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ON ANALYTIC MAPPINGS AMONG ALGEBROID SURFACES

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§ 1. Introduction.

Let R_n $(n\geqq 2)$ and S_m $(m\geqq 2)$ be algebroid surfaces defined by irreducible equations

$$y^n+A_1(z)y^{n-1}+\cdots+A_n(z)=0 \tag{1.1}$$

and

$$u^m+B_1(w)u^{m-1}+\cdots+B_m(w)=0, \tag{1.2}$$

respectively, where $A_1, \cdots, A_n$, $B_1, \cdots, B_{m-1}$ and B_m are meromorphic functions in the finite plane. In this case we shall extend every boundary component of R_n (resp. S_m) over $z=\infty$ (resp. $w=\infty$) if it has an algebraic character, that is, for a certain large value r_0 it has no branch point over $|z|\geqq r_0$ (resp. $|w|\geqq r_0$) with the exception of points over $z=\infty$ (resp. $w=\infty$). Here we assume that R_n and S_m have an infinite number of branch points.

Let $\mathfrak{p}_{R_n}$ (resp. $\mathfrak{p}_{S_m}$) be the projection map $(z, y)\to z$ (resp. $(w, u)\to w$). Let φ be a non-trivial analytic mapping of R_n into S_m. In the sequel when we speak of an analytic mapping of R_n into S_m we shall always mean a non-trivial one. If φ preserves the projection maps, that is

$$\mathfrak{p}_{S_m}\varphi(p)=\mathfrak{p}_{S_m}\varphi(q) \qquad \text{whenever} \quad \mathfrak{p}_{R_n}p=\mathfrak{p}_{R_n}q,$$

then φ is called a rigid analytic mapping of R_n into S_m. Otherwise we say that φ is a non-rigid analytic mapping of R_n into S_m.

In the subsequent lines we make use of the inverse mapping $\mathfrak{p}_{R_n}^{-1}$, as an n-valued analytic branch, of the z sphere onto R_n. We set

$$h(z)=\mathfrak{p}_{S_m}\circ\varphi\circ\mathfrak{p}_n^{-1}(z).$$

Then $h(z)$ reduces to a single-valued function of z when and only when φ is rigid.

In the present paper we shall study analytic mappings of R_n into S_m. In § 3 we give two sufficient conditions for the rigidity of any existing analytic mapping

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of R_n into S_m. In the case of $n=m=2$ and $n=m=3$, Ozawa [8] and Mutō [5] showed that every analytic mapping of R_n into S_m is rigid. Hiromi and Mutō [4] gave a sufficient condition for the rigidity of any existing analytic mapping of R_n into S_m. In § 4 we give several non-existence criteria of analytic mappings. In § 5 we study the multivaluedness of $h(z)$, when φ is non-rigid. In § 6 we give a relation between the orders of branch points of R_n and S_m, when there exists a rigid analytic mapping of R_n into S_m. In § 7 we give a theorem on the growth of analytic mappings. Some of the results in § 4, § 6 and § 7 contain earlier results in Ozawa [7], [8], Mutō [5] and Hiromi and Mutō [3], [4]. In § 8 we discuss analytic mappings of R_n into itself. Our basic tool is an elegant result obtained by Heins [2].

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§ 2. Preliminary.

Let φ be an analytic mapping of R_n into S_m and $h(z)$ the corresponding function of z. We assume that the function h is a k-valued algebroid function. Let R_n' be the proper existence domain of $h(z)$. Let p_0, $\mathfrak{p}_{R_n} p_0 = z_0$ be a point on R_n whose order of ramification being counted with respect to R_n' is $\mu_0 - 1$. Let q_0, $\mathfrak{p}_{S_m} q_0 = w_0$ be the φ-image of p_0 on S_m whose order of ramification is $\lambda_0 - 1$. Then we have the following expansion in a neighborhood of z_0:

$$h(z) = w_0 + a_\tau (\sqrt[\mu_0]{z - z_0})^\tau + \cdots \qquad a_\tau \neq 0 \tag{2. 1}$$

or

$$h(z) = \frac{a_{-\tau}}{(\sqrt[\mu_0]{z - z_0})^\tau} + \cdots \qquad a_{-\tau} \neq 0. \tag{2. 2}$$

