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A REMARK ON A TURNING POINT PROBLEM

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§ 1. Introduction.

As a turning point problem, there are two questions: 1) is there any non-singular transformation which reduces a given differential equation to a simpler form and then what types of typical equations arise by this transformation, 2) can we construct asymptotic expansions of a solution in the full neighborhood of the turning point. For the first problem, there are many references, for example Sibuya [5], Wasow [6], and Hanson [2]. For the second problem, the author developed a matching method to a considerably general type of equations [3]. To apply the matching method, we must calculate the inner solution, outer solution and connection matrix between them. In general the inner domain where the asymptotic expansion of the inner solution exists depends on a parameter ε and shrinks when ε tends to zero to one point that is a turning point itself. But in some cases, this inner domain does not shring to one point, a finite number of the inner domains cover a full neighborhood of the turning point, and then it is unnecessary to construct the outer solution if we consider only a sufficiently small neighborhood of the turning point. In this note we give a sufficient condition so as to give this case for the two examples of equations.

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§ 2. Example 1.

Let be given a differential equation of the form

$$\varepsilon^h \frac{dy}{dx} = A(x, \varepsilon) y, \tag{2.1}$$

where ε is a small complex parameter, y is an n-dim vector, x is a complex independent variable, h is a positive integer, and $A(x, \varepsilon)$ is an n-by-n matrix such that

$$A(x, \varepsilon) = \begin{bmatrix} 0 & 1 & & & \mathbf{0} & \\ & & \ddots & & & \\ & \mathbf{0} & & & \ddots & \\ & & & & & 1 \\ p_n(x, \varepsilon), & p_{n-1}(x, \varepsilon), & \cdots, & & p_2(x, \varepsilon), & 0 \end{bmatrix}. \tag{2.2}$$

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The functions $p_k(x, \varepsilon)$ $(k=2, \cdots, n)$ are holomorphic in the domain of the x, ε-space defined by the inequalities,

$$(2.3)\qquad |x| \leqq x_0, \qquad 0<|\varepsilon| \leqq \varepsilon_0, \qquad |\arg \varepsilon| \leqq \delta_0,$$

and each of the functions $p_k(x, \varepsilon)$ has a uniformly asymptotic expansion in power series of ε such that

$$(2.4)\qquad p_k(x, \varepsilon) \simeq \sum_{\nu=0}^{\infty} p_{k\nu}(x) \varepsilon^{\nu}$$

with holomorphic coefficients $p_{k\nu}(x)$:

$$(2.5)\qquad p_{k\nu}(x) = \sum_{\mu=m_{k\nu}}^{\infty} x^{\mu} p_{k\nu\mu}, \qquad p_{k\nu m_{k\nu}} \neq 0$$

where $p_{k\nu\mu}$ are constant and $m_{k\nu}$ are non negative integers. We suppose here that $m_{k0} \geqq 1$ $(k=2, \cdots, n)$, in particular $m_{n0}=q \geqq 1$, and then the equation (2.1) has a turning point at the origin.

For nonzero coefficients of the expansions (2.4), (2.5) we associate a characteristic polygon according to Iwano and Sibuya [1].

ASSUMPTION I. *The characteristic polygon consists of only one segment, that is, for nonzero constants $p_{k\nu\mu}$ we must have*

$$(2.6)\qquad \frac{m_{k\nu}}{k} \geqq \frac{q}{n} - \frac{(n+q)\nu}{nhk} \qquad \textit{for} \quad \nu=0, 1, \cdots, \quad k=2, \cdots, n.$$

ASSUMPTION II. *The characteristic roots $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the matrix*

$$(2.7)\qquad B=\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ p_{n0q}, & \cdots, p_{k0\mu_k}, & \cdots, & 0 \end{bmatrix}$$

are mutually distinct, here the indices of the nonzero elements $p_{k0\mu_k}$ satisfy $\mu_k/k=q/n$.

ASSUMPTION III. *The functions $p_{k0}(x)$ have the forms*

$$p_{n0}(x)=p_{n0q}x^q, \qquad p_{k0}(x)=p_{k0\mu_k}x^{\mu_k} \quad (k=n-1, \cdots, 2),$$

where μ_k satisfies $\mu_k/k=q/n$.

Under the assumptions I, II, we can apply the matching method to obtain asymptotic expansions of the solution of (2. 1) in the neighborhood of turning point [3]. Moreover if the assumptions I, II, III are satisfied and if $h=1$ we can construct inner solutions in the neighborhood of the turning point as stated in the introduction.

