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A CONVEXITY IN METRIC SPACE AND NONEXPANSIVE MAPPINGS, I

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1. Introduction.

In this paper, we shall discuss convexity and fixed point theorems in certain metric space which are described in an abstract form. At first we shall introduce a concept of convexity in a metric space and study the properties of the space which we call a convex metric space. Futhermore, we formulate some fixed point theorems for nonexpansive mappings (i.e. mappings which do not increase distances) in the space. Consequently, these generalize fixed point theorems which have been previously proved by Browder [1], Kirk [6] and the author [7] in a Banach space.

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2. Definitions and propositions.

Throughout this paper, we consider a metric space X with a *convex structure* such that there exists a mapping W from $X\times X\times[0,1]$ to X (i.e. $W(x, y; \lambda)$ defined for all pairs $x, y\in X$ and λ $(0\leqq\lambda\leqq1)$) and valued in X satisfying

$$(*)\qquad d(u, W(x, y; \lambda))\leqq\lambda d(u, x)+(1-\lambda)d(u, y)$$

for all $u\in X$ and call this space X a *convex metric space*. A Banach space and each of its convex subsets are, of course, convex metric spaces. But a Fréchet space is not necessary a convex metric space. There are many examples of convex metric spaces which are not imbedded in any Banach space. We give two preliminary examples here.

EXAMPLE 1. Let I be the unit interval $[0,1]$ and X be the family of closed intervals $[a_i, b_i]$ such that $0\leqq a_i\leqq b_i\leqq1$. For $I_i=[a_i, b_i]$, $I_j=[a_j, b_j]$ and λ $(0\leqq\lambda\leqq1)$, we define a mapping W by $W(I_i, I_j; \lambda)=[\lambda a_i+(1-\lambda)a_j,\ \lambda b_i+(1-\lambda)b_j]$ and define a metric d in X by the Hausdorff distance, i.e.

$$d(I_i, I_j)=\sup_{a\in I}\{|\inf_{b\in I_i}\{|a-b|\}-\inf_{c\in I_j}\{|a-c|\}|\}.$$

EXAMPLE 2. We consider a linear space L which is also a metric space with

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the following properties:

(1) For $x, y \in L$, $d(x, y)=d(x-y, 0)$;

(2) For $x, y \in L$ and λ $(0 \leqq \lambda \leqq 1)$,

$$d(\lambda x+(1-\lambda)y, 0) \leqq \lambda d(x, 0)+(1-\lambda)d(y, 0).$$

A subset K of a convex metric space X is said to be *convex* if $W(x, y; \lambda) \in K$ for all $x, y \in K$ and λ $(0 \leqq \lambda \leqq 1)$. The following three Propositions are easy.

PROPOSITION 1. *Let $\{K_\alpha: \alpha \in A\}$ be a family of convex subsets of X, then $\bigcap_{\alpha \in A} K_\alpha$ is also a convex subset of X.*

PROPOSITION 2. *The open spheres $S(x, r)$ and the closed spheres $\bar{S}(x, r)$ in X are convex subsets of X.*

Proof. For $y, z \in S(x, r)$ and λ $(0 \leqq \lambda \leqq 1)$, there exists $W(y, z; \lambda) \in X$. Since X is a convex metric space,

$$\begin{aligned} d(x, W(y, z; \lambda)) &\leqq \lambda d(x, y)+(1-\lambda)d(x, z) \\ &< \lambda r+(1-\lambda)r=r. \end{aligned}$$

Therefore $W(y, z; \lambda) \in S(x, r)$. Similarly, $\bar{S}(x, r)$ is a convex subset of X.

PROPOSITION 3. *For $x, y \in X$ and λ $(0 \leqq \lambda \leqq 1)$,*

$$d(x, y)=d(x, W(x, y; \lambda))+d(W(x, y; \lambda), y).$$

Proof. Since X is a convex metric space, we obtain

$$\begin{aligned} d(x, y) &\leqq d(x, W(x, y; \lambda))+d(W(x, y; \lambda), y) \\ &\leqq \lambda d(x, x)+(1-\lambda)d(x, y)+\lambda d(x, y)+(1-\lambda)d(y, y) \\ &= \lambda d(x, y)+(1-\lambda)d(x, y)=d(x, y) \end{aligned}$$

for $x, y \in X$ and λ. Therefore,

$$d(x, y)=d(x, W(x, y; \lambda))+d(W(x, y; \lambda), y)$$

for $x, y \in X$ and λ.