We define

$$N(r; q_0, S_m) = \frac{1}{n\lambda_0} \int_0^r (n(t; q_0, S_m) - n(0; q_0, S_m)) \frac{dt}{t} + \frac{n(0; q_0, S_m)}{n\lambda_0} \log r,$$

where

$$n(r; q_0, S_m) = \sum_{\varphi(p) = q_0,\ |\mathfrak{p}_{R_n} p| \leq r} \tau.$$

Suppose that the point q_0 whose order of ramification is $\lambda_0 - 1$ lies over w_0. Then we define a function $u_0(q)$ as follows: In the case of $w_0 \neq \infty$

$$u_0(q) = \frac{1}{\lambda_0} \log \frac{\delta_0}{|w - w_0|} \qquad |w - w_0| \leq \delta_0,$$

$$= 0 \qquad \text{otherwise;}$$

In the case of $w_0=\infty$

$$u_0(q)=\frac{1}{\lambda_0}\log\frac{|w|}{\delta_0} \qquad |w|\geqq\delta_0,$$
$$=0 \qquad \text{otherwise.}$$

By making use of this function we define $m(r; q_0, S_m)$ in the following manner:

$$m(r; q_0, S_m)=\frac{1}{2n\pi}\int_{|z|=r} u_0(\varphi(\mathfrak{p}_{R_n}^{-1}(re^{i\theta})))d\theta.$$

For another point q_1 on S_m we define $N(r; q_1, S_m)$ and $m(r; q_1, S_m)$ analogously. It is well known that there exists a function $u(q; q_0, q_1)$, which is harmonic in q on S_m save at q_0 and q_1 where it has a positive normalized logarithmic singularity and a negative normalized logarithmic singularity respectively, and which is bounded in the complement of some compact neighborhood of $\{q_0, q_1\}$. Using this function we can obtain a simple relation between the sum $m(r; q, S_m)+N(r; q, S_m)$ and $T(r, h)$, where $T(r, h)$ is the Nevanlinna-Selberg characteristic function for $h(z)$. That is

$$m(r; q, S_m)+N(r; q, S_m)=\frac{1}{m}T(r, h)+O(1). \tag{2. 3}$$

Let $R_n(r)$ be the part of R_n which lies over $|z|\leqq r$. Put

$$N(r; S_h)=\frac{1}{n}\int_0^r(n(t; S_h)-n(0; S_h))\frac{dt}{t}+\frac{n(0; S_h)}{n}\log r,$$

$$n(r; S_h)=\sum_{R_n(r)}(\tau-1),$$

where τ has already been defined in (2. 1) and (2. 2). By the Nevanlinna-Selberg second fundamental theorem for $h(z)$ we have

$$(l-2k)T(r, h)\leqq\sum_{\nu=1}^{l}N(r; w_\nu)-N(r; S_h)+O(\log rT(r, h)) \tag{2. 4}$$

outside a set of finite measures, [6], [10]. Using (2. 3), we have

$$\left(l-\frac{1}{m}\sum_{\nu=1}^{l'}\lambda_\nu\right)T(r, h)+O(1)\geqq\sum_{\nu=1}^{l}N(r; w_\nu)-\sum_{\nu=1}^{l'}\lambda_\nu N(r; q_\nu, S_m). \tag{2. 5}$$

Hence, by (2. 4) and (2. 5), we have

$$\left(\frac{1}{m}\sum_{\nu=1}^{l'}\lambda_\nu-2k\right)T(r, h)\leqq\sum_{\nu=1}^{l'}\lambda_\nu N(r, q_\nu, S_m)-N(r; S_h)+O(\log rT(r, h)) \tag{2. 6}$$

outside a set of finite measure. Hiromi and Mutō gave this relation in [4].

§ 3. Sufficient conditions for the rigidity.

First, we introduce two counting functions. We denote by $n(r; B, R_n)$ the number of branch points of R_n and $n^*(r; B, R_n)$ the number of branch points of R_n whose order of ramification is $n-1$, which lie over $|z|\leqq r$, respectively. Correspondingly we define

$$N(r; B, R_n)=\frac{1}{n}\int_0^r (n(t; B, R_n)-n(0; B, R_n))\frac{dt}{t}+\frac{n(0; B, R_n)}{n}\log r,$$

$$N^*(r; B, R_n)=\frac{1}{n}\int_0^r (n^*(t; B, R_n)-n^*(0; B, R_n))\frac{dt}{t}+\frac{n^*(0; B, R_n)}{n}\log r.$$

We obtain the following

THEOREM 1. *Assume that the inequality*

$$\frac{N^*(r; B, R_n)}{N(r; B, R_n)}\geqq\varepsilon>0$$

holds for a set of r of infinite measure. Then every analytic mapping of R_n into S_m is rigid whenever it exists.

