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INTEGRAL INEQUALITIES IN A COMPACT ORIENTABLE MANIFOLDS, RIEMANNIAN OR KÄHLERIAN

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Introduction. Obata [2] has recently obtained some integral inequalities satisfied by a function f in a compact orientable Riemannian manifold. In this paper, we study integral inequalities satisfied by a vector field in a compact orientable Riemannian manifold and in a compact Kählerian manifold.

§ 1. Integral inequalities in a compact orientable Riemannian manifold.

Let M be an n-dimensional compact orientable Riemannian manifold. We denote by d the operator which operates on a skew symmetric tensor of degree p, u: $u_{i_1\cdots i_p}$ and gives a skew symmetric tensor of degree $p+1$,

$$du: \quad \nabla_i u_{i_1\cdots i_p} - \nabla_{i_1} u_{i i_2\cdots i_p} \cdots - \nabla_{i_p} u_{i_1\cdots i_{p-1} i},$$

by δ the operator which operates on u and gives a skew symmetric tensor of degree $p-1$,

$$\delta u: \quad g^{ji} \nabla_j u_{i i_2\cdots i_p},$$

and by D the operator which operates on u and gives a skew symmetric tensor of degree $p+1$,

$$Du: \quad \nabla_i u_{i_1\cdots i_p} + \nabla_{i_1} u_{i i_2\cdots i_p} + \cdots + \nabla_{i_p} u_{i_1\cdots i_{p-1} i},$$

∇_j being the operator of covariant differentiation with respect to the Christoffel symbols $\{{}_j{}^h{}_i\}$ formed with the fundamental tensor g_{ji} of M. Furthermore we denote by Δ the operator $\delta d + d\delta$ and by $\square$ the operator $\delta D - D\delta$. For a vector u, we have

$$\Delta u: \quad g^{ij} \nabla_i \nabla_j u_h - K_h{}^i u_i,$$

and

$$\square u: \quad g^{ij} \nabla_i \nabla_j u_h + K_h{}^i u_i,$$

$K_h{}^i$ being the Ricci tensor. We define the global inner product of two tensors $a_{i_1\cdots i_p}$ and $b_{i_1\cdots i_p}$ of the same order p by

$$(a, b) = \frac{1}{p!} \int_M a_{i_1\cdots i_p} b^{i_1\cdots i_p} d\sigma,$$

$d\sigma$ being the volume element of the manifold M.

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At first we get by virtue of Ricci identity, the following equation in M for any vector field v^i,

$$\begin{aligned} \nabla_i[(\nabla_j v^j)v^i] &= (\nabla_i\nabla_j v^j)v^i + (\nabla_j v^j)(\nabla_i v^i) \\ &= (\nabla_j\nabla_i v^j)v^i - K_{ij}v^i v^j + (\nabla_i v^i)^2. \end{aligned} \tag{1}$$

Similarly, we have

$$\nabla_j[(\nabla_i v^j)v^i] = (\nabla_j\nabla_i v^j)v^i + (\nabla_i v^j)(\nabla_j v^i). \tag{2}$$

Thus, applying Green's theorem [3] to (1) and (2), we get

$$\int_M [(\nabla_j\nabla_i v^j)v^i - K_{ij}v^i v^j + (\nabla_i v^i)^2]d\sigma = 0, \tag{A}$$

and

$$\int_M [(\nabla_j\nabla_i v^j)v^i + (\nabla_i v^j)(\nabla_j v^i)]d\sigma = 0, \tag{B}$$

respectively, and, subtracting (A) from (B), the integral formula

$$\int_M [K_{ij}v^i v^j + (\nabla_i v_j)(\nabla^j v^i) - (\nabla_i v^i)^2]d\sigma = 0, \tag{C}$$

where $\nabla^j = g^{ij}\nabla_i$.