For $E \subset X$, we denote

$$R_x(E)=\sup\{d(x, y): y \in E\},$$

$$R(E)=\inf\{R_x(E): x \in E\},$$

$$E_c=\{x \in E: R_x(E)=R(E)\}$$

and denote the diameter of E by

$$\delta(E)=\sup\{d(x, y):\ x, y\in E\}.$$

A point $x\in E$ is a *diametral point* of E provided

$$\sup\{d(x, y):\ y\in E\}=\delta(E).$$

A convex metric space X will be said to have *Property* (C) if every bounded decreasing net of nonempty closed convex subsets of X has a nonempty intersection. By Šmulian's theorem, every weakly compact convex subset of a Banach space has Property (C) [cf. 5, p. 433]. We obtain the following Proposition from the definition of Property (C), Propositions 1 and 2.

PROPOSITION 4. *If X has Property* (C), *then E_c is nonempty, closed and convex.*

Proof. Let $E_n(x)=\{y\in E:\ d(x, y)\leqq R(E)+1/n\}$ for $n=1, 2, 3, \cdots$ and $x\in X$. It is easily seen that the sets $C_n=\bigcap_{x\in X}E_n(x)$ form a decreasing sequence of nonempty closed convex sets, and hence $\bigcap_{n=1}^{\infty}C_n$ is nonempty, closed and convex. Since $E_c=\bigcap_{n=1}^{\infty}C_n$, it satisfies the conclusion.

PROPOSITION 5. *Let M be a nonempty compact subset of X and let K be the least closed convex set containing M. If the diameter $\delta(M)$ is positive, then there exists an element $u\in K$ such that* $\sup\{d(x, u):\ x\in M\}<\delta(M)$.

Proof. Since M is compact, we may find $x_1, x_2\in M$ such that $d(x_1, x_2)=\delta(M)$. Let $M_0\subset M$ be maximal so that $M_0\supset\{x_1, x_2\}$ and $d(x, y)=0$ or $\delta(M)$ for all $x, y\in M_0$. It is obvious that M_0 is finite. Let us assume $M_0=\{x_1, x_2, \cdots, x_n\}$. Since X is a convex metric space, we can define

$$y_1=W(x_1, x_2; 1/2),$$

$$y_2=W(x_3, y_1; 1/3),$$

$$\cdots,$$

$$y_{n-2}=W(x_{n-1}, y_{n-3}; 1/n-1),$$

$$y_{n-1}=W(x_n, y_{n-2}; 1/n)=u.$$

Since M is compact, we can find $y_0\in M$ such that

$$d(y_0, u)=\sup\{d(x, u):\ x\in M\}.$$

Now, by using the condition (*) of convex metric space, we obtain

$$d(y_0, u)\leqq\frac{1}{n}d(y_0, x_n)+\frac{n-1}{n}d(y_0, y_{n-2})$$

$$\leqq\frac{1}{n}d(y_0, x_n)+\frac{n-1}{n}\left(\frac{1}{n-1}d(y_0, x_{n-1})+\frac{n-2}{n-1}d(y_0, y_{n-3})\right)$$

$$=\frac{1}{n}d(y_0, x_n)+\frac{1}{n}d(y_0, x_{n-1})+\frac{n-2}{n}d(y_0, y_{n-3})$$

$$\cdots,$$

$$\leqq\frac{1}{n}\sum_{k=1}^{n}d(y_0, x_k)\leqq\delta(M).$$

Therefore if $d(y_0, u)=\delta(M)$, then we must have $d(y_0, x_k)=\delta(M)>0$ for all $k=1, 2, \cdots, n$, which means that $y_0 \in M_0$ by definition of M_0. But, then we must have $y_0=x_k$ for some $k=1, 2, \cdots, n$. This is a contradiction. Therefore