Proof. Let φ be an analytic mapping of R_n into S_m and $h(z)$ corresponding function. A theorem in [4] implies that the proper existence domain of $h(z)$ is not R_n. So, let R_n' be the proper existence domain of $h(z)$. Suppose that $h(z)$ is $k(\geqq 2)$-valued function of z, then R_n' has an infinite number of branch points whose order of ramification is $k-1$. Hence $h(z)$ is an algebroid function of z. Therefore we can apply the Nevanlinna-Selberg second fundamental theorem, [6], [10].

Let $n_2(r; q_0, S_m)$ be the number of simple q_0 points of φ, that is, $\tau=1$ in (2. 1) or (2. 2), $n_3(r; q_0, S_m)$ the number of multiple q_0 points of φ, that is, $\tau\geqq 2$ in (2. 1) or (2. 2), being counted multiply and $\bar{n}_1(r; q_0, S_m)$ the number of distinct multiple q_0 points of φ, which lie over $|z|\leqq r$, respectively. Correspondingly we define

$$N_2(r; q_0, S_m)=\frac{1}{n\lambda_0}\int_0^r (n_2(t; q_0, S_m)-n_2(0; q_0, S_m))\frac{dt}{t}+\frac{n_2(0; q_0, S_m)}{n\lambda_0}\log r,$$

$$N_3(r; q_0, S_m)=\frac{1}{n\lambda_0}\int_0^r (n_3(t; q_0, S_m)-n_3(0; q_0, S_m))\frac{dt}{t}+\frac{n_3(0; q_0, S_m)}{n\lambda_0}\log r,$$

$$\bar{N}_1(r; q_0, S_m)=\frac{1}{n\lambda_0}\int_0^r (\bar{n}_1(t; q_0, S_m)-\bar{n}_1(0; q_0, S_m))\frac{dt}{t}+\frac{\bar{n}_1(0; q_0, S_m)}{n\lambda_0}\log r,$$

where λ_0-1 is the order of ramification of q_0. Let $\{q_\nu\}$ be the branch points of S_m. By (2. 6), we have

$$\left(\frac{1}{m}\sum_{\nu=1}^{l'}\lambda_\nu-2k\right)T(r, h)$$

$$\leqq \sum_{\nu=1}^{l'}\lambda_\nu N(r; q_\nu, S_m)-N(r; S_h)+O(\log r\,T(r, h))$$

$$\leqq \sum_{\nu=1}^{l'}\lambda_\nu N_2(r; q_\nu, S_m)+\sum_{\nu=1}^{l'}\lambda_\nu N_3(r; q_\nu, S_m)-N(r; S_h)+O(\log r\,T(r, h))$$

$$\leqq \sum_{\nu=1}^{l'}\lambda_\nu N_2(r; q_\nu, S^m)+\sum_{\nu=1}^{l'}\lambda_\nu \bar{N}_1(r; q_\nu, S_m)+O(\log r\,T(r, h))$$

$$\leqq \sum_{\nu=1}^{l'}\lambda_\nu N_2(r; q_\nu, S_m)+\sum_{\nu=1}^{l'}\frac{\lambda_\nu}{2m}T(r, h)+O(\log r\,T(r, h))$$

outside a set of finite measure where $\lambda_\nu-1$ is the order of ramification of q_ν. Since only the branch points of R_n can be simple q_ν points, by $\varepsilon N(r; B, R_n)\leqq N^*(r; B, R_n)$, we have

$$\left(\sum_{\nu=1}^{l'}\frac{\lambda_\nu}{2m}-2k\right)T(r, h)$$

$$\leqq \sum_{\nu=1}^{l'}\lambda_\nu N_2(r; q_\nu, S_m)+O(\log r\,T(r, h)) \tag{3. 1}$$

$$\leqq N(r; B, R_n)+O(\log r\,T(r, h))$$

$$\leqq C_1N^*(r; B, R_n)+O(\log r\,T(r, h))$$

for a set of infinite measure, where C_1 is a positive constant. On the other hand the ramification theorem implies the following relation:

$$N^*(r; B, R_n)\leqq N(r; B, R_n')\leqq(2k-2)T(r, h).$$

Consequently we have

$$\left(\sum_{\nu=1}^{l'}\frac{\lambda_\nu}{2m}-C_2\right)T(r, h)\leqq O(\log r\,T(r, h))$$

for a set of infinite measure, where C_2 is a positive constant. This inequality is untenable, since S_m has an infinite number of branch points for which $\lambda_\nu\geqq 2$. Thus $h(z)$ must be an entire function of z, that is, the mapping φ is rigid. This completes the proof of theorem 1.

