y)(\nabla_X \eta^v)(\xi_u)+4(\nabla_X \eta^v)(Y)=0. \tag{3.8}$$

Putting $Y=\xi_w$ in this equation, we find

$$\eta^u(X)(\nabla_{\xi_w} \eta^v)(\xi_u)-(\nabla_X \eta^v)(\xi_w)+4(\nabla_X \eta^v)(\xi_w)=0, \tag{3.9}$$

from which, puttiug $X=\xi_x$,

$$(\nabla_{\xi_w} \eta^v)(\xi_x)-(\nabla_{\xi_x} \eta^v)(\xi_w)+4(\nabla_{\xi_x} \eta^v)(\xi_w)=0,$$

that is,

$$(\nabla_{\xi_u} \eta^v)(\xi_w)=0$$

by virtue of (3. 6), and consequently, (3. 9) gives

$$(\nabla_X \eta^v)(\xi_w)=0.$$

Thus, from (3. 8), we have

$$(\nabla_X \eta^v)(Y)=0. \tag{3.10}$$

From (3.7) and (3.10), we find

$$-\eta^u(X)\nabla_Y\xi_u+\eta^u(Y)\nabla_X\xi_u-4f(\nabla_X f)Y=0, \tag{3.11}$$

from which, putting $Y=\xi_v$, we obtain

$$-\eta^u(X)\nabla_{\xi_v}\xi_u+\nabla_X\xi_v-4f(\nabla_X f)\xi_v=0,$$

$$-\eta^u(X)\nabla_{\xi_v}\xi_u+\nabla_X\xi_v+4f^2\nabla_X\xi_v=0,$$

$$-\eta^u(X)\nabla_{\xi_v}\xi_u+\nabla_X\xi_v-4(\nabla_X\xi_v)+4\eta^u(\nabla_X\xi_v)\xi_u=0,$$

or

$$\eta^u(X)\nabla_{\xi_v}\xi_u-3(\nabla_X\xi_v)=0, \tag{3.12}$$

by virtue of $\eta^u(\nabla_X\xi_v)=-(\nabla_X\eta^u)(\xi_v)=0$. Putting $X=\xi_w$ in (3.12), we find

$$\nabla_{\xi_v}\xi_w-3\nabla_{\xi_w}\xi_v=0,$$

from which

$$\nabla_{\xi_v}\xi_w=0$$

by virtue of (3.4), and consequently (3.12) gives

$$\nabla_X\xi_v=0. \tag{3.13}$$

Thus, from (3.7), we find

$$f(\nabla_X f)Y=0,$$

from which, applying f,

$$-(\nabla_X f)Y+\eta^u\{(\nabla_X f)Y\}\xi_u=0,$$

$$(\nabla_X f)Y=0.$$

Thus the proof is complete.

COROLLARY 3.2. *Suppose M^{2n+1} has an affine almost contact structure (ϕ, ξ, η) such that it is normal and ϕ and η are affine Killing. Then the structure is an affine cosymplectic structure, that is,*

$$\nabla\phi=0, \qquad \nabla\xi=0, \qquad \nabla\eta=0.$$

4. Hypersurfaces of almost complex manifolds.

Let M^{2n+2} be an almost complex manifold with structure tensor J. If there exists an affine connection ∇ without torsion such that

$$(\nabla_X J)Y+(\nabla_Y J)X=0 \tag{4.1}$$

for any vector fields X and Y, we call M^{2n+2} an *affine almost Tachibana mani-*

fold [7].

Let M^{2n+2} be an almost complex manifold with structure tensor J and M^{2n+1} an orientable differentiable hypersurface of M^{2n+2}. We denote by B the differential of the imbedding. We can choose a vector field C along M^{2n+1} such that the transform JC of C by J is tangent to the hypersurface M^{2n+1}. Indeed, suppose first of all that JBX is tangent to M^{2n+1} for every vector field X on M^{2n+1}, then $JBX=BfX$ for some tensor field f of type (1, 1) on M^{2n+1}. Applying J to this equation we find $f^2=-1$, that is, f defines an almost complex structure on M^{2n+1}, which is impossible. Thus, there exists a vector field ξ on M^{2n+1} such that $JB\xi$ is not tangent to M^{2n+1}. Now setting $C=JB\xi$ we have

$$JC=-B\xi, \tag{4.2}$$

and hence C is as desired.

