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A REMARK ON DERIVED SPACES

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Let G be a locally compact Abelian group, and let μ be the Haar measure on G. If $p \geqq 1$, the space of all μ-measurable functions f such that $|f|^p$ is summable is designated by $L_p(G)$.

We denote by $L_p^0(G)$ the set of all $f \in L_p(G)$ such that $\|f\|_0 = \sup\{\|f*h\|_p : h \in L_1(G), \|\hat{h}\|_\infty \leqq 1\} < \infty$. Clearly $L_p^0(G)$ is the linear subspace of $L_p(G)$. We call $L_p^0(G)$ the *derived space of* $L_p(G)$ ([5], p. 72).

The following results were showed by Helgason [3] and Figa-Talamanca [1].

Let G be a locally compact Abelian group.

1) *If G is a compact, then* $L_p^0(G)$ $(1 \leqq p \leqq 2)$ *is algebraically and topologically isomorphic to* $L_2(G)$.

2) *If G is non-compact and connected, then* $L_p^0(G) = \{0\}$ $(1 \leqq p < 2)$.

3) *If G is non-compact and separable, then* $L_1^0(G) = \{0\}$.

4) *If G is an infinite discrete group, then* $L_p^0(G) = \{0\}$ $(1 \leqq p < 2)$.

We shall show the following theorem in this short note.

THEOREM A. *Let G be a non-compact locally compact Abelian group, then* $L_p^0(G) = \{0\}$ $(1 \leqq p < 2)$.

Proof. From the structure theorem of locally compact Abelian groups ([6, p. 95]), we know that G has an open subgroup H which is the direct sum of a compact group and an Euclidean space R^m.

a) Suppose $m \geqq 1$. If we shall show that for any compact set K of G there exists an element d of G such that $\{K + kd\}$, $k = 0, \pm 1, \pm 2, \cdots$, are pairwise disjoint, then we can prove this theorem by using the similar argument in [1].

Since K is compact, there is a finite set $\{x_i : i = 1, 2, \cdots, n\}$ of G such that $K \subset \bigcup_{i=1}^n (x_i + H)$. Put $K_0 = \bigcup_{i=1}^n (((x_i + H) \cap K) - x_i)$. Then K_0 is a compact subset of H. Since $m \geqq 1$, there is an element $d \in H$ such that $\{K_0 + kd\}$, $k = 0, \pm 1, \pm 2, \cdots$, are pairwise disjoint. Clearly, $\{K + kd\}$, $k = 0, \pm 1, \pm 2, \cdots$, are also pairwise disjoint.

b) Suppose now $m = 0$. Let $\dot{\mu}$ and μ_0 be the Haar measures on G/H and H respectively such that

$$\int_G f(x)\, d\mu(x) = \int_{G/H} \left(\int_H f(x+y)\, d\mu_0(x) \right) d\dot{\mu}(\dot{y})$$

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and $||\mu_0||=1$. Let

$$\Phi g(\dot{y})=\int_H g(x+y)\, d\mu_0(x)$$

for each $g\in L_p(G)$, since $||\mu_0||=1$, we have that

$$\int_{G/H} |\Phi g(\dot{y})|^p\, d\dot{\mu}(\dot{y})=\int_{G/H}\left|\int_H g(x+y)\, d\mu_0(x)\right|^p d\dot{\mu}(\dot{y})$$

$$\leqq\int_{G/H}\left(\int_H |g(x+y)|^p\, d\mu_0(x)\right) d\dot{\mu}(\dot{y})=\int_G |g(x)|^p\, d\mu(x).$$

Therefore, Φ is a norm-decreasing linear operator of $L_p(G)$ into $L_p(G/H)$. On the other hand, if φ is the canonical homomorphism of G to G/H, then $\Psi g(x)=g(\varphi(x))$ belongs to $L_p(G)$ for any $g\in L_p(G/H)$. Indeed,

$$\int_G |g(\varphi(x))|^p\, d\mu(x)=\int_{G/H}\left(\int_H |g(\varphi(x+y))|^p\, d\mu_0(x)\right) d\dot{\mu}(\dot{y})$$

$$=\int_{G/H} |g(\dot{y})|^p\, d\dot{\mu}(\dot{y})<\infty.$$

It is evident that $\Phi\Psi g=g$ for all $g\in L_p(G/H)$.

For f, $g\in C_c(G)$, let

$$F(f,\, g,\, x\, ;\, y,\, \dot{z})=\int_H f(x+y-u-z)\, g(u+z)\, d\mu_0(u),\quad (x\in G,\ y\in H,\ \dot{z}\in G/H).$$

Suppose that K_g is a compact support of g, then $F(f, g, x\,; y, \dot{z})=0$ for all $z\notin(K_g+H)$. Since K_g+H is compact, there exists a finite subset $\{z_1, \cdots, z_n\}$ of G such that $K_g+H\subset\cup_{i=1}^n(z_i+H)$. Thus, we have that if $\dot{z}\notin\{\dot{z}_1, \cdots, z_n\}$, then $F(f, g, x\,; y, \dot{z})=0$ for all $y\in H$ and $\Phi g(\dot{z})=0$. Hence, it follows that

$$\Phi(f*g)(\dot{x})=\int_H\int_G f(x+y-z)\, g(z)\, d\mu(z)\, d\mu_0(y)$$

$$=\int_H\int_{G/H}\int_H f(x+y-z-u)\, g(z+u)\, d\mu_0(u)\, d\dot{\mu}(\dot{z})\, d\mu_0(y)$$