REMARK. There exists a pair of algebroid surfaces R_n and S_m for which there exists a non-rigid analytic mapping, even if the surface R_n has an infinite number of branch points whose order of ramification is $n-1$. In fact, let R_4 and S_2 be algebroid surfaces defined by

$$y^4=(\exp z-1)(\exp(\exp z-1)-1)^2,$$

$$u^2=w(\exp w^2-1),$$

respectively. Then there exists a non-rigid analytic mapping φ of R_4 into S_2 induced by $h(z)=\sqrt{\exp z-1}$, that is $\varphi=\mathfrak{p}_{S_2}^{-1}\circ h\circ\mathfrak{p}_{R_4}$.

However we have the following

THEOREM 2. *Assume that the surface S_m has an infinite number of branch points whose order of ramification is $m-1$ and that $n\leqq m$. Then every analytic mapping of R_n into S_m is rigid whenever it exists.*

Proof. Let φ be an analytic mapping of R_n into S_m and $h(z)$ the corresponding function. Let $\{q_\nu\}$ be the branch points of S_m whose order of ramification is $m-1$. Suppose that φ is not rigid. Then, since $n\leqq m$, φ has no simple q_ν points. Let $\bar{n}(r; q_\nu, S_m)$ be the number of distinct q_ν points of φ over $|z|\leqq r$. Put

$$\bar{N}(r; q_\nu, S_m)=\frac{1}{n\lambda_\nu}\int_0^r(\bar{n}(t; q_\nu, S_m)-\bar{n}(0; q_\nu, S_m))\frac{dt}{t}+\frac{\bar{n}(0; q_\nu, S_m)}{n\lambda_\nu}\log r,$$

where $\lambda_\nu-1$ is the order of ramification of q_ν. Further put $T(r,\varphi)=T(r,h)/m$. Then we have

$$\Theta(q_\nu)=1-\varlimsup_{r\to\infty}\frac{\bar{N}(r; q_\nu, S_m)}{T(r,\varphi)}\geqq K,$$

where K is a positive constant independent of ν. Thus $h(z)$ must be transcendental. On the other hand we have, by (2. 6),

$$\sum\lambda_\nu\Theta(q_\nu)\leqq 2km.$$

It is untenable. Therefore the mapping φ must be rigid. This completes the proof of theorem 2.

§ 4. Non-existence criteria for analytic mappings.

THEOREM 3. *Assume that S_m is the same as in theorem 2 and that $n<m$, Then there is no analytic mapping of R_n into S_m.*

Proof. Theorem 2 asserts that every analytic mapping of R_n into S_m is rigid whenever it exists. Since $n<m$, we have $\Theta(q_\nu)\geqq K$, where K is a positive constant independent of ν. By the same procedure as in theorem 2 we can see that it is untenable. Thus there is no analytic mapping of R_n into S_m. This completes the proof of theorem 3.

THEOREM 4. *Let S_m be the same as in theorem* 2. *Assume that n is a prime number and that $n>m$. Then there is no analytic mapping of R_n into S_m.*

To prove this theorem we use the following theorem, [4].

THEOREM A. *Assume that there exists an analytic mapping φ of R_n into S_m. If n is a prime number, then φ is rigid. If n is not a prime number, then the corresponding function $h(z)$ of φ is k-valued, where k is a proper divisor of n and φ may or may not be rigid.*

Proof of theorem 4. By theorem A every analytic mapping of R_n into S_m is rigid whenever it exists. Let $\{q_\nu\}$ be the branch points of S_m whose order of ramification is $m-1$. Since n is a prime number, we have $\Theta(q_\nu)\geqq K$, where K is a positive constant independent of ν. By the same procedure as in theorem 2 we can conclude that there is no analytic mapping of R_n into S_m.

REMARK. We can obtain the same assertions as in theorems 3 and 4 if the number of branch points of S_m whose order of ramification is $m-1$ is greater than a constant dependent on n and m.