We now define a tensor field ϕ of type (1, 1) and a 1-form η on M^{2n+1} by

$$JBY=B\phi Y+\eta(Y)C. \tag{4.3}$$

It is easily checked that ϕ, ξ, η satisfy equations (3. 1), that is the hypersurface M^{2n+1} admits an affine almost contact structure (cf. [6]).

The equations of Gauss and Weingarten are

$$\begin{aligned} \nabla_{BX}BY&=B\bar{\nabla}_XY+h(X, Y)C,\\ \nabla_{BX}C&=-BHX+l(X)C, \end{aligned} \tag{4.4}$$

where $\bar{\nabla}$ is the affine connection without torsion on M^{2n+1} induced from the affine connection ∇ on M^{2n+2} with respect to the affine normal C. h is the second fundamental tensor which is symmetric, H is a tensor field of type (1, 1) representing the Weingarten map with respect to the affine normal C, and l is the third fundamental tensor [8].

THEOREM 4. 1. *Let M^{2n+2} be an affine almost Tachibana manifold and M^{2n+1} an orientable differentiable hypersurface. Let (ϕ, ξ, η) denote an induced affine almost contact structure on M^{2n+1}. Then ϕ is affine Killing if and only if the second fundamental form h has the form*

$$h(X, Y)=\eta(X)h(Y, \xi)-\eta(Y)h(X, \xi)-h(\xi, \xi)\eta(X)\eta(Y)$$

and the Weingarten map H has the form

$$HX=(2h(X, \xi)-h(\xi, \xi)\eta(X))\xi.$$

Proof. Applying the operator ∇_{BX} to equation (4. 3), we find

$$\begin{aligned} &(\nabla_{BX}J)BY+Jh(X, Y)C\\ &=B(\bar{\nabla}_X\phi)Y+h(X, \phi Y)C+(\bar{\nabla}_X\eta)(Y)C+\eta(Y)(-BHX+l(X)C), \end{aligned}$$

from which, M^{2n+2} being an affine almost Tachibana manifold,

(4. 5) $$-2h(X, Y)\xi=(\bar{\nabla}_X\phi)Y+(\bar{\nabla}_Y\phi)X-\eta(Y)HX-\eta(X)HY$$

and

(4. 6) $$h(X, \phi Y)+h(\phi X, Y)+(\bar{\nabla}_X\eta)(Y)+(\bar{\nabla}_Y\eta)(X) +\eta(Y)l(X)+\eta(X)l(Y)=0.$$

Thus in order for ϕ to be affine Killing, it is necessary and sufficient that

(4. 7) $$2h(X, Y)\xi=\eta(X)HY+\eta(Y)HX$$

for any vector fields X and Y.

Putting $Y=\xi$ in (4. 7), we find

$$HX=2h(X, \xi)\xi-\eta(X)H\xi,$$

from which, putting $X=\xi$, $H\xi=h(\xi, \xi)\xi$. Thus we have

(4. 8) $$HX=(2h(X, \xi)-h(\xi, \xi)\eta(X))\xi.$$

On the other hand, we have from (4. 7)

$$2h(X, Y)=\eta(X)\eta(HY)+\eta(Y)\eta(HX)$$

or using (4. 8)

(4. 9) $$h(X, Y)=\eta(X)h(Y, \xi)+\eta(Y)h(X, \xi)-h(\xi, \xi)\eta(X)\eta(Y).$$

Conversely if (4. 8) and (4. 9) are satisfied, then equation (4. 7) is satisfied completing the proof.

Finally we note that if $h(X, Y)$ has the form (4. 9), equation (4. 6) becomes

$$\eta(X)h(\phi Y, \xi)+\eta(Y)h(\phi X, \xi)+\bar{\nabla}_X\eta)(Y)+(\bar{\nabla}_Y\eta)(X) +\eta(X)l(Y)+\eta(Y)l(X)=0$$

which shows after a short computation that then η is affine Killing if and only if

$$h(\phi X, \xi)+l(X)=0.$$

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