Let $x=s\cdot\varepsilon^{n/(n+q)}$, $\varepsilon=\rho^{n+q}$, and let S be a sector of the s-plane of central angle less than $n\pi/(n+q)$ whose boundary lines do not coincide with any singular direction $\operatorname{Re}(\lambda_j-\lambda_k)s^{(n+q)/n}=0$ and contains them in its inside for every $j, k=1,\cdots,n$. Then we have

THEOREM 1. *For every positive integer r, there exists a domain D of the s, ρ-plane defined by*

$$\arg s\in S,\qquad 0<|\rho|\leqq\rho_1,\qquad |\arg\rho|\leqq\delta_1,\qquad |s^{1/n}\rho|\leqq c_1, \tag{2. 8}$$

where ρ_1, δ_1 and c_1 are some constants depending on r, and there exists a fundamental matrix solution $Y(s,\rho)$ such that

$$\begin{aligned}&Y(s,\rho)-\Omega(\rho^n\cdot s^{k(s)})\sum_{\nu=1}^{r}Y_\nu(s)(s^{k(s)\cdot 1/n}\rho)^\nu s^{k(s)R}\exp[Q(s)]\\ &=\Omega(\rho^n\cdot s^{k(s)})E_r(s,\rho)[s^{k(s)\cdot 1/n}\rho]^{r+1}\cdot s^{k(s)R}\exp[Q(s)],\end{aligned} \tag{2. 9}$$

where $Y_\nu(s)$ and $E_r(s,\rho)$ are bounded in D. Here the matrices $\Omega(t)$ is of the form

$$\Omega(t)=\begin{bmatrix}1 & & & 0\\ & t^{q/n} & & \\ & & \ddots & \\ 0 & & & t^{(n-1)q/n}\end{bmatrix},$$

R is a diagonal constant matrix, $Q(s)$ is also a diagonal matrix with polynomial elements of $s^{1/n}$ of the degree $n+q$, and $k(s)$ denotes a function such that

$$k(s)=\begin{cases}0, & |s|\leqq s_0,\\ 1, & |s|>s_0,\end{cases} \tag{2. 10}$$

for some large s_0. A finite number of domains of type D cover a full neighborhood of the turning point.

Proof. If we transform the equation (2. 1) by

$$x=s\cdot\varepsilon^{n/(n+q)},\qquad \varepsilon=\rho^{n+q},\qquad y=\Omega(\rho^n)\cdot v$$

we have

$$\frac{dv}{ds}=H(s,\rho)\cdot v,$$

where

$$H(s,\rho)=\begin{bmatrix} 0 & 1 & & & 0 \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & 1 \\ \rho^{-nq}p_n(x,\varepsilon), & \cdots, & & \rho^{-2q}p_2(x,\varepsilon), & 0 \end{bmatrix}.$$

If the assumptions I, II, and III are satisfied, and if $h=1$, $H(s,\rho)$ is holomorphic in s and ρ for (2. 3) and formally $H(s,\rho)$ is reordered in power series of ρ such that

$$H(s,\rho)\sim\sum_{\nu=0}^{\infty}H_\nu(s)\rho^\nu.$$

Here

$$H_0(s)=\begin{bmatrix} 0 & 1 & & & 0 \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & 1 \\ p_{n0q}s^q, & \cdots, & p_{k0\mu_k}s^{\mu_k}, & \cdots, & 0 \end{bmatrix},$$

$$H_\nu(s)=\begin{bmatrix} & & 0 & & \\ & & & & \\ h_{n\nu}(s), & \cdots, & h_{k\nu}(s), & \cdots, & 0 \end{bmatrix},$$

where $h_{k\nu}$ is a polynomial of s of degree at most $\{\nu+kq-(n+q)\}/n$. Then we can prove that for every $m\geqq 1$, there exists a matrix function $E_{m+1}(s,\rho)$, bounded in (2. 3) and $|s|\geqq s_0$ such that

$$H(s,\rho)-\sum_{\nu=0}^{m}H_\nu(s)\rho^\nu=s^{-1}\Omega(s)\cdot E_{m+1}(s,\rho)\cdot\Omega(s^{-1})\cdot[s^{1/n}\rho]^{m+1}.$$

If we construct an inner solution by the same method as in [3] and apply the above estimate, we obtain the desired results by using the number $e=1/n$ in place of $e=1+q/n+1/nh$ in [3].

§ 3. Example 2. Hydrodynamic type [4].

We consider here an n-th order equation of the type

$$\varepsilon^{n-m}L_n(y)+L_m(y)=0, \tag{3.1}$$

where $n-2\geqq m\geqq 0$, and

$$L_n(y)=-y^{(n)}+\sum_{\nu=m+1}^{n-1}R_{\nu+1}(x,\varepsilon)y^{(\nu)},$$

$$L_m(y)=\sum_{\nu=0}^{m}(P_{\nu+1}(x)+\varepsilon R_{\nu+1}(x,\varepsilon))y^{(\nu)}.$$

The functions $P_j(x)$ are holomorphic in x and in particular $P_{m+1}(x)=x^q$ $(q\geqq 1)$, and $R_j(x,\varepsilon)$ satisfy the same conditions as that of $p_j(x,\varepsilon)$ in the preceding section.