By the same method as above we can prove the following two non-existence criteria for rigid analytic mappings.

THEOREM 5. *Assume that S_m has at least three branch points whose order of ramification is $m-1$ and that $n<m$. Then there is no rigid analytic mapping of R_n into S_m.*

THEOREM 6. *Assume that S_m has at least three branch points whose order of ramification is $m-1$ and that n is not an integral multiple of m. Then there is no rigid analytic mapping of R_n into S_m.*

§ 5. Non-rigidity of analytic mappings.

Let R_6 and S_2 be algebroid surfaces defined by irreducible equations

$$y^6=G(z), \qquad u^2=g(w),$$

respectively, where G and g are entire functions having an infinite number of zeros whose orders are less than 6 and 2, respectively.

Let φ be an analytic mapping of R_6 into S_2 and h the corresponding function of φ. Then theorem A asserts that $h(z)$ must be k-valued, where k is a proper divisor of 6.

Furthermore we have the following

THEOREM 7. *Assume that there exists an analytic mapping φ of R_6 into S_2. Then the corresponding function $h(z)$ is either single-valued or three-valued, that is, the case where it is two-valued does not occur.*

Proof. Let u^* be the analytic mapping of S_2 into the finite plane defined by $u^*=u\circ\mathfrak{p}_{S_2}$. Then $u^*\circ\varphi$ gives an analytic mapping of R_6 into the finite plane. Thus we have

$$u^*\circ\varphi\circ\mathfrak{p}_{R_6}^{-1}=f_0+f_1y+\cdots+f_5y^5,$$

where $f_0,\cdots,f_4$ and f_5 are meromorphic functions of z in the finite z plane. Further

$$u^*\circ\varphi\circ\mathfrak{p}_{R_6}^{-1}=u\circ\mathfrak{p}_{S_2}\circ\varphi\circ\mathfrak{p}_{R_6}^{-1}=u\circ h.$$

Hence

$$u\circ h=f_0+f_1y+\cdots+f_5y^5.$$

Since $u^2=g(w)$, we have

$$g\circ h=(f_0+f_1y+\cdots+f_5y^5)^2.$$

By $y^6=G(z)$, we have

$$\begin{aligned} g\circ h=&f_0^2+f_3^2G+2f_1f_5G+2f_2f_4G\\ &+2(f_0f_1+f_2f_5G+f_3f_4G)y\\ &+(f_1^2+f_4^2G+2f_0f_2+2f_3f_5G)y^2\\ &+2(f_0f_3+f_1f_2+f_4f_5G)y^3\\ &+(f_2^2+f_5^2G+2f_0f_4+2f_1f_3)y^4\\ &+2(f_0f_5+f_1f_4+f_2f_3)y^5. \end{aligned}$$

Suppose that $h(z)$ is two-valued function of z. Then we have

$$f_0f_5+f_1f_4+f_2f_3=0, \tag{5.1}$$

$$f_0f_1+f_2f_5G+f_3f_4G=0, \tag{5.2}$$

(5. 3) $$f_1^2+f_4^2G+2f_0f_2+2f_3f_5G=0,$$

(5. 4) $$f_2^2+f_5^2G+2f_0f_4+2f_1f_3=0.$$

By (5. 1) and (5. 3), we have

(5. 5) $$(f_1+f_4y^3)^2=-2(f_0+f_3y^3)(f_2+f_5y^3).$$

By (5. 2) and (5. 4), we have

(5. 6) $$(f_2+f_5y^3)^2=-2(f_0+f_3y^3)(f_1+f_4y^3)/y^3.$$

From (5. 5) and (5. 6), we have

$$y^3(f_2+f_5y^3)^3=(f_1+f_4y^3)^3.$$

Therefore we can see that

$$f_1=f_2=f_4=f_5=0.$$

Hence

$$g\circ h=(f_0+f_3y^3)^2.$$

Thus every zero of g must be a perfectively branched value of h. It is untenable, since g has an infinite number of zeros. This completes the proof of theorem 7.

Let S_4 be an algebroid surface defined by irreducible equation

$$u^4=g(w),$$

where g has an infinite number of zeros whose orders are less than 4. Then, by the same method as above we obtain

THEOREM 8. *Let R_6 be the same as in theorem 7. Then for every analytic mapping of R_6 into S_4 the corresponding function $h(z)$ is either single-valued or three-valued, that is, the case where it is two-valued does not occur.*

§ 6. Necessary condition for the existence of analytic mappings.