$$\sup\{d(x, u): x\in M\}=d(y_0, u)<\delta(M).$$

The above Proposition gives us the following definition. A convex metric space is said to have *normal structure* if for each closed bounded convex subset E of X which contains at least two points, there exists $x\in E$ which is not a diametral point of E. It is obvious that a compact convex metric space has normal structure. Every bounded closed convex subset of uniformly convex Banach space has normal structure, too. As an extension of the case in Banach space, we introduce a concept of strict convexity in a convex metric space. A convex metric space X is said to be *strictly convex* if for any $x, y\in X$ and λ $(0\leqq\lambda\leqq1)$, there exists a unique element $z\in X$ such that $\lambda d(x, y)=d(z, y)$ and $(1-\lambda)d(x, y)=d(x, z)$. We have seen from Proposition 3 that

$$d(x, y)=d(x, W(x, y; \lambda))+d(W(x, y; \lambda), y)$$

for each pair of elements x and y of a convex metric space and all real number λ $(0\leqq\lambda\leqq1)$. Futhermore, from

$$d(x, W(x, y; \lambda))\leqq\lambda d(x, x)+(1-\lambda)d(x, y)$$

$$=(1-\lambda)d(x, y)$$

and $d(W(x, y; \lambda), y)\leqq\lambda d(x, y)$, it is obvious that $W(x, y; \lambda)$ is an element of X such that satisfies

$$(1-\lambda)d(x, y)=d(x, W(x, y; \lambda)) \quad \text{and} \quad \lambda d(x, y)=d(W(x, y; \lambda), y).$$

3. Fixed point theorems.

Let X be a metric space and K be a subset of X. A mapping T of K into X is said to be *nonexpansive* (cf. Browder [1]) if for each pair of elements x and y of K, we have $d(Tx, Ty)\leqq d(x, y)$. Now, we will prove fixed point theorems for nonexpansive mappings in convex metric spaces. The following Theorem can be proved by a modification of the method of Kirk [6].

THEOREM 1. *Suppose that X has Property* (C). *Let K be a nonempty bounded closed convex subset of X with normal structure. If T is a nonexpansive mapping* *of K into itself, then T has a fixed point in K.*

Proof. Let Φ be a family of all nonempty closed and convex subsets of K, each of which is mapped into itself by T. By Property (C) and Zorn's lemma, Φ has a minimal element E. We show that E consists of a single point. Let $x \in E_c$. Then $d(Tx, Ty) \leqq d(x, y) \leqq R_x(E)$ for all $y \in E$, and hence $T(E)$ is contained in the spherical ball $\bar{S}(T(x), R(E))$. Since $T(E \cap \bar{S}) \subset E \cap \bar{S}$, the minimality of E implies $E \subset \bar{S}$. Hence $R_{T(x)}(E) \leqq R(E)$. Since $R(E) \leqq R_x(E)$ for all $x \in E$, $R_{T(x)}(E) = R(E)$. Hence $T(x) \in E_c$ and $T(E_c) \subset E_c$. By Proposition 4, $E_c \in \Phi$. If $z, w \in E_c$, then $d(z, w) \leqq R_z(E) = R(E)$. Hence, by normal structure, $\delta(E_c) \leqq R(E) < \delta(E)$. Since this contradicts the minimality of E, $\delta(E) = 0$ and E consists of a single point.

We prove the following:

THEOREM 2. *Suppose X being strictly convex with Property* (C). *Let K be a nonempty bounded closed convex subset of X with normal structure. If $\mathcal{F}$ is a commuting family of nonexpansive mappings of K into itself, then the family has a common fixed point in K.*

Proof. If T is a nonexpansive mapping in a strictly convex metric space, the set F of fixed points of T is a nonempty closed convex set. In fact, as $W(x, y; \lambda) \in K$ for $x, y \in F$ and λ $(0 \leqq \lambda \leqq 1)$, by Proposition 3

$$d(Tx, Ty) \leqq d(Tx, T(W(x, y; \lambda))) + d(T(W(x, y; \lambda)), Ty)$$
$$\leqq d(x, W(x, y; \lambda)) + d(W(x, y, \lambda); y) = d(x, y)$$