On the other hand, we have the following identities;

$$(\nabla_j v_i)(\nabla^i v^j) = (\nabla_j v_i)(\nabla^j v^i) - \frac{1}{2}(\nabla_j v_i - \nabla_i v_j)(\nabla^j v^i - \nabla^i v^j), \tag{3}$$

$$(\nabla_j v_i)(\nabla^i v^j) = -(\nabla_j v_i)(\nabla^j v^i) + \frac{1}{2}(\nabla_j v_i + \nabla_i v_j)(\nabla^j v^i + \nabla^i v^j). \tag{4}$$

Substituting (3) and (4) into (C), we get respectively

$$\int_M \left[K_{ij}v^i v^j + (\nabla_i v_j)(\nabla^i v^j) - \frac{1}{2}(\nabla_i v_j - \nabla_j v_i)(\nabla^i v^j - \nabla^j v^i) - (\nabla_i v^i)^2\right]d\sigma = 0, \tag{D}$$

$$\int_M \left[K_{ij}v^i v^j - (\nabla_i v_j)(\nabla^i v^j) + \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)(\nabla^i v^j + \nabla^j v^i) - (\nabla_i v^i)^2\right]d\sigma = 0. \tag{E}$$

From these equations we obtain the following

PROPOSITION 1. *In an $n(\geqq 2)$ dimensional compact orientable Riemannian manifold M, we have the following integral inequalities for any vector field v:*

1) $$(Kv, v) \geqq -(\nabla v, \nabla v).$$

The equality occurs if and only if the vector field v is harmonic.

2) $$(Kv, v) \leqq (dv, dv) + (\delta v, \delta v).$$

The equality occurs if and only if the vector field v is parallel.

3) $$(Kv, v) \leqq (dv, dv) + \frac{n-1}{n}(\delta v, \delta v).$$

The equality occurs if and only if the vector field v is concircular.

Proof. From (D), we immediately obtain inequalities 1) and 2) and in 1) the equality occurs if and only if

$$\frac{1}{2}(\nabla_j v_i - \nabla_i v_j)(\nabla^j v^i - \nabla^i v^j) + (\nabla_i v^i)^2 = 0,$$

from which we have $\nabla_j v_i - \nabla_i v_j = 0$ and $\nabla_i v^i = 0$, which mean that the vector v^i is harmonic. In 2) the equality occurs if and only if $(\nabla_i v_j)(\nabla^i v^j) = 0$, which shows that the vector v^i is parallel.

On the other hand, the inequality

$$\left[\nabla_i v_j - \frac{1}{n}(\nabla_a v^a) g_{ij}\right]\left[\nabla^i v^j - \frac{1}{n}(\nabla_a v^a) g^{ij}\right] \geqq 0$$

that is,

$$(\nabla_i v_j)(\nabla^i v^j) \geqq \frac{1}{n}(\nabla_a v^a)^2$$

is valid for any vector v^i and the equality occurs if and only if the vector v^i is concircular. Substituting the above inequality into (D), we obtain 3), and we complete the proof.

On the other hand, we know the following theorem proved by Ishihara and Tashiro [1]:

THEOREM A. *If an n-dimensional compact Riemannian manifold admits a non-homothetic concircular transformation, then the manifold is conformally diffeomorphic to a sphere in an* $(n+1)$*-dimensional Euclidean space and vice versa.*

Using theorem A, we can restate 3) of Proposition 1 as follows:

COROLLARY 1. *In an* $n(\geqq 2)$ *dimensional compact simply connected orientable Riemannian manifold M, we have*

$$(Kv, v) \leqq (dv, dv) + \frac{n-1}{n}(\delta v, \delta v)$$

for any vector field v. For a non-parallel vector field v, the equality occurs if and only if the manifold M is conformally diffeomorphic to a sphere in an $(n+1)$*-dimensional Euclidean space.*

REMARK. If $v_i = \nabla_i f$, Corollary 1 reduces to

COROLLARY 2. *Let M be an* $n(\geqq 2)$ *dimensional compact Riemannian manifold. If f is a function over M, we have*

$$(K \operatorname{grad} f, \operatorname{grad} f) \leqq \frac{n-1}{n}(\Delta f, \Delta f).$$

For non-constant function f, the equality occurs if and only if the manifold is conformally diffeomorphic to a sphere in an $(n+1)$-dimensional Euclidean space.

Corollary 2 was proved by Obata [2].

PROPOSITION 2. *In an $n(\geqq 2)$ dimensional compact simply connected orientable Riemannian manifold, we have, for any vector field v, the following integral inequalities:*

1) $$(Kv, v) \leqq (\nabla v, \nabla v) + (\delta v, \delta v).$$

The equality occurs if and only if the vector field v is a Killing vector.

2) $$(Kv, v) \leqq (\nabla v, \nabla v) + \frac{n-2}{n}(\delta v, \delta v).$$

The equality occurs if and only if the vector field v is a conformal Killing vector.