Let R_n and S_m be general algebroid surfaces. First, we define the order of branch points of R_n as follows:

$$\rho_{N(B,R_n)}=\overline{\lim_{r\to\infty}}\frac{\log N(r; B, R_m)}{\log r}$$

We also define $\rho_{N(B,S_m)}$ analogously.

We obtain the following theorem:

THEOREM 9. *Assume that* $\rho_{N(B,R_n)}<\infty$ *and* $0<\rho_{N(B,S_m)}<\infty$ *and that there exists a rigid analytic mapping* φ *of* R_n *into* S_m. *Then*

$$\rho_{N(B,R_n)}=\nu\rho_{N(B,S_m)},$$

where ν *is a positive integer.*

To prove this theorem we use the following theorem [1]:

THEOREM B. *Let* $E(z)$ *and* $F(z)$ *be transcendental entire functions. Assume that the zeros of* $E(z)$ *have a positive exponent of convergence. Then the zeros of* $E\circ F(z)$ *cannot have a finite exponent of convergence.*

Proof of theorem 9. Let $h(z)$ be the corresponding function of φ. It is sufficient to prove that $h(z)$ is a polynomial. For we can prove our assertion by the same argument as in [3].

Assume that $h(z)$ is an entire function of infinite order. Let $\{q_\nu\}$ be the branch points of S_m. Then we have

$$N(r; B, \varphi, S_m)=N_2(r; B, \varphi, S_m)+N_3(r; B, \varphi, S_m),$$

$$N_3(r; B, \varphi, S_m)\leqq 2N(r; 0, h')\leqq 6T(r, h)$$

outside a set of finite measure, where

$$N(r; B, \varphi, S_m)=\sum_{\nu=1}^{\infty} N(r; q_\nu, S_m),$$

$$N_2(r; B, \varphi, S_m)=\sum_{\nu=1}^{\infty} N_2(r; q_\nu, S_m),$$

$$N_3(r; B, \varphi, S_m)=\sum_{\nu=1}^{\infty} N_3(r; q_\nu, S_m).$$

On the other hand

$$\sum_{\nu=1}^{p} N(r; w_\nu, h)\leqq N(r; B, \varphi, S_m)$$

for an arbitrary but fixed number p of the projections $\{w_\nu\}_{\nu=1}^{\infty}$ of all the branch points of S_m and for all r. By the Nevanlinna second fundamental theorem we have

$$(p-3)T(r, h) \leqq \sum_{\nu=1}^{p} N(r; w_\nu, h).$$

outside a set of finite measure. Thus we have

$$KT(r, \varphi) \leqq N_2(r; B, \varphi, S_m)$$

for an arbitrary but fixed number K and for all r outside a set of finite measure. Only the branch points of R_n can be simple q points for a branch point q of S_m. Hence

$$KT(r, \varphi) \leqq N_2(r; B, \varphi, S_m) \leqq N(r; B, R_n) \tag{6. 1}$$

holds outside a set of finite measure. It is a contradiction [cf. 3, pp. 239-240]. Therefore $h(z)$ must be of finite order.

Next we shall show that $h(z)$ must be a polynomial. Assume that $h(z)$ is a transcendental entire function of finite order. Then, by theorem B the order of $N(r; B, \varphi, S_m)$ must be infinite. This contradicts the following relation which holds for all $r \geqq r_0$:

$$\begin{aligned} N(r; B, \varphi, S_m) &= N_2(r; B, \varphi, S_m) + N_3(r; B, \varphi, S_m) \\ &\leqq N(r; B, R_n) + 6T(r, h). \end{aligned}$$

Therefore $h(z)$ must be a polynomial.

§ 7. Growth of analytic mappings.

THEOREM 10. *Assume that there exists an analytic mapping φ of R_n into S_m, then it satisfies*

$$\overline{\lim_{r\to\infty}} \frac{N(r; B, R_n)}{T(r, \varphi)} = \infty.$$

Proof. First, we assume that the corresponding function $h(z)$ is a k-valued algebroid function of z. Suppose that our assertion does not hold. Then we have

$$N(r; B, R_n) < MT(r, \varphi)$$

for all sufficiently large r, where M is a positive constant. By the relation (3. 1) we have

$$\left(\sum_{\nu=1}^{p'} \lambda_\nu - 2km\right) T(r, \varphi) \leqq MT(r, \varphi) + O(\log rT(r, \varphi))$$

outside a set of finite measure. Since S_m has an infinite number of branch points, it is untenable. That is, our assertion holds.