ASSUMPTION I′. *The characteristic polygon associated with the equation* (3. 1) *consists of only one segment.*

This condition implies that the reduced equation (which is obtained from (3. 1) by letting $\varepsilon\to 0$)

$$\sum_{\nu=0}^{m}P_{\nu+1}(x)y^{(\nu)}=0 \tag{3.2}$$

has a regular singular point at the origin. Let $u_1,\cdots,u_m$ be a characteristic roots of (3. 2) at the regular singular point $x=0$.

ASSUMPTION II′. $(n-m)(u_j-u_k)$ *is not an integer for every* $j\neq k$.

ASSUMPTION III′. $P_{\nu+1}(x)=p_{\nu+1}x^{q+\nu-m}$ *with constant* $p_{\nu+1}$ $(\nu=0,1,\cdots,m)$.

Under the assumptions I′ and II′, the asymptotic nature of solutions of (3. 1) can be analyzed by the matching method. If the assumption III′ is satisfied in addition we get an inner solution similar to that of Theorem 1. Let $x=s\cdot\varepsilon^{(n-m)/(n-m+q)}$, $\varepsilon=\rho^{n-m+q}$, S be a sector of the s-plane of central angle less than $(n-m)\pi/(n-m+q)$ whose boundary lines do not coincide with any singular direction $\mathrm{Re}(\lambda_j-\lambda_k)s^{(n-m+q)/(n-m)}=0$ and contain them in its inside for every j,k $(j\neq k)$ where each λ_j is zero or a root of $x^{n-m}=1$, and let Y be a column vector consists of $(y_1,\cdots,y_m,y_{m+1},\cdots,y_n)=(y,y'\cdots,y^{(m-1)},y^{(m)},\varepsilon y^{(m+1)},\cdots,\varepsilon y^{(n-1)})$. Then by a little changes of the construction of inner solution in [4] we can prove a following theorem.

THEOREM 2. *For every positive integer* r, *there exists a domain* D *of the* s,ρ-*plane defined by*

$$\arg s\in S,\qquad 0<|\rho|\leqq\rho_1,\qquad |\arg\rho|\leqq\delta_1,\qquad |s^{1/(n-m)}\rho|\leqq c_1 \tag{3.3}$$

and there exists a fundamental matrix solution $Y(s, \rho)$ *of the form*

$$(3.4)\qquad \begin{aligned} &Y(s, \rho)-\Omega(\rho^{n-m})\cdot s^{k(s)})\cdot \sum_{\nu=0}^{r} Y_\nu(s)[s^{k(s)\cdot 1/(n-m)}\rho]^\nu\cdot s^{k(s)R}\exp[Q(s)] \\ &=\Omega(\rho^{n-m}\cdot s^{k(s)})E_r(s, \rho)[s^{k(s)1/(n-m)}\cdot\rho]^{m+1}\cdot s^{k(s)R}\exp[Q(s)], \end{aligned}$$

where $Y_\nu(s)$ *and* $E_r(s, \rho)$ *are bounded in* D. *Here the matrix* R *is constant and diagonal,* $Q(s)$ *is a diagonal matrix whose diagonal elements are* 1 *or polynomials of* $s^{1/(n-m)}$ *of degree* $n-m+q$, *the fuuction* $k(s)$ *is the same function as in* (2.10), *and the matrix* $\Omega(t)$ *is*

$$\Omega(t)=\begin{bmatrix} t^m & & & & & \\ & \ddots & & & 0 & \\ & & t_1 & & & \\ & & & t^{q/(n-m)} & & \\ & 0 & & & \ddots & \\ & & & & & t^{(n-m-1)q/(n-m)} \end{bmatrix}.$$

REMARK. When the condition II′ does not satisfied, it may be possible to give a sufficient condition to ensure that the inner domains cover a full neighborhood of the turning point, but we do not go into the details.

REEERENCES

[1] IWANO, M., AND Y. SIBUYA, Reduction of the order of a linear ordinary differential equations containing a parameter. Kōdai Math. Sem. Rep. **15** (1963), 1–28.

[2] HANSON, R. J., Reduction and classification of certain turning point problems for systems. Thesis, Univ. of Wisconsin (1964).

[3] NISHIMOTO, T., On a matching method for a linear ordinary differential equation containing a parameter, I; II; III. Kōdai Math. Sem. Rep. **17** (1965), 307–328; **18** (1966), 61–86; **19** (1967), 80–94.

[4] NISHIMOTO, T., A turning point problem of an n-th order differential equation of hydrodynamic type. Kōdai Math. Sem. Rep. **20** (1968), 218–256.

[5] SIBUYA, Y., Formal solutions of a linear ordinary differential equation of the n-th order at a turning point. Funkc. Ekvac. **4** (1962), 115–139.

[6] WASOW, W., Simplification of turning point problems for systems of linear differential equations. Trans. Amer. Math. Soc. **106** (1963), 100–114.

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