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ON THE GROWTH OF ALGEBROID FUNCTIONS WITH SEVERAL DEFICIENCIES

BY MITSURU OZAWA

In this paper we shall generalize some results on the growth of meromorphic functions with several dedicient values to n-valued algebroid functions. Edrei-Fuchs [1] had given several striking results on the growth of meromorphic functions.

Let $y(z)$ be an n-valued transcendental algebroid function defined by an irreducible equation

$$F(z, y) \equiv A_n y^n + A_{n-1} y^{n-1} + \cdots + A_1 y + A_0 = 0,$$

where $A_n, \cdots, A_0$ are entire functions. Here we assume that there is no common zero for all A_j, $j=0, \cdots, n$. If $A_n \equiv 1$, then $y(z)$ is called an n-valued entire algebroid function.

THEOREM 1. *Let $y(z)$ be an n-valued transcendental entire algebroid function with n finite deficient values a_j, $j=1, \cdots, n$. Then the lower order of $y(z)$ is positive.*

THEOREM 2. *Let $y(z)$ be an n-valued transcendental algebroid function. Assume that there is a constant α such that*

$$\alpha m(r, A) \leqq n\mu(r, A), \qquad 0 < \alpha \leqq 1.$$

Assume that there are $n+1$ deficient values a_j, $j=1, \cdots, n+1$ of y. Then the lower order of y is positive.

Here $A = \max(|A_n|, |A_{n-1}|, \cdots, |A_1|, |A_0|)$ and

$$n\mu(r, A) = \frac{1}{2\pi} \int_0^{2\pi} \log A \, d\theta, \qquad m(r, A) = \frac{1}{2\pi} \int_0^{2\pi} \overset{+}{\log} A \, d\theta.$$

THEOREM 3. *Let $y(z)$ be an n-valued transcendental entire algebroid function of finite order λ. Assume that $y(z) = a_j$, $j=1, \cdots, n$ have their roots only on the negative real axis. Then the lower order μ of $y(z)$ satisfies $\mu \leqq \lambda \leqq \mu + 1$.*

Theorems 1 and 2 are generalizations of Edrei-Fuchs' theorem 4, Corollary 4.1 in [1]. Theorem 3 is an extension of Shea's theorem 2 in [3].

The following fact gives several extensions of Edrei-Fuchs' theorems to entire algebroid functions: For an entire algebroid function y

$$\delta(a, y) \geqq \delta(0, F(z, a)).$$

This is almost trivial. By Valiron's theorem [5]

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$$|T(r, y)-\mu(r, A)|=O(1).$$

Further we have

$$n\mu(r, A)=m(r, A)$$

for y and hence

$$n\mu(r, A)=m(r, A)\geqq m(r, F(z, a)).$$

Therefore

$$\begin{aligned}1-\delta(a, y) &= \varlimsup_{r\to\infty}\frac{N(r; a, y)}{T(r, y)}\\ &\leqq \varlimsup_{r\to\infty}\frac{N(r; 0, F(z, a))}{m(r, F(z, a))}\ \varlimsup_{r\to\infty}\frac{m(r, F(z, a))}{m(r, A)}\\ &\leqq 1-\delta(0, F(z, a)).\end{aligned}$$

§1. **Proof of Theorem 1.** Let $F(z, a_j)$ denote by g_j. Edrei-Fuchs' argument does work in our case. They proved the following inequality: For $z=re^{i\theta}$ and $R=\sigma r$, $\sigma>1$

$$\text{(A)}\qquad \log|g_j(z)|\leqq N(R; 0, g_j)+\frac{2}{\sigma-1}\left(m(R, g_j)+m\left(R, \frac{1}{g_j}\right)\right)+O(\log r).$$

Starting from (A) we have

$$\log|g_j(z)|\leqq N(R; 0, g_j)+\frac{4}{\sigma-1}m(R, g_j)+O(\log r).$$

Hence

$$\begin{aligned}&\max_{1\leqq j\leqq n}(\log|g_j(z)|, 0)\\ &\leqq\frac{4}{\sigma-1}\max_{1\leqq j\leqq n}m(R, g_j)+\max_{1\leqq j\leqq n}N(R; 0, g_j)+O(\log r).\end{aligned}$$