Next we assume that $h(z)$ is a k-valued algebraic function of z. Then we have

$$T(r, \varphi)=O(\log r).$$

On the other hand, since R_n has an infinite number of branch points we have

$$\log r=o(N(r; B, N_n)).$$

Hence, in both cases we have

$$\overline{\lim_{r\to\infty}}\frac{N(r; B, R_n)}{T(r, \varphi)}=\infty.$$

This completes the proof of theorem 10.

REMARK. By the same method as above, we have

$$\lim_{r\to\infty}\frac{N(r; B, R_n)}{T(r, \varphi)}=\infty,$$

when the order of φ is finite.

§ 8. Analytic mappings of R_n into itself.

Our surface R_n is of parabolic type and its universal covering surface is of hyperbolic type. Under this abstract situation Heins [2] discussed an analytic mapping of an open Riemann surface into itself. One of his interesting results may be stated as follows:

THEOREM C. *The analytic mappings of an open Riemann surface R with non-abelian fundamental group and of parabolic type into itself are univalent. If R does not have any planer boundary elements, the maps in question are onto.*

By making use of this theorem we can obtain the following theorem which is an extension of earlier results in [3], [4], [9]. The earlier result in [3] and [4] was proved by making use of the Nevanlinna value distribution theory.

THEOREM 11. *Let R_n be an algebroid surface and let φ be a rigid analytic mapping of R_n into itself. Then the mapping φ is an analytic mapping of R_n onto itself and the corresponding function $h(z)$ must be of the form $e^{2\pi i p/q}z+b$ with a suitable rational number p/q.*

Proof. Since φ is rigid, the corresponding function $h(z)$ is a meromorphic function of z in the finite z-plane. If $h(z)$ is transcendental, then there is at least one point w such that the equation $h(z)=w$ has an infinite number of roots. Hence there is at least one point q over w which is covered by φ infinitely often. It contradicts the univalency of φ.

Suppose that $h(z)$ is a rational function. Then, as above we can see that $h(z)$ must be of the form

$$h(z)=\frac{az+b}{cz+d}, \qquad ad-bc\neq 0.$$

Since φ is an analytic mapping of R_n into itself, we have $c=0$. That is, φ is onto and $h(z)=Az+B$. Considering the iterations of φ we can prove our assertion.

In general, there exists an algebroid surface R_n admitting a non-rigid analytic mapping of R_n onto itself. In fact, let R_4 be an algebroid surface defined by

$$y^4=z^2(1-z^2)\prod_{n=1}^{\infty}\left(\left(1-\frac{z^2}{a_n}\right)\left(1-\frac{1-z^2}{a_n}\right)\right)^2,$$

where $\{a_n\}$ are suitable complex numbers. Then there is an analytic mapping of R_4 onto itself induced by $h^2+z^2=1$, that is, $\varphi=\mathfrak{p}_{R_4}^{-1}\circ h\circ\mathfrak{p}_{R_4}$.

However we can prove

THEOREM 12. *If the part of R_n which lies over $|z|\geqq r$ is connected for all r, then every analytic mapping of R_n into itself is rigid.*

Proof. Let φ be an analytic mapping of R_n into itself and h the corresponding function of φ. By theorem A, h satisfies

$$h^k+p_1(z)h^{k-1}+\cdots+p_k(z)=0,$$

where k is a proper divisor of n. Since φ is univalent, the coefficients $p_1,\cdots,p_{k-1}$ and p_k are rational functions. Let R_n' be an algebraic surface defined by

$$y^k+p_1(z)\,y^{k-1}+\cdots+p_k(z)=0$$

and R_n'' be an algebraic surface defined by

$$z^k+p_1(y)z^{k-1}+\cdots+p_k(y)=0.$$

Then R_n'' is also k-sheeted. We can extend the mapping φ over $z=\infty$. Then the extended mapping induces an analytic mapping φ' of R_n' into R_n''. Suppose that $h(z)$ is k-valued, then R_n' must have a branch point whose order of ramification is $k-1$ over $z=\infty$, since R_n' has only one boundary component. The fact that φ is onto asserts that R_n'' also has a branch point whose order of ramification is $k-1$ over $z=\infty$. Using the analyticity of φ' around the point which lies over $z=\infty$, we can see that $k=1$. This completes the proof.

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