KŌDAI MATH. SEM. REP.
21 (1969), 463—471

INTEGRAL FORMULAS FOR SUBMANIFOLDS OF CODIMENSION 2 AND THEIR APPLICATIONS

BY KENTARO YANO AND MASAFUMI OKUMURA

§ 1. Introduction

Various integral formulas for hypersurfaces of a Riemannian manifold have been found and applied to the study of closed hypersurfaces with constant mean curvature.

Integral formulas for submanifolds of codimension greater than 1 was first obtained by Okumura [6] for the case of submanifolds of codimension 2 of an odd dimensional sphere. He made use of the natural contact structure of the odd dimensional sphere. Integral formulas for general submanifolds of a Riemannian manifold have been obtained by Katsurada [1], [2], [3], Kôjyô [2], Nagai [3], [4], and Yano [9].

In a recent paper [7], Okumura obtained integral formulas for a submanifold of codimension 2, invariant under the curvature transformation, of a Riemannian manifold admitting an infinitesimal conformal transformation and used them to prove that, under certain conditions, the submanifold in question is totally umbilical.

In the present paper, we study a problem similar to that treated in [7]. In [7], the ambient Riemannian manifold was supposed to admit an infinitesimal conformal transformation, but in this paper, we assume instead that there exists a vector field along the submanifold whose covariant differential is proportional to the displacement. We do not assume that the submanifold is invariant under the curvature transformation but instead we put a condition on the integral of a quantity depending on the curvature.

We moreover study the case in which the ambient Riemannian manifold admits a scalar function v such that $\nabla_j \nabla_i v = f(v) g_{ji}$ and prove that the submanifold satisfying certain conditions is isometric to a sphere by a method used in [8].

§ 2. Submanifolds of codimension 2.

We consider an $(n+2)$-dimensional orientable Riemannian manifold M^{n+2} of differentiability class C^∞ covered by a system of coordinate neighborhoods $\{U; x^h\}$, where and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, n, n+1, n+2\}$. We denote by g_{ji}, $\{ {}_j{}^h{}_i \}$, ∇_i, and $K_{kji}{}^h$, the metric tensor, the Christoffel symbols formed

Received June 26, 1969.

with g_{ji}, the operator of covariant differentiation with respect to $\{{}_j{}^h{}_i\}$, and the curvature tensor of M^{n+2} respectively.

We consider an n-dimensional orientable submanifold M^n differentiably imbedded in M^{n+2} and denote by

$$x^h = x^h(u^a) \tag{2.1}$$

its parametric representation, where and in the sequel the indices a, b, c, d, e run over the range $\{1, 2, \cdots, n\}$. If we put

$$B_b{}^h = \partial_b x^h, \qquad (\partial_b = \partial/\partial u^b) \tag{2.2}$$

then $B_b{}^h$, for each fixed b, is a vector field tangent to M^n and $B_b{}^h$ are linearly independent. A Riemannian metric

$$g_{cb} = g_{ji} B_c{}^j B_b{}^i \tag{2.3}$$

is induced on M^n. We denote by $\{{}_c{}^a{}_b\}$, ∇_c and $K_{dcb}{}^a$, the Christoffel symbols formed with g_{cb}, the operator of covariant differentiation with respect to $\{{}_c{}^a{}_b\}$ and the curvature tensor of M^n respectively.

Now, the so-called van der Waerden-Bortolotti covariant derivative of $B_b{}^h$ is given by

$$\nabla_c B_b{}^h = \partial_c B_b{}^h + \left\{ {h \atop j\ i} \right\} B_c{}^j B_b{}^i - B_a{}^h \left\{ {a \atop c\ b} \right\}. \tag{2.4}$$

Since $\nabla_c B_b{}^h$, as vectors of M^{n+2}, are normal to M^n, the vector field

$$H^h = \frac{1}{n} g^{cb} \nabla_c B_b{}^h \tag{2.5}$$

is normal to the submanifold M^n and is called the mean curvature vector of M^n.

We assume throughout the paper that the mean curvature vector never vanishes and take the first unit normal C^h to M^n in the direction of the mean curvature vector. We take the second unit normal D^h in such a way that $B_1{}^h, B_2{}^h, \cdots, B_n{}^h, C^h$ and D^h give the positive orientation of M^{n+2}.