Let g be

$$\max_{1\leqq j\leqq n}(|g_j(z)|, 1).$$

Then

$$\begin{aligned}\log g &\leqq c+\max_{0\leqq j\leqq n-1}(\log|A_j|, 0)=c+\log A\\ &\leqq d+\log g.\end{aligned}$$

Hence

$$m(r, g)\leqq n\mu(r, A)+O(1)\leqq m(r, g)+O(1).$$

Further by Valiron's theorem

$$T(r, y)-\mu(r, A)=O(1).$$

Hence

$$m(r, g) \leqq \frac{4}{\sigma-1} \max_{1 \leqq j \leqq n} m(R, g_j) + \max_{1 \leqq j \leqq n} N(R; 0, g_j) + O(\log r).$$

Since

$$m(R, g_j) \leqq m(R, g),$$

$$m(r, g) \leqq \frac{4}{\sigma-1} m(R, g) + \max_{1 \leqq j \leqq n} N(R; 0, g_j) + O(\log r).$$

Let γ be the maximum of $1-\delta\ (a_j, y)$, $j=1, \cdots, n$. Let c', c satisfy

$$\gamma < c' < c < 1.$$

Then for $r \geqq r_0$

$$\frac{1}{n} N(R; 0, g_j) < c' T(R, y)$$

$$\leqq c' \frac{1}{n} m(R, g) + O(1).$$

Thus for $r \geqq r_1 \geqq r_0$

$$m(r, g) \leqq \left(\frac{4}{\sigma-1} + c' + \frac{A \log r}{m(R, g)} \right) m(R, g)$$

$$< \left(\frac{4}{\sigma-1} + c \right) m(R, g).$$

Hence

$$\frac{m(\sigma r, g)}{m(r, g)} > \frac{1}{\dfrac{4}{\sigma-1} + c}$$

Here we put

$$\sigma = 1 + \frac{4}{c(1-c)}.$$

Then

$$\frac{m(\sigma r, g)}{m(r, g)} > \frac{1}{c(2-c)}.$$

This implies

$$\frac{m(\sigma^n r^*, g)}{m(r^*, g)} = \prod_{k=1}^{n} \frac{m(\sigma^k r^*, g)}{m(\sigma^{k-1} r^*, g)} > \left\{ \frac{1}{c(2-c)} \right\}^n.$$

Let r satisfy $\sigma^n r^* \leqq r < \sigma^{n+1} r^*$. Then

$$\frac{\log m(r, g)}{\log r} > \frac{\log m(\sigma^n r^*, g)}{\log \rho^{n+1} r^*}$$

$$> \frac{n \log \dfrac{1}{c(2-c)} + \log m(r^*, g)}{(n+1) \log \sigma + \log r^*},$$

$$\mu = \varlimsup_{r\to\infty} \frac{\log T(r, y)}{\log r} = \varlimsup_{r\to\infty} \frac{\log m(r, g)}{\log r}$$

$$\geqq \frac{\log \dfrac{1}{c(2-c)}}{\log \sigma}.$$

Here by letting $c \to \gamma$ Theorem 1 follows.

§ 2. **Proof of Theorem 2.** For a general n-valued algebroid function it appears a new difficulty. As in § 1 we have

$$\max_{1\leqq j\leqq n+1} \log |g_j(z)| \leqq \max_{1\leqq j\leqq n+1} N(R; 0, g_j) + \frac{4}{\sigma-1} \max_{1\leqq j\leqq n+1} m(R, g_j) + O(\log r).$$

Hence for $r \geqq r_0$

$$n\mu(r, A) \leqq n\mu(r, g) + O(1)$$
$$\leqq c' n\mu(R, A) + \frac{4}{\sigma-1} m(R, g) + O(\log r),$$

where

$$A = \max_{0\leqq j\leqq n} |A_j|, \qquad g = \max_{1\leqq j\leqq n+1} |g_j|.$$

Since

$$n\mu(r, A) \leqq m(r, A) \leqq n\mu(r, A)/\alpha,$$
$$m(r, g) \leqq m(r, A) + O(1) \leqq n\mu(r, A)/\alpha + O(1).$$

Hence

$$n\mu(r, A) \leqq c' n\mu(R, A) + \frac{4}{\sigma-1} \frac{1}{\alpha} n\mu(R, A) + O(\log r).$$

Now the same process as in § 1 does work. Hence we have the desired result.

