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SURFACES OF CURVATURE $\lambda_N=0$ IN E^{2+N}

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1.[1),2)] In [3], Prof. Ōtsuki introduced some kinds of curvature, $\lambda_1, \lambda_2, \cdots, \lambda_N$, for surfaces in a $(2+N)$-dimensional Euclidean space E^{2+N}. These curvatures play a main rôle for the surfaces in higher dimensional Euclidean space.

In [5], Shiohama proved that a complete, oriented surface M^2 in E^{2+N} with the curvatures $\lambda_1=\lambda_2=\cdots=\lambda_N=0$ is a cylinder.

In this note, we shall prove the following theorem:

THEOREM 1. *Let f: $M^2 \to E^{2+N}$ $(N \geqq 2)$ be an immersion of a compact, oriented surface M^2 in a $(2+N)$-dimensional Euclidean space E^{2+N}. Then*

(I) *The last curvature $\lambda_N=0$ if and only if M^2 is imbedded as a convex surface in a 3-dimensional linear subspace of E^{2+N}, and*

(II) *The first curvature $\lambda_1=a=constant$ and the last curvature $\lambda_N=0$ if and only if M^2 is imbedded as a sphere in a 3-dimensional linear subspace of E^{2+N} with radius $1/\sqrt{a}$.*

2. Lemmas. In order to prove Theorem 1, we first prove the following two lemmas.

LEMMA 1. *Let f: $M^2 \to E^{2+N}$ be an immersion given as in Theorem 1. Then the last curvature $\lambda_N \geqq 0$ if and only if M^2 is imbedded as a convex surface in a 3-dimensional linear subspace of E^{2+N}.*

Proof. Let f: $M^2 \to E^{2+N}$ be an immersion given as in Theorem 1, and let $(p, e_1, e_2, \cdots, e_{2+N})$ be a Frenet-frame in the sense of Ōtsuki [2], then we have the following:

(2. 1) $$dp=\omega_1 e_1+\omega_2 e_2,$$

(2. 2) $$de_A=\sum_B \omega_{AB} e_B, \qquad \omega_{AB}+\omega_{BA}=0,$$

(2. 3) $$\omega_{ir}=\sum_r A_{rij}\omega_j, \qquad A_{rij}=A_{rji},$$

(2. 4) $$\omega_{ir}\wedge\omega_{2r}=\lambda_{r-2}\omega_1\wedge\omega_2 \qquad \lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_N,$$

(2. 5) $$G(p)=\sum_r \lambda_{r-2}(p),$$

$$A, B=1, \cdots, 2+N, \quad r=3, \cdots, 2+N, \quad i, j=1, 2,$$

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2) This work was supported in part by the SDF at University of Notre Dame, 1968–1970.

where ω_1, ω_2 and ω_{12} are the basic forms, and the connection form of M^2 with respect to the induced metric, and $G(p)$ denotes the Gaussian curvature at p.

Let B_ν denote the normal bundle of the immersion f: $M^2 \to E^{2+N}$, then for any $(p, e) \in B_\nu$, we can write

$$e = e_3 \cos\theta_1 + \cdots + e_{2+N} \cos\theta_N, \qquad -\frac{\pi}{2} \leqq \theta_i \leqq \frac{\pi}{2}. \tag{2.6}$$

As in [3], we know that the Lipschitz-Killing curvature $K(p, e)$ satisfies

$$K(p, e) = \lambda_1(p) \cos^2\theta_1 + \cdots + \lambda_N(p) \cos^2\theta_N. \tag{2.7}$$

Now, suppose that $\lambda_N \geqq 0$, then by (2.4) and (2.7) we know that $K(p, e) \geqq 0$ for all $(p, e) \in B_\nu$. Hence, the total absolute curvature $T(f)$ of the immersion f: $M^2 \to E^{2+N}$ satisfies

$$\begin{aligned} T(f) &= \int_{B_\nu} |K(p, e)|\, dV \wedge d\sigma_{N-1} = \int_{B_\nu} K(p, e) dV \wedge d\sigma_{N-1} \\ &= \int_{B_\nu} (\lambda_1(p) \cos^2\theta_1 + \cdots + \lambda_N(p) \cos^2\theta_N) dV \wedge d\sigma_{N-1} \\ &= \frac{c_{N+1}}{2\pi} \int_{M^2} G(p) dV = (2-2g) c_{N+1}. \end{aligned} \tag{2.8}$$

Therefore by a result due to Chern-Lashof [2], we know that $T(f) \geqq (2+2g)c_{N+1}$, hence we know that f is a minimal imbedding and the genus $g=0$. Hence, also by a result due to Chern-Lashof [2], M^2 is imbedded as a convex surface in a 3-dimensional linear subspace of E^{2+N}.

Conversely, if M^2 is imbedded as a convex surface in a 3-dimensional linear subspace of E^{2+N}. Then we have

$$T(f) = \int_{B_\nu} |K(p, e)|\, dV \wedge d_{N-1} = 2c_{N+1} \qquad \text{and} \quad g=0. \tag{2.9}$$

On the other hand, by the last three equalities of (2.8), we have

$$\int_{B_\nu} K(p, e) dV \wedge d\sigma_{N-1} = 2c_{N+1}. \tag{2.10}$$

Hence, by (2.9) and (2.10) we know that the Lipschitz-Killing curvature $K(p, e) \geqq 0$ for all $(p, e) \in B_\nu$. Therefore by (2.4) and (2.7), we can easily verify that the last curvature $\lambda_N \geqq 0$. This completes the proof of the Lemma.

