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ON THE SOLUTION OF THE FUNCTIONAL EQUATION $f\circ g(z)=F(z)$, II

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In this note we shall prove some extensions of our previous results on the functional equation $f\circ g(z)=F(z)$. We have always assumed that F is an entire function of finite order in [3]. In this note we shall be mainly concerned with a special type of entire functions of infinite order as F.

THEOREM 1. *Assume that $F(z)$ has the form $P(z)e^{H(z)}$ with a polynomial $P(z)$, which is not a constant, and with an entire function $H(z)$ of order less than* 1 *(which does not exclude a polynomial). Then the functional equation $f\circ g(z)=F(z)$ has no transcendental entire solutions f and g.*

Proof. Let $\{w_n\}$ be the set of solutions of $f(w)=0$. If the set $\{w_n\}$ contains at least two points w_1 and w_2, at least one of $g(z)=w_j$, $j=1, 2$, contains an infinite number of solutions, which must be the zeros of $P(z)$. This is a contradiction. Hence $f(w)=0$ has at most one solution. If there is no solution of $f(w)=0$, then $P(z)$ must be a constant, which is absurd. Hence $f(w)=0$ has just one solution. Therefore $f(w)=A(w-w_1)^n e^{L(w)}$, $L(0)=0$. Next consider the equation $g(z)=w_1$. This has only a finite number of solutions. Hence $g(z)=w_1+Q(z)e^{M(z)}$, $M(0)=0$, where $Q(z)