$$=\int_H\sum_{i=1}^n F(f,\, g,\, x\,;\, y,\, \dot{z}_i)\, d\mu_0(y)$$

$$=\sum_{i=1}^n\int_H F(f,\, g,\, x\,;\, y,\, \dot{z}_i)\, d\mu_0(y)$$

$$=\sum_{i=1}^n\int_H\int_H f(x+y-z_i-u)\, g(z_i+u)\, d\mu_0(u)\, d_0\mu(y)$$

$$=\sum_{i=1}^n\int_H\int_H f(x+y-z_i-u)\, g(z_i+u)\, d\mu_0(y)\, d\mu_0(u)$$

$$=\sum_{i=1}^{n}\int_{H}f(x+y-z_i-u)\,d\mu_0(y)\int_{H}g(z_i+u)\,d\mu_0(u)$$

$$=\sum_{i=1}^{n}\Phi f(\dot{x}-\dot{z}_i)\Phi g(\dot{z}_i)$$

$$=\int_{G/H}\Phi f(\dot{x}-\dot{z})\Phi g(\dot{z})\,d\dot{\mu}(\dot{z})$$

$$=\Phi f*\Phi g(\dot{x}).$$

Since $C_c(G)$ is dense in $L_p(G)$ $(1\leqq p<2)$, we have that if $f\in L_p(G)$ and $g\in L_1(G)$, then $\Phi(f*g)=\Phi f*\Phi g$.

Let Γ be the dual group of G, and let Λ be the anihilator of H. For any $h\in L_1(G/H)$, it is evident that $\widehat{\Psi h}(\gamma)=\hat{h}(\gamma)$ for any $\gamma\in\Lambda$ and $\widehat{\Psi h}(\gamma)=0$ for any $\gamma\in\Gamma\backslash\Lambda$. Therefore, if $f\in L_p^0(G)$ and $h\in L_1(G/H)$, then

$$\|\Phi f*h\|_p=\|\Phi(f*\Psi h)\|_p\leqq\|f*\Psi h\|_p\leqq\|f\|_0\|\widehat{\Psi h}\|_\infty=\|f\|_0\|\hat{h}\|_\infty.$$

Consequently, $\Phi f\in L_p^0(G/H)$ for any $f\in L_p^0(G)$.

Let $f_\gamma(x)=(-x,\gamma)f(x)$ for any $f\in L_p(G)$ and $\gamma\in\Gamma$. Clearly, if $f\in L_p^0(G)$, then $f_\gamma\in L_p^0(G)$. Suppose that there exists a non-zero element $f\in L_p^0(G)$. If γ is an element of Γ such that $\hat{f}(\gamma)\neq 0$, then

$$\widehat{\Phi f_\gamma}(0)=\int_{G/H}\Phi f_\gamma(\dot{x})\,d\dot{\mu}(\dot{x})$$

$$=\int_{G/H}\int_{H}(-(x+y),\gamma)f(x+y)\,d\mu_0(y)\,d\dot{\mu}(\dot{x})=\hat{f}(\gamma)\neq 0.$$

Thus, we have that $L_p^0(G/H)\neq\{0\}$. But, since G/H is infinite discrete, this is impossible. Therefore, $L_p^0(G)=\{0\}$. This completes the proof.

The following theorem was proved by Gaudry in the case of a locally compact Abelian group with an infinite discrete subgroup.

THEOREM B. *Let G be a non-compact locally compact Abelian group. If g is a function on Γ such that $\varphi g\in\bigcup_{1\leq p<2}L_p(G)^\wedge$, where $L_p(G)^\wedge=\{\hat{f}:f\in L_p(G),\ \hat{f}$ is the Fourier transform of $f\}$, for each $\varphi\in C_0(\Gamma)$, then g is zero locally almost evyeywhere.*

Proof. From the hypothesis, we can assume g has a compact support K. Then, there is a number p_0, $1<p_0<2$, such that $\varphi g\in L_{p_0}(G)^\wedge$ for all $\varphi\in C_0(\Gamma)$ ([2], p. 486). Let $\varphi_0\in C_0(\Gamma)$ such that $\varphi_0\equiv 1$ on K, then $\varphi_0 g=g\in L_{p_0}(G)^\wedge$. Let $f\in L_{p_0}(G)$ such that $\hat{f}=g$ locally almost everywhere. Then $f\in L_{p_0}^0(G)$ ([1]). Therefore, theorem A shows f is zero. Thus g is zero locally almost everywhere. This completes the proof.

For $s\in G$, τ_s will denote the translation operator defined by $(\tau_s f)(t)=f(ts^{-1})$. A continuous linear operator T from $L_p(G)$ to $L_p(G)$ is called a *multiplier* for $L_p(G)$ whenever $T\tau_s=\tau_s T$ for each $s\in G$. The collection of all mutipliers for $L_p(G)$ will be denoted by $M(L_p(G))$.

Combining the theorem A with the proof of theorem 5 in [1] we can prove the next theorem.

THEOREM C. *Let G be a non-compact locally compact Abelian group and suppose* $1<p<\infty$, $p\neq 2$. If $\varphi\in L_\infty(\Gamma)$ *corresponds a multiplier T in* $M(L_p(G))$ (*i.e.* $\widehat{(Tf)}=\varphi\hat{f}$ *and has the property that whenever* ψ *is a function for which* $|\psi(\gamma)|\leqq|\varphi(\gamma)|$ *for each* $\gamma\in\Gamma$ *then* ψ *corresponds to a multiplier in* $M(L_p(G))$, *then* $\varphi=0$.

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