and hence, by strict convexity of the space, $T(W(x, y; \lambda)) = W(x, y; \lambda)$. This implies that F is convex. Let F_α be the fixed point sets of $T_\alpha \in \mathcal{F}$. If $u \in F_\alpha$, then for any α', $T_\alpha T_{\alpha'} u = T_{\alpha'} T_\alpha u = T_{\alpha'} u$ i.e., $T_{\alpha'} u$ lies in F_α and each $T_{\alpha'}$ maps F_α into itself. If we are given a finite sequence $\alpha_1, \alpha_2, \cdots, \alpha_m$, and consider T_{α_m} as a nonexpansive mapping of $F_{\alpha_1} \cap F_{\alpha_2} \cap \cdots \cap F_{\alpha_m}$ into itself, it follows from Theorem 1 that $\cap_{k=1}^{m} F_{\alpha_k} \neq \phi$. Hence by Property (C), the family $\{F_\alpha\}$ has a nonempty intersection, but this consists of the common fixed point.

In the general case, the set of fixed points of nonexpansive mapping is not necessary convex. However, we will prove the following Theorem by assuming compactness. Before the proof of Theorem, we define the following: Let K be a compact convex metric space. Then a family $\mathcal{F}$ of nonexpansive mappings T of K into itself is said to have *invariant property* in K if for any compact convex subset $E \subset K$ such that $TE \subset E$ for each $T \in \mathcal{F}$, there exists a compact subset $M \subset E$ such that $TM = M$ for each $T \in \mathcal{F}$.

THEOREM 3. *Let K be a compact convex metric space. If $\mathcal{F}$ is a family of nonexpansive mappings with invariant property in K, then the family $\mathcal{F}$ has a common fixed point.*

Proof. By using Zorn's lemma, we can find a minimal nonempty convex compact set $E \subset K$ such that E is an invariant under each $T \in \mathfrak{F}$. If E consists of a single point, then Theorem is proved. By hypothesis, there exists a compact subset M of E such that $M = \{T(x): x \in M\}$ for each $T \in \mathfrak{F}$. If M contains more than one point, by Proposition 5 there exists an element u in the least convex set containing M such that

$$\rho = \sup\{d(u, x): x \in M\} < \delta(M),$$

where $\delta(M)$ is the diameter of M. Let us define

$$E_0 = \bigcap_{x \in M} \{y \in E: d(x, y) \leqq \rho\},$$

then E_0 is the nonempty closed convex proper subset of E invariant under each T in $\mathfrak{F}$. This is a contradiction to the minimality of E.

De Marr [4] showed that a commutative family of nonexpansive mappings of K into itself has invariant property in K. The following theorem asserts that this is true even if one considers a left amenable semigroup (cf. Day [2]) of nonexpansive mappings.

THEOREM 4. *Let K be a compact convex metric space. If $\mathfrak{F}$ is a left amenable semigroup of nonexpansive mappings T of K into K, then the family $\mathfrak{F}$ has invariant property in K.*

Proof. Let E be a compact convex subset of K such that E is invariant under each T in $\mathfrak{F}$. By using Zorn's lemma, we can find a minimal nonempty compact set $M \subset E$ such that M is invariant under each T in $\mathfrak{F}$. Let $C(M)$ be the space of bounded continuous real valued functions on M and $C(M)^*$ be the dual space of $C(M)$ and

$$K[C(M)] = \{L \in C(M)^*: L(e) = ||L|| = 1\}$$

where e denotes the constant 1 function on M. Since the semigroup of restrictions of mappings T to M is left amenable, by [3] there exists an element $L \in K[C(M)]$ such that $L(U_T f) = L(f)$ for all $f \in C(M)$ and $T \in \mathfrak{F}$ where $U_T f$ denotes an element in $C(M)$ such that $(U_T f)(x) = f(Tx)$ for each x in M. The Riesz's theorem asserts that to the element L, there corresponds a unique probability measure m on M such that

$$L(f) = \int_M f\,dm$$

for each f in $C(M)$ and m is invariant under each $T \in \mathfrak{F}$. Therefore it is obvious that $m(TM) = m(T^{-1}TM) = m(M) = 1$. Let $F(\subset M)$ be the support of the measure m. Then F, TF and $T^{-1}F$ are closed and contained in M. Since $1 = m(F) = m(T^{-1}F)$, it is obvious that F is contained in $T^{-1}F$ for each T in $\mathfrak{F}$. Therefore F is invariant under each T in $\mathfrak{F}$. This implies $TM = M$ for each T in $\mathfrak{F}$.