By Tsuzuki's example [4] there is an algebroid function satisfying

$$\varliminf_{r\to\infty} \frac{n\mu(r, A)}{m(r, A)} = 0.$$

In this case we cannot carry out our discussion as in the above. However it is still conjectured that Theorem 2 without the assumption $n\mu(r, A) \geqq \alpha m(r, A)$, $0 < \alpha \leqq 1$ does hold.

In our previous paper [2] we gave several sufficient conditions for $\alpha m(r, A) \leqq n\mu(r, A)$, $0 < \alpha \leqq 1$.

§ 3. **Proof of Theorem 3.** If $\lambda \leqq 1$, then the result remains true trivially. Assume that $\lambda > 1$. Let

$$F(z, a_j)=e^{Q_j(z)}g_j(z),$$

where $g_j(z)$ is the canonical product formed by the zeros of $F(z, a_j)$. Let q_j be the genus of $F(z, a_j)$ and d_j the degree of $Q_j(z)$. Put $q=\max q_j$, $d=\max d_j$. Let s_j be the genus of $g_j(z)$. Put $s=\max s_j$. Firstly we have

$$\lambda=\overline{\lim_{r\to\infty}}\frac{\log m(r, y)}{\log r}=\overline{\lim_{r\to\infty}}\frac{\log m(r, \max|F(z, a_j)|)}{\log r}$$

$$\leqq\overline{\lim_{r\to\infty}}\frac{\log \sum_{j=1}^{n} m(r, F(z, a_j))}{\log r}$$

$$\leqq\max\overline{\lim_{r\to\infty}}\frac{\log m(r, F(z, a_j))}{\log r}\equiv\max\lambda_j.$$

Further evidently $\lambda_j\leqq\lambda$. Hence $\max\lambda_j=\lambda$. By the well-known theorem $q_j\leqq\lambda_j\leqq q_j+1$. Hence $q\leqq\lambda\leqq q+1$. By the definition of genus

$$q_j=\max(d_j, s_j).$$

Thus

$$q=\max(d, s)$$

Assume that $d\leqq s$. Hence $q=s$. Since $\lambda>1$ and q is an integer, $s=1$. Then there is an index k such that $s_k=s$. Applying Edrei-Fuchs' argument in [1], Theorem 2, we have

$$\lim_{r\to\infty}\frac{m(r, F(z, a_k))}{r^{s_k}}=\infty,$$

$$\lim_{r\to\infty}\frac{m(r, F(z, a_k))}{r^{s_k+1}}=0.$$

This implies that, putting μ_k the lower order of $m(r, F(z, a_k))$,

$$\lambda_k\leqq s_k+1, \qquad \mu_k\geqq s_k.$$

Evidently

$$\mu\geqq\max\mu_j\geqq\mu_k\geqq s_k=s.$$

Hence

$$q=s\leqq\mu\leqq\lambda\leqq q+1.$$

Therefore

$$\mu\leqq\lambda\leqq\mu+1.$$

Next assume that $d>s$. Then $q=d>s\geqq 0$. Evidently $\lambda=d$. Since s and d are integers, $d=s+1$. However by the well-known property of canonical product

$$m(r, g_j)=o(r^{s+1}), \qquad j=1, \cdots, n.$$

On the other hand

$$m(r, e^{Q_j})=\frac{|A_j|}{\pi}r^{d_j}(1+o(r)),$$

where $A_j z^{d_j}+\cdots=Q_j(z)$. Thus for some k

$$m(r, e^{Q_k})=cr^d(1+o(r)).$$

Thus

$$m(r, g_j)=o(m(r, e^{Q_k})).$$

Therefore

$$\mu=d=\lambda.$$

In this case Theorem 3 follows.

Shea's formulation is somewhat different. However his formulation is equivalent to ours. Further theorem 1 implies the positivity of μ.

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DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.