Then the equations of Gauss and those of Weingarten are written as

$$\nabla_c B_b{}^h = h_{cb} C^h + k_{cb} D^h \tag{2.6}$$

and

$$\begin{cases} \nabla_c C^h = -h_c{}^a B_a{}^h + l_c D^h, \\ \nabla_c D^h = -k_c{}^a B_a{}^h - l_c C^h \end{cases} \tag{2.7}$$

respectively, where h_{cb} and k_{cb} are the second fundamental tensors with respect to the normals C^h and D^h respectively and l_c the third fundamental tensor, $h_c{}^a$ and $k_c{}^a$ being defined by

$$h_c{}^a = h_{cb} g^{ba}, \qquad k_c{}^a = k_{cb} g^{ba}.$$

The normals C^h and D^h being chosen intrinsically, the quantities h, k and l are all intrinsic quantities of M^n.

Since $(1/n) g^{cb} \nabla_c B_b{}^h$ is in the direction of C^h, we see from (2.6) that

$$g^{cb} k_{cb} = k_c{}^c = 0. \tag{2.8}$$

Now the equations of Gauss, those of Codazzi and those of Ricci are respectively written as

$$K_{kjih} B_d{}^k B_c{}^j B_b{}^i B_a{}^h = K_{dcba} - h_{da} h_{cb} + h_{ca} h_{db} - k_{da} k_{cb} + k_{ca} k_{db}, \tag{2.9}$$

$$\begin{cases} K_{kjih} B_d{}^k B_c{}^j B_b{}^i C^h = \nabla_d h_{cb} - \nabla_c h_{db} - l_d k_{cb} + l_c k_{db}, \\ K_{kjih} B_d{}^k B_c{}^j B_b{}^i D^h = \nabla_d k_{cb} - \nabla_c k_{db} + l_d h_{cb} - l_c h_{db}, \end{cases} \tag{2.10}$$

$$K_{kjih} B_d{}^k B_c{}^j C^i D^h = \nabla_d l_c - \nabla_c l_d + h_{da} k_c{}^a - h_{ca} k_d{}^a. \tag{2.11}$$

§ 3. Vector fields along the submanifold of codimension 2.

Take a normal vector field

$$V^h = \lambda C^h + \mu D^h. \tag{3.1}$$

Then, using equations of Weingarten, we have

$$\nabla_c V^h = (-\lambda h_c{}^a - \mu k_c{}^a) B_a{}^h + (\partial_c \lambda - l_c \mu) C^h + (\partial_c \mu + l_c \lambda) D^h,$$

and consequently the connection induced in the normal bundle from the Riemannian connection of M^{n+2} is given by

$$\nabla_c{}' \lambda = \partial_c \lambda - l_c \mu, \qquad \nabla_c{}' \mu = \partial_c \mu + l_c \lambda. \tag{3.2}$$

Thus in order that a normal vector field $\lambda C^h + \mu D^h$ be parallel with respect to the connection induced in the normal bundle, it is necessary and sufficient that

$$\partial_c \lambda - l_c \mu = 0, \qquad \partial_c \mu + l_c \lambda = 0. \tag{3.3}$$

These equations show that

$$\lambda^2 + \mu^2 = \text{constant},$$

that is, a normal vector field parallel with respect to the connection induced in the normal bundle is of constant length.

If $\lambda C^h (\neq 0)$ is parallel with respect to the connection induced in the normal bundle, then we have

$$\lambda = \text{const. and } l_c = 0,$$

and conversely. If $\mu D^h(\neq 0)$ is parallel, we have

$$\mu=\text{const. and } l_c=0,$$

and conversely. Thus, in order that the mean curvature vector $(1/n)g^{cb}\nabla_c B_b{}^h(\neq 0)$ be parallel with respect to the connection induced in the normal bundle, it is necessary and sufficient that

$$h_a{}^a=\text{const.}\neq 0, \qquad l_c=0.$$

Take next a vector field X^h defined along the submanifold M^n and assume that the covariant differential of this vector field is always proportional to the displacement along the manifold. For such a vector field we have

$$\nabla_b X^h=fB_b{}^h, \tag{3.4}$$

f being a scalar function of M^n.