3) $$(Kv, v) \geqq -(\underset{v}{\mathcal{L}} g, \underset{v}{\mathcal{L}} g) + (\delta v, \delta v).$$

The equality occurs if and only if the vector field v is parallel.

4) $$(Kv, v) \geqq -(\underset{v}{\mathcal{L}} g, \underset{v}{\mathcal{L}} g) + \frac{n+1}{n}(\delta v, \delta v).$$

The equality occurs if and only if the vector field v is concircular and for non-parallel vector field v the equality occurs if and only if the manifold is conformally diffeomorphic to a sphere in an $(n+1)$-dimensional Euclidean space.

Proof. The inequality 1) is obvious by (E) and the equality occurs if and only if $(\nabla_i v_j + \nabla_j v_i)(\nabla^i v^j + \nabla^j v^i) = 0$, which shows that the vector v^i is a Killing vector.

On the other hand, we have the inequality

$$\left[\nabla_i v_j + \nabla_j v_i - \frac{2}{n}(\nabla_a v^a) g_{ij}\right]\left[\nabla^i v^j + \nabla^j v^i - \frac{2}{n}(\nabla_a v^a) g^{ij}\right] \geqq 0,$$

that is,

$$(\nabla_i v_j + \nabla_j v_i)(\nabla^i v^j + \nabla^j v^i) \geqq \frac{4}{n}(\nabla_a v^a)^2,$$

which is valid for any vector and the equality occurs if and only if the vector v^i is a conformal Killing vector. Substituting the above inequality into (E), we get 2). Using (E) instead of (D) in 2) and 3) of Proposition 1, we can easily prove 3) and 4).

§ 2. Integral inequalities on a compact Kählerian manifold.

Let M be a Kählerian manifold, that is, M is an even-dimensional space with a mixed tensor $F_i{}^j$ and with a Riemannian metric g_{ij} which satisfies the following conditions:

$$F_j{}^i F_i{}^h = -\delta_j^h,$$

$$F_j{}^t F_i{}^s g_{ts} = g_{ji},$$

and

$$\nabla_j F_i{}^h = 0.$$

First, we recall some important formulas in the theory of Kählerian manifold [3]:

$$K_i{}^a F_a{}^h = -\frac{1}{2} K_{kji}{}^h F^{kj}, \tag{5}$$

$$K_i{}^a F_a{}^h = F_i{}^a K_a{}^h. \tag{6}$$

We assume that the Kählerian manifold M is compact in the discussion which follows.

REMARK. For a harmonic vector in a compact Kählerian manifold, we know the following theorem [3]:

THEOREM B. *A necessary and sufficient conditions for a vector v_i in a compact Kählerian manifold to be covariant analytic is that the vector v_i is harmonic.*

Combining 1) of Proposition 1 and Theorem B, we obtain

COROLLARY. *In a compact Kählerian manifold, for any vector field v we have the following integral inequality:*

$$(Kv, v) \geqq -(\nabla v, \nabla v).$$

The equality occurs if and only if the vector field v is covariant analytic.

Next, in a compact orientable Riemannian manifold M applying Green's formula

$$\int_M g^{ij} \nabla_i \nabla_j f d\sigma = 0$$

to $f = (1/2) v^i v_i$, we find that

$$\int_M [(g^{ij} \nabla_i \nabla_j v^h) v_h + (\nabla^j v^i)(\nabla_j v_i)] d\sigma = 0. \tag{F}$$

Forming the difference (F)$-$(D) and the sum (F)$+$(E), we obtain respectively

$$\int_M \left[(g^{ij} \nabla_i \nabla_j v^h - K_i{}^h v^i) v_h + \frac{1}{2} (\nabla_i v_j - \nabla_j v_i)(\nabla^i v^j - \nabla^j v^i) + (\nabla_i v^i)^2 \right] d\sigma = 0, \tag{G}$$

$$\int_M \left[(g^{ij} \nabla_i \nabla_j v^h + K_i{}^h v^i) v_h + \frac{1}{2} (\nabla_i v_j + \nabla_j v_i)(\nabla^i v^j + \nabla^j v^i) - (\nabla_i v^i)^2 \right] d\sigma = 0. \tag{H}$$