LEMMA 2. *Let f: $M^2 \to E^{2+N}$ $(N \geqq 1)$ be an immersion given as in Theorem* 1, *and let $\bar{f}$: $M^2 \to E^{3+N}$ be the immersion given by $\bar{f}(p) = f(p)$ for all $p \in M^2$. Then the Lipschitz-Killing curvature $K(p, e)$ and $\bar{K}(p, e)$ of the immersions f and $\bar{f}$ satisfy the following*:

$$\bar{K}(p, e) = \cos^2\theta K(p, e'), \qquad (p, e) \in \bar{B}_\nu, \tag{2.11}$$

where e' denotes the unit vector of the projection of e in E^{2+N}, and θ denotes the angle between e and e'.

Proof. We consider the bundle of all frames $p, e_1', e_2', \cdots, e_{2+N}'$, such that $p\in M^2$, e_1', e_2' are tangent vectors and $e_3', \cdots, e_{2+N}'$ are normal vectors to $f(M^2)$ at $f(p)$. If we set

$$\omega_{2+N,A}'=de_{2+N}'\cdot e_A' \tag{2.12}$$

and let ω_1', ω_2' denote the basic forms, then the Lipschitz-Killing curvature $K(p, e_{2+N}')$ of the immersion f is given by

$$\omega_{2+N,1}'\wedge\omega_{2+N,2}'=K(p, e_{2+N}')\omega_1\wedge\omega_2. \tag{2.13}$$

Now, let a be the one of the two unit vectors perpendicular to E^{2+N} in E^{3+N}. A unit normal vector at $f(p)$ can be written uniquely in the form:

$$\bar{e}_{3+N}=(\cos\theta)e_{2+N}'+(\sin\theta)a, \qquad -\frac{\pi}{2}\leqq\theta\leqq\frac{\pi}{2},$$

where e_{2+N}' is the unit vector in the direction of its projection in E^{2+N}. Let

$$\bar{e}_{2+N}=(\sin\theta)e_{2+N}'-(\cos\theta)a, \qquad \bar{e}_s=e_s', \qquad 1\leqq s\leqq 1+N,$$

and

$$\bar{\omega}_{3+N,A}=d\bar{e}_{3+N}\cdot\bar{e}_A.$$

Then we have

$$\bar{\omega}_{3+N,s}=\cos\theta\,\omega_{2+N,s}'.$$

Therefore by (2.13) we can easily get

$$\bar{K}(p, e)=\cos^2\theta K(p, e')$$

where e' is the unit vector in the direction of the projection of e in E^{2+N}.

3. Proof of Theorem 1. The necessity of Part (I) in Theorem 1 follows immediately from Lemma 1. On the other hand, suppose that M^2 is imbedded as a convex surface in a 3-dimensional linear subspace E of E^{2+N}. Without loss of generality, we can suppose that $E\subset E^{1+N}$. Now, let

$$f'\colon M^2\rightarrow E^{1+N}$$

be the immersion of M^2 into E^{1+N} given by $f'(p)=f(p)$ for all $p\in M^2$. Then by Lemma 2, we know that for all $(p, e)\in B_\nu$, we have

$$K(p, e)=\cos^2\theta K'(p, e') \qquad -\frac{\pi}{2}<\theta\leqq\frac{\pi}{2}.$$

Hence

$$K(p, e)=0, \qquad \theta=\frac{\pi}{2}.$$

Now, by Lemma 1, we know that $K'(p, e')\geqq 0$ for all $(p, e')\in B'_\nu$. Hence by (2. 4) and (2. 7), we know that last curvature $\lambda_N=0$.

Now, suppose that not only the last curvature $\lambda_N=0$ but the first curvature $\lambda_1=a=$constant. Then by the fact that M^2 is imbedded as a convex surface in a 3-dimensional linear subspace E, we can easily see, from Lemma 2, that

$$\lambda_1(p)=K(p, e)$$

where e is a unit normal vector at $f(p)$ in E. Furthermore we can easily verify that the Lipschitz-Killing curvature $K(p, e)$ for such e is equal to the Gaussian curvature $G(p)$ of the immersion $\bar{f}$: $M^2\to E$ which is induced by f in a natural way. Hence by the fact that M^2 is compact, we know that M^2 is imbedded in E with constant Gaussian curvature $G(p)=a$. Therefore M^2 is imbedded in E as a sphere with radius $1/\sqrt{a}$.

Conversely, suppose that M^2 is imbedded as a sphere in a 3-dimensional linear subspace E with radius $1/\sqrt{a}$. Then we know that the Gaussian curvature $G(p)=\bar{K}(p, e)=a$ for all (p, e) in the normal bundle of the immersion $\bar{f}$: $M^2\to E$. Hence by Lemma 2, (2. 4) and (2. 7) we can easily verify that the first curvature $\lambda_1=a$ and the last curvature $\lambda_N=0$. This completes the proof of Theorem 1.

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Added in Print. A recent paper of author generalizes Lemma 1 to even-dimensional manifolds in Euclidean spaces.