4. Application.

Let K be a compact convex subset of a Banach space and X be the set of all nonexpansive mappings of K into itself. Then for each pair of elements A and B of X, define a metric d by $d(A,B)=\sup\{\|Ax-Bx\|: x\in K\}$ whence X is a metric space with d. Define a mapping W from $X\times X\times[0,1]$ to X by

$$W(A, B; \lambda)(x)=\lambda Ax+(1-\lambda)Bx$$

for $x\in K$ and $\lambda\in[0,1]$.

LEMMA. *The set X is a compact convex metric space with respect to metric d and the convex structure W.*

Proof. For $A, B, C\in X$ and real number $\lambda\in[0,1]$,

$$\begin{aligned} d(A, W(B,C;\lambda)) &= \sup\{\|Ax-W(B,C;\lambda)x\|: x\in K\} \\ &\leqq \sup\{\lambda\cdot\|Ax-Bx\|+(1-\lambda)\|Ax-Cx\|: x\in K\} \\ &\leqq \lambda d(A,B)+(1-\lambda)d(A,C) \end{aligned}$$

and hence W is a convex structure in X. We will show that the set X with metric d is compact. If $\varepsilon>0$ is given, a subset E of K is called an *ε-net* if E is finite and $K=\cup_{a\in E}S(a,\varepsilon)$. Let $\{U_n\}$ be a sequence of nonexpansive mappings of X. Then we show that there exists a subsequence $\{U_k\}$ of $\{U_n\}$ such that U_k converges to a point of X. Let N_n be a $1/n$-net of K and $N=\cup\{N_n: n=1,2,\cdots\}$. Then it is obvious that there exists a subsequence $\{U_k\}$ of $\{U_n\}$ such that U_kx for every x in N converges to a point of K. We show that U_kz converges to a point for every z of K. Let z any point of K and ε be any given positive number, then there exists $x\in N$ such that $\|z-x\|\leqq\varepsilon/3$. Now, since U_kx converges, there exists a positive integer k_0 such that

$$\begin{aligned} \|U_{k_1}z-U_{k_2}z\| &\leqq \|U_{k_1}z-U_{k_1}x\|+\|U_{k_1}x-U_{k_2}x\|+\|U_{k_2}x-U_{k_2}z\| \\ &\leqq \|z-x\|+\|U_{k_1}x-U_{k_2}x\|+\|x-z\| \\ &\leqq \varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon \end{aligned}$$

if $k_1, k_2>k_0$. Hence U_kz converges to a point of K. Let us define $Uz=\lim U_kz$ for every z of K. Then it is obvious that U is a nonexpansive mapping. We will show that the convergence is uniform. Let ε be any positive number and choose n_0 such that $1/n_0\leqq\varepsilon/3<\varepsilon$, then N_{n_0} contains a point x such that $\|x-z\|\leqq\varepsilon/3$. Now k_0 exists such that $\|U_{k_1}x-U_{k_2}x\|\leqq\varepsilon/3$ for all x in N_{n_0} when $k_1, k_2>k_0$. Thus k_0 is independent of z and $\|U_{k_1}z-U_{k_2}z\|\leqq\varepsilon$ when $k_1, k_2>k_0$. This shows uniformity of the convergence. Therefore, there exists a subsequence $\{U_k\}$ of $\{U_n\}$ such that U_k converges. This completes the proof.

From Theorem 3 and Lemma, we obtain the following:

THEOREM 5. *Let K be a compact convex subset of a Banach space and X be the compact convex metric space of all nonexpansive mappings of K into itself and $\mathcal{F}$ be a family of nonexpansive mappings of X into X. If $\mathcal{F}$ has invariant property in X, then $\mathcal{F}$ has a common fixed nonexpansive mapping in X.*

REMARK. Let T be an element of X. Then T defines a nonexpansive mapping of X into itself. Therefore if $\mathcal{F}\subset X$ has invariant property in X, there exists $U\in X$ such that $TU=U$ for each T in $\mathcal{F}$. Fixed points of U are common fixed points of the family $\mathcal{F}\subset X$.

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