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ON AN ELEMENTARY PROOF OF LOCAL MAXIMALITY FOR THE COEFFICIENT a_8

BY MITSURU OZAWA

0. Let $f(z)$ be a normalized regular function univalent in the unit circle $|z|<1$

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

Recently several authors proved the local maximality for the coefficient a_8 at the Koebe function. Our previous method in [1] has some further possibility for the global study in principle. However there were two difficulties in order to give any effective global study. We started from Golusin's inequality and then made use of Golusin's as an essential lemma in [1]. If we express the starting inequality fully in our notations, then it is so big that it is almost impossible to control it. This is the first difficulty. The second difficulty for the global study is that the proof given in [1] does not lie on the elementary level. In this paper we shall show that the first difficulty can be overcome to a great extent and the second one can be eliminated completely. We shall start from Grunsky's inequality and then use Golusin's as a lemma. Our result is the following theorem:

THEOREM. *There are positive constants* ε *and* A *satisfying*

$$\Re a_8 \leqq 8 - A(2-\Re a_2)$$

for $0 \leqq 2-\Re a_2 \leqq \varepsilon$. *If* $\Re a_8=8$ *there, then* $f(z)$ *reduces to the Koebe function* $z/(1-z)^2$.

By the method given here there appears a big chance of proving the Bieberbach conjecture for a_8, although it is still required to perform a tremendous amount of calculation. Another support is given by the method in [2], which is essentially the same as the one given here. In [2] we proved the following fact: If a_2 is real non-negative, then $\Re a_8 \leqq 8$ with equality holding only for the Koebe function. The present author believes that a little variant of this method leads to the final goal in a quite near future.

We make use of the same notations as in [1].

1. LEMMA 1.

$$11(\tau^2+\tau'^2)+9(\varphi^2+\varphi'^2)+7(\xi^2+\xi'^2)+5(\eta^2+\eta'^2)+3(y^2+y'^2)+x'^2 \leqq 4x-x^2.$$

By Lemma 1 we have that $\tau, \tau', \varphi, \varphi', \xi, \xi', \eta, \eta', y, y'$ and x' are $O(x^{1/2})$. We can prove that

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$$y=O(x), \qquad \eta=O(x), \qquad \xi=O(x).$$

However we need several non-elementary results in order to prove these precise results. From the methodological point of view we shall not make use of these sharper results. In the sequel we shall omit the terms whose orders are higher than x. This omission does not make any trouble for local maximality by continuity.

LEMMA 2. $\eta+\frac{13}{3}y$

$$\begin{aligned}
&\leqq \frac{73}{9}x+\frac{1}{2}x'^2-\frac{1}{2}x'y'-\frac{1}{4}y^2-\frac{1}{4}\left(y'+\frac{8}{3}x'\right)^2\\
&-\frac{3}{4}\left(\eta+\frac{5}{3}y\right)^2-\frac{3}{4}\left(\eta'+\frac{5}{3}y'+x'\right)^2-\frac{5}{4}\left(\xi+\frac{5}{3}\eta+y\right)^2\\
&-\frac{5}{4}\left(\xi'+\frac{5}{3}\eta'+y'\right)^2-\frac{7}{4}\left(\varphi+\frac{5}{3}\xi+\eta\right)^2-\frac{7}{4}\left(\varphi'+\frac{5}{3}\xi'+\eta'\right)^2\\
&+O(x^{3/2}).
\end{aligned}$$

Proof. By Golusin's inequality

$$\begin{aligned}
|b_{1,1}x_1+b_{3,1}x_3|^2+3|b_{1,3}x_1+b_{3,3}x_3|^2+5|b_{1,5}x_1+b_{3,5}x_3|^2\\
+7|b_{1,7}x_1+b_{3,7}x_3|^2\leqq|x_1|^2+3|x_3|^2,
\end{aligned}$$

we have the desired result by putting $x_1=4p/3$, $x_3=2/3$.

2. We start from Grunsky's inequality with $m=7$, $x_2=x_4=x_6=0$, $x_1=\gamma$, $x_3=\delta/3$, $x_5=p/5$, $x_7=1/7$. Then we have

$$\begin{aligned}
\Re a_8 \leqq\ & \frac{2}{7}+\frac{2}{5}p^2+\frac{27}{7\cdot 64}p^7-\frac{1}{80}p^7+x\gamma^2+\frac{\delta^2}{12}(8-p^3)\\
&+\left(\frac{5}{4}p^2-2\delta\right)\varphi+\left(\frac{11}{8}p^3-p\delta-2\gamma\right)\xi+\left(\frac{7}{8}p^4+\frac{p^2}{2}\delta-\delta^2-2p\gamma\right)\eta\\
&+\left(\frac{11}{16}p^5-\frac{p^3}{2}\delta+\frac{p}{2}\delta^2-2\delta\gamma\right)y+\left(\frac{3}{2}p^3+\frac{p}{2}\delta\right)y^2\\
&+\left(\frac{27}{8}p^2+2\delta\right)y\eta+\frac{9}{4}p\eta^2+\frac{9}{2}py\xi+4\eta\xi+3y\varphi\\
&-\left(\frac{73}{64}p^5-\frac{p}{4}\delta^2\right)x'^2-\left(\frac{49}{16}p^4-p^2\delta+\frac{1}{2}\delta^2\right)x'y'-\left(\frac{3}{2}p^3+\frac{p}{2}\delta\right)y'^2\\
&-\frac{15}{4}p^3x'\eta'-\left(\frac{27}{8}p^2+2\delta\right)y'\eta'-\frac{9}{4}p\eta'^2-\left(\frac{29}{8}p^2+\delta\right)x'\xi'\\
&-\frac{9}{2}py'\xi'-4\eta'\xi'-\frac{7}{2}px'\varphi'-3y'\varphi'-2x'\tau'+O(x^{3/2}).
\end{aligned}$$