If we put

$$X^h=z^a B_a{}^h+\alpha C^h+\beta D^h, \tag{3.5}$$

we have

$$\begin{aligned}\nabla_b X^h=&(\nabla_b z^a-\alpha h_b{}^a-\beta k_b{}^a)B_a{}^h\\&+(\partial_b\alpha-l_b\beta+h_{ba}z^a)C^h\\&+(\partial_b\beta+l_b\alpha+k_{ba}z^a)D^h.\end{aligned}$$

Thus if we assume that the covariant differential of X^h is proportional to the displacement along M^n, then we have

$$\nabla_b z^a=f\delta_b^a+\alpha h_b{}^a+\beta k_b{}^a \tag{3.6}$$

or

$$\nabla_b z_a=fg_{ba}+\alpha h_{ba}+\beta k_{ba} \tag{3.7}$$

and

$$\begin{cases}\partial_b\alpha-l_b\beta+h_{ba}z^a=0,\\ \partial_b\beta+l_b\alpha+k_{ba}z^a=0.\end{cases} \tag{3.8}$$

§ 4. Integral formulas for a closed submanifold of codimension 2.

We consider an $(n+2)$-dimensional Riemannian manifold M^{n+2} and a closed orientable submanifold M^n of codimension 2 imbedded in it. We assume that there exists a vector field

(4. 1) $$X^h = z^a B_a{}^h + \alpha C^h + \beta D^h$$

along M^n whose covariant differential along M^n is always proportional to the displacement:

(4. 2) $$\nabla_c X^h = f B_c{}^h.$$

Then we have

(4. 3) $$\nabla_c z_b = f g_{cb} + \alpha h_{cb} + \beta k_{cb},$$

from which

$$g^{cb} \nabla_c z_b = nf + \alpha h_a{}^a.$$

Thus, integrating over M^n, we find

(4. 4) $$\int_{M^n} (nf + \alpha h_a{}^a) dV = 0,$$

where dV denotes the volume element of M^n.

We now compute $\nabla_a(h_b{}^a z^b)$:

$$\begin{aligned} \nabla_a(h_b{}^a z^b) &= (\nabla_a h_b{}^a) z^b + h^{ba} \nabla_b z_a \\ &= (\nabla_a h_b{}^a) z^b + h^{ba}(f g_{ba} + \alpha h_{ba} + \beta k_{ba}) \\ &= (\nabla_a h_b{}^a) z^b + f h_a{}^a + \alpha h^{ba} h_{ba} + \beta h^{ba} k_{ba}. \end{aligned}$$

But, from the first of equations (2. 10) of Codazzi, we have

$$K_{kjih} B_d{}^k B^{ji} C^h = \nabla_d h_a{}^a - \nabla_a h_d{}^a + l_a k_d{}^a,$$

where

$$B^{ji} = g^{cb} B_c{}^j B_b{}^i,$$

and consequently we have

$$\begin{aligned} \nabla_a(h_b{}^a z^b) = &-K_{kjih} B_d{}^k z^d B^{ji} C^h + z^d \nabla_d h_a{}^a + l_a k_d{}^a z^d \\ &+ f h_a{}^a + \alpha h^{ba} h_{ba} + \beta h^{ba} k_{ba}. \end{aligned}$$

Thus, integrating over M^n, we obtain

(4. 5) $$\begin{aligned} &\int_{M^n} K_{kjih} B_d{}^k z^d B^{ji} C^h dV \\ &= \int_{M^n} (z^d \nabla_d h_a{}^a + l_a k_d{}^a z^d + f h_a{}^a + \alpha h^{ba} h_{ba} + \beta h^{ba} k_{ba}) dV. \end{aligned}$$

§ 5. Closed submanifolds with mean curvature vector parallel with respect to the connection induced in the normal bundle.

We consider a closed orientable submanifold M^n of codimension 2 of an $(n+2)$-dimensional Riemannian manifold M^{n+2} and assume that M^n admits a vector field X^h whose covariant differential along M^n is always proportional to the displacement:

$$\nabla_c X^h = f B_c^h \tag{5.1}$$

and that the mean curvature vector $(1/n)g^{cb}\nabla_c B_b^h (\neq 0)$ is parallel with respect to the connection induced in the normal bundle:

$$h_a{}^a = \text{const.} \neq 0, \qquad l_c = 0. \tag{5.2}$$

Then we have first of all

$$\int_{M^n} (nf + \alpha h_a{}^a) dV = 0. \tag{5.3}$$

We next have from (4.5)

$$\begin{aligned} &\int_{M^n} K_{kjih} B_d{}^k z^d B^{ji} C^h dV \\ &= \int_{M^n} (f h_a{}^a + \alpha h^{ba} h_{ba} + \beta h^{ba} k_{ba}) dV. \end{aligned} \tag{5.4}$$