On the other hand, we get the following two pairs of equations which are valid for an arbitrary vector v^i in a Kählerian manifold, by virtue of (5), (6) and the Ricci identity:

$$\begin{cases} \dfrac{1}{2}(F^{js}\nabla_s v^i - F^{is}\nabla^j v_s)(F_j{}^r\nabla_r v_i - F_i{}^r\nabla_j v_r) = (\nabla^j v^i)(\nabla_j v_i) - F^{js}F^{ir}(\nabla_s v_i)(\nabla_j v_r), \\ \nabla_j[(\nabla^j v_i)v^i - (F^{js}F^{ir}\nabla_s v_i)v_r] = (g^{ts}\nabla_t\nabla_s v_i - K_i{}^h v_h)v^i + (\nabla^j v^i)(\nabla_j v_i) - F^{js}F^{ir}(\nabla_s v_i)(\nabla_j v_r), \end{cases}$$

$$\begin{cases} \dfrac{1}{2}(F^{js}\nabla_s v^i - F_s{}^i\nabla^j v^s)(F_j{}^r\nabla_r v_i - F_{ri}\nabla_j v^r) = (\nabla^j v^i)(\nabla_j v_i) - F^{js}F^{ri}(\nabla_s v_i)(\nabla_j v_r), \\ \nabla_j[(\nabla^j v_i)v^i - (F^{js}F^{ri}\nabla_s v_i)v_r] = (g^{ts}\nabla_t\nabla_s v_i + K_i{}^h v_h)v^i + (\nabla^i v^j)(\nabla_i v_j) - F^{js}F^{ri}(\nabla_s v_i)(\nabla_j v_r). \end{cases}$$

Accordingly, we obtain

$$\begin{aligned} &\nabla_j[(\nabla^j v_i)v^i - (F^{js}F^{ir}\nabla_s v_i)v_r] \\ &\quad = (g^{ts}\nabla_t\nabla_s v_i - K_i{}^h v_h)v^i + \frac{1}{2}(F^{js}\nabla_s v^i - F^{is}\nabla^j v_s)(F_j{}^r\nabla_r v_i - F_i{}^r\nabla_j v_r), \end{aligned} \tag{7}$$

$$\begin{aligned} &\nabla_j[(\nabla^j v_i)v^i - (F^{js}F^{ri}\nabla_s v_i)v_r] \\ &\quad = (g^{ts}\nabla_t\nabla_s v_i + K_i{}^h v_h)v^i + \frac{1}{2}(F^{js}\nabla_s v^i - F_s{}^i\nabla^j v^s)(F_j{}^r\nabla_r v_i - F_{ri}\nabla_j v^r). \end{aligned} \tag{8}$$

Applying Green's theorem, we obtain the following integral formulas in a compact Kählerian manifold M:

$$\int_M\Big[(g^{ts}\nabla_t\nabla_s v_i - K_i{}^h v_h)v^i + \frac{1}{2}(F^{js}\nabla_s v^i - F^{is}\nabla^j v_s)(F_j{}^r\nabla_r v_i - F_i{}^r\nabla_j v_r)\Big]d\sigma = 0, \tag{I}$$

$$\int_M\Big[(g^{ts}\nabla_t\nabla_s v_i + K_i{}^h v_h)v^i + \frac{1}{2}(F^{js}\nabla_s v^i - F_s{}^i\nabla^j v^s)(F_j{}^r\nabla_r v_i - F_{ri}\nabla_j v^r)\Big]d\sigma = 0. \tag{J}$$

Forming the difference (H)−(I) and (J)−(G), we obtain respectively

$$\begin{aligned} &\int_M\Big[2K_{ij}v^i v^j + \frac{1}{2}(\nabla^i v^j + \nabla^j v^i)(\nabla_i v_j + \nabla_j v_i) - (\nabla_i v^i)^2 \\ &\qquad\qquad - \frac{1}{2}(F^{js}\nabla_s v^i - F^{is}\nabla^j v_s)(F_j{}^r\nabla_r v_i - F_i{}^r\nabla_j v_r)\Big]d\sigma = 0, \end{aligned} \tag{K}$$

$$\begin{aligned} &\int_M\Big[2K_{ij}v^i v^j - \frac{1}{2}(\nabla_i v_j - \nabla_j v_i)(\nabla^i v^j - \nabla^j v^i) - (\nabla_i v^i)^2 \\ &\qquad\qquad + \frac{1}{2}(F^{js}\nabla_s v^i - F_s{}^i\nabla^j v^s)(F_j{}^r\nabla_r v_i - F_{ri}\nabla_j v^r)\Big]d\sigma = 0. \end{aligned} \tag{L}$$

From equations (K) and (L), we obtain the following Propositions.