Further we put

$$\delta=\frac{5}{8}p^2+\frac{1}{2}y, \qquad \gamma=\frac{3}{8}p^3+\frac{3}{2}py+\eta.$$

Here remembering $p=2-x$ we have

$$\begin{aligned}\Re a_8 \leqq 8-\frac{31}{4}x+\frac{3}{4}\left(\eta+\frac{13}{3}y\right)\\ -3y^2+\frac{1}{2}\eta^2+2y\xi+2\eta\xi+2y\varphi\\ -\left\{\frac{267}{8}x'^2+\frac{337}{8}x'y'+\frac{29}{2}y'^2+30x'\eta'+\frac{37}{2}y'\eta'+\frac{9}{2}\eta'^2\right.\\ \left.+17x'\xi'+9y'\xi'+4\eta'\xi'+7x'\varphi'+3y'\varphi'+2x'\tau'\right\}\\ +O(x^{3/2}).\end{aligned}$$

By making use of Lemma 2 we have

$$\begin{aligned}\Re a_8 \leqq 8-\frac{5}{3}x-3y^2+\frac{1}{2}\eta^2+2y\xi+2\eta\xi+2y\varphi\\ -\left\{\frac{267}{8}x'^2+\frac{337}{8}x'y'+\frac{29}{2}y'^2+30x'\eta'+\frac{37}{2}y'\eta'+\frac{9}{2}\eta'^2\right.\\ \left.+17x'\xi'+9y'\xi'+4\eta'\xi'+7x'\varphi'+3y'\varphi'+2x'\tau'\right\}\\ -\frac{3}{8}x'y'+\frac{3}{8}x'^2-\frac{3}{16}\left(y'+\frac{8}{3}x'\right)^2-\frac{9}{16}\left(\eta'+\frac{5}{3}y'+x'\right)^2\\ -\frac{15}{16}\left(\xi'+\frac{5}{3}\eta'+y'\right)^2-\frac{21}{16}\left(\varphi'+\frac{5}{3}\xi'+\eta'\right)^2+O(x^{3/2}).\end{aligned}$$

We apply Lemma 1 to $-5x/3$. Then we have

$$\begin{aligned}\Re a_8 \leqq 8-\frac{Q}{12}-\frac{R}{48}+O(x^{3/2}),\\ Q=51y^2+19\eta^2-24y\xi-24\eta\xi+35\xi^2-24y\varphi+45\varphi^2,\\ R=1695x'^2+2178x'y'+885y'^2+1494x'\eta'+1128y'\eta'+531\eta'^2\\ +816x'\xi'+522y'\xi'+552\eta'\xi'+360\xi'^2+336x'\varphi'\\ +144y'\varphi'+126\eta'\varphi'+210\xi'\varphi'+243\varphi'^2+96x'\tau'+220\tau'^2.\end{aligned}$$

Now Q is positive definite, which is very easy to verify. Next R is positive definite. Since $R_1 \equiv 10.5x'^2 + 96x'\tau' + 220\tau'^2 \geqq 0$, we may consider the symmetric matrix associated with $R' = R - R_1$ with variables $(x', y', 3\eta', \xi', 3\varphi')$, which is

$$\begin{pmatrix} 1684.5 & 1089 & 249 & 408 & 56 \\ 1089 & 885 & 188 & 261 & 24 \\ 249 & 188 & 59 & 92 & 7 \\ 408 & 261 & 92 & 360 & 35 \\ 56 & 24 & 7 & 35 & 27 \end{pmatrix}.$$

Now we make its principal diagonal minor determinants. They are

$$27, \quad \begin{vmatrix} 360 & 35 \\ 35 & 27 \end{vmatrix}, \quad \begin{vmatrix} 59 & 92 & 7 \\ 92 & 360 & 35 \\ 7 & 35 & 27 \end{vmatrix},$$

$$\begin{vmatrix} 885 & 188 & 261 & 24 \\ 188 & 59 & 92 & 7 \\ 261 & 92 & 360 & 35 \\ 24 & 7 & 35 & 27 \end{vmatrix}, \quad \begin{vmatrix} 1684.5 & 1089 & 249 & 408 & 56 \\ 1089 & 885 & 188 & 261 & 24 \\ 249 & 188 & 59 & 92 & 7 \\ 408 & 261 & 92 & 360 & 35 \\ 56 & 24 & 7 & 35 & 27 \end{vmatrix}.$$

All of them are positive. Thus R' is positive definite and hence so is R. By continuity we have the desired result.

References

[1] Jenkins, J. A., and M. Ozawa, On local maximality for the coefficient a_8. Ill. Journ. Math. **11** (1967), 596–602.

[2] Ozawa, M., and Y. Kubota, On the eighth coefficient of univalent functions. To appear.

Department of Mathematics,
Tokyo Institute of Technology.