Now, forming (5.4)$-$(5.3)$\times(1/n)h_a{}^a$, we find

$$\begin{aligned} &\int_{M^n} K_{kjih} B_d{}^k z^d B^{ji} C^h dV \\ &= \int_{M^n} \left[\alpha \left(h^{ba} h_{ba} - \frac{1}{n} h_b{}^b h_a{}^a \right) + \beta h^{ba} k_{ba} \right] dV, \end{aligned}$$

or

$$\begin{aligned} &\int_{M^n} K_{kjih} B_d{}^k z^d B^{ji} C^h dV \\ &= \int_{M^n} \left[\alpha \left\{ \left(h^{ba} - \frac{1}{n} h_e{}^e g^{ba} \right) \left(h_{ba} - \frac{1}{n} h_d{}^d g_{ba} \right) + k^{ba} k_{ba} \right\} + (h^{ba} k_{ba} \beta - k^{ba} k_{ba} \alpha) \right] dV. \end{aligned} \tag{5.5}$$

We denote by X'^h and X''^h the tangential part and normal part of X^h respectively.

Suppose that

$$\int_{M^n} K_{kjih} X'^k B^{ji} C^h dV \leqq 0,$$

$$\alpha > 0,$$

$$h^{ba} k_{ba} \beta - k^{ba} k_{ba} \alpha \geqq 0,$$

that is, the vector

$$Y^h = h^{ba} k_{ba} C^h + k^{ba} k_{ba} D^h$$

vanishes or this vector and

$$X''^h = \alpha C^h + \beta D^h$$

have positive orientation in the normal bundle, or

$$\int_{M^n} K_{kjih} X'^k B^{ji} C^h dV \geqq 0,$$

$$\alpha < 0,$$

$$h^{ba} k_{ba} \beta - k^{ba} k_{ba} \alpha \leqq 0,$$

that is, the vector Y^h vanishes or Y^h and X''^h have negative orientation in the normal bundle, then we have

$$h_{cb} - \frac{1}{n} h_d{}^d g_{cb} = 0, \qquad k_{cb} = 0,$$

that is, the submanifold under consideration is totally umbilical. Thus we have

THEOREM 5. 1. *Let M^n be a closed orientable submanifold of codimension 2 of an $(n+2)$-dimensional Riemannian manifold M^{n+2} and assume that M^n admits a vector field X^h whose covariant differential along M^n is always proportional to the displacement. If*

(i) *the mean curvature vector field* ($\neq 0$) *is parallel with respect to the connection induced in the normal bundle,*

(ii) $\int_{M^n} K_{kjih} X'^k B^{ji} C^h dV \leqq 0 \quad (\geqq 0)$,

(iii) $\alpha > 0 \quad (< 0)$,

(iv) $Y^h = 0$ *or* Y^h *and* X''^h *have positive* (*negative*) *orientation in the normal bundle,*

then the submanifold is totally umbilical.

If the submanifold is invariant under the curvature transformation, then we have

$$K_{kjih} X'^k B^{ji} C^h = 0$$

and consequently the second condition of Theorem 5.1 is automatically satisfied.

If the ambient Riemannian manifold M^{n+2} admits a scalar function v such that

$$\nabla_j \nabla_i v = f(v) g_{ji}, \tag{5. 6}$$

then we have

$$\nabla_c v^h = f(v) B_c{}^h \tag{5.7}$$

along any submanifold, where we have put

$$v^h = v_i g^{ih}, \qquad v_i = \nabla_i v.$$

This equation shows that the vector field v^h defined along M^n has covariant differential always proportional to the displacement along M^n.

Thus, under 4 conditions of Theorem 5.1, we have

$$h_{cb} = \lambda g_{cb}, \qquad k_{cb} = 0, \qquad l_c = 0, \tag{5.8}$$

λ being a constant different from zero, and consequently, (3.7) and (3.8),

$$\nabla_b z_a = (f + \alpha\lambda) g_{ba} \tag{5.9}$$

and

$$\partial_b \alpha + \lambda z_b = 0. \tag{5.10}$$

But

$$z_b = B_b{}^i v_i = \partial_b v$$

and consequently, we have from (3.7) and (5.10),

$$\alpha + \lambda v = c \text{ (constant)}.$$

Thus, from (5.9),

$$\nabla_b \nabla_a v = (f + c\lambda - \lambda^2 v) g_{ba}. \tag{5.10'}$$

We examine two cases,

(I) $f = kv$, $k = \text{const.} \neq 0$, $v \neq \text{const.}$ along M^n.

In this case, we have

$$\nabla_b \nabla_a v = [-(\lambda^2 - k) v + \lambda c] g_{ba}. \tag{5.11}$$

Here, $\lambda^2 - k \neq 0$, because if $\lambda^2 - k = 0$, then we have $\nabla_b \nabla_a v = \lambda c g_{ba}$, from which $g^{ba} \nabla_b \nabla_a v = n\lambda c$, which, the submanifold being closed, is impossible unless $v = \text{constant}$ on M^n.

Thus, $\lambda^2 - k$ being different from zero, we have from (5.11)

$$\nabla_b \nabla_a \left(v - \frac{\lambda c}{\lambda^2 - k} \right) = -(\lambda^2 - k) \left(v - \frac{\lambda c}{\lambda^2 - k} \right) g_{ba}, \tag{5.12}$$

from which

$$g^{ba} \nabla_b \nabla_a \left(v - \frac{\lambda c}{\lambda^2 - k} \right) = -n(\lambda^2 - k) \left(v - \frac{\lambda c}{\lambda^2 - k} \right)$$

which shows that $\lambda^2-k>0$. Thus, by a famous theorem of Obata [5], the submanifold is isometric to a sphere.

(II) $f=k$, k=constant, $v\neq$const. along M^n.

In this case, we have

$$\nabla_b\nabla_a v=(-\lambda^2 v+k+c\lambda)g_{ba}. \tag{5.13}$$

Here $\lambda\neq 0$, because if $\lambda=0$, then we have v=const. along M^n. Thus we have

$$\nabla_b\nabla_a\left(v-\frac{k+c\lambda}{\lambda^2}\right)=-\lambda^2\left(v-\frac{k+c\lambda}{\lambda^2}\right)g_{ba}, \tag{5.14}$$

from which we conclude that the submanifold is isometric to a sphere. Thus we have

THEOREM 5.2. *Let M^n be a closed orientable submanifold of codimension* 2 *of an* $(n+2)$*-dimensional orientable Riemannian manifold M^{n+2} which admits a scalar function v such that $\nabla_j\nabla_i v=f(v)g_{ji}$, where $f(v)=kv$, or k, k being a constant, and $v\neq$const. along M^n. Then under* 4 *conditions of Theorem* 5.1 *where $X^h=(\nabla_i v)g^{ih}$, the submanifold is totally umbilical and is isometric to a sphere.*

BIBLIOGRAPHY

[1] KATSURADA, Y., Closed submanifolds with constant ν-th mean curvature related with a vector field in a Riemannian manifold. J. of Fac. Sci., Hokkaido University, Ser. I, **21** (1969), 171–181.

[2] KATSURADA, Y., AND H. KÔJYÔ, Some integral formulas for closed submanifolds in a Riemann space. J. of Fac. Sci., Hokkaido University, Ser. I, **20** (1968), 90–100.

[3] KATSURADA, Y., AND T. NAGAI, On some properties of a submanifold with constant mean curvature in a Riemann space. J. of Fac. Sci., Hokkaido University, Ser. I, **20** (1968), 79–89.

[4] NAGAI, T., On certain conditions for a submanifold in a Riemannian space to be isometric to a sphere. J. of Fac. Sci., Hokkaido University, Ser. I, **21** (1969), 135–159.

[5] OBATA, M., Certain conditions for a Riemannian manifold to be isometric with a sphere. J. Math. Soc. Japan **14** (1962), 333–340.

[6] OKUMURA, M., Compact orientable submanifold of codimension 2 in an odd dimensional sphere. Tôhoku Math. J. **20** (1968), 8–20.

[7] OKUMURA, M., Submanifolds of codimension 2 with certain properties. To appear.

[8] YANO, K., Notes on hypersurfaces in a Riemannian manifold. Canadian J. of Math. **19** (1967), 439–446.

[9] YANO, K., Integral formulas for submanifolds and their applications. To appear in Canadian J. of Math.

TOKYO INSTITUTE OF TECHNOLOGY, AND
SAITAMA UNIVERSITY.