PROPOSITION 3. *In a compact Kählerian manifold, for any vector field v, we have*

$$(Kv, v) \geqq \frac{1}{2}[(\delta v, \delta v) - (\underset{v}{\mathcal{L}} g, \underset{v}{\mathcal{L}} g)].$$

The equality occurs if and only if the vector field v is covariant analytic.

Proof. From (K), we can easily obtain the above inequality and we can see that the equality occurs if and only if

$$(F^{js}\nabla_s v^i - F^{is}\nabla^j v_s)(F_j{}^r\nabla_r v_i - F_i{}^r\nabla_j v_r) = 0,$$

which means that the vector v^i is covariant analytic.

PROPOSITION 4. *In a compact Kählerian manifold, for any vector field v, we have*

$$(\Box v, v) \leqq 0.$$

The equality occurs if and only if the vector field v is contravariant analytic.

Proof. From (L), we obtain

$$(Kv, v) \leqq \frac{1}{2}[(dv, dv) + (\delta v, \delta v)].$$

On the other hand we know that the equation $(\Delta v, v) + (dv, dv) + (\delta v, \delta v) = 0$ is valid for any vector v [3], so we have

$$(Kv, v) + \frac{1}{2}(\Delta v, v) \leqq 0, \qquad \text{that is,} \quad (2Kv + \Delta v, v) \leqq 0,$$

which shows that the above inequality is valid. The equality occurs if and only if

$$(F^{js}\nabla_s v^i - F_s{}^i\nabla^j v^s)(F_j{}^r\nabla_r v_i - F_{ri}\nabla_j v^r) = 0,$$

which means that the vector v^i is contravariant analytic.

COROLLARY 1. *In a compact Kählerian manifold, for any scalar function f, we have*

$$(K \operatorname{grad} f, \operatorname{grad} f) \leqq \frac{1}{2}(\Delta f, \Delta f).$$

The equality occurs if and only if the grad $f = f_i$ *is contravariant analytic.*

COROLLARY 2. *In an n-dimensional compact Kähler-Einstein space M, if f is a proper function of Δ corresponding to the eigenvalue λ (=constant), then we have*

$$\lambda \leqq -\frac{2K}{n},$$

where $K = g^{ij}K_{ij}$ is a scalar curvature. The equality occurs if and only if $\nabla_i f = f_i$ is contravariant analytic.

Proof. Substituting the conditions that $\Delta f=\lambda f$ and the space M is an Einstein space ($K_{ij}=(K/n)g_{ij}$, K=constant) into the inequality

$$\int_M K_{ji}f^i f^j d\sigma \leqq \frac{1}{2}\int_M (\Delta f)^2 d\sigma$$

which is valid for any scalar function f by Corollary 1, we have

$$\frac{K}{n}\int_M f_i f^i d\sigma \leqq \frac{\lambda^2}{2}\int_M f^2 d\sigma.$$

On the other hand, applying Green's theorem to $\nabla_i(f\cdot f^i)=f_i f^i+f(\Delta f)$, we get

$$\int_M f_i f^i d\sigma = -\int_M f(\Delta f)d\sigma = -\lambda\int_M f^2 d\sigma.$$

Therefore we have

$$-\frac{K}{n}\lambda\int_M f^2 d\sigma \leqq \frac{\lambda^2}{2}\int_M f^2 d\sigma,$$

for any scalar function f. Accordingly, we have

$$-\frac{K}{n}\lambda \leqq \frac{\lambda^2}{2}, \qquad \text{that is,} \quad \lambda\left(\frac{K}{n}+\frac{\lambda}{2}\right)\geqq 0.$$

We see by easy computations that the constant λ appearing in $\Delta f=\lambda f$ is necessarily negative, we have $\lambda\leqq -2K/n$.

REMARK. We know the following theorem of Yano [3]:
If a compact Kähler-Einstein space with $K>0$ admits a Killing vector field v^i, then the equation $\Delta f=-(2K/n)f$ admits a solution other than zero given by $f=(n/2K)F^{ji}\nabla_j v_i$ and vice versa.

Thus we conclude that $-2K/n$ in $\lambda\leqq -2K/n$ of Corollary 2 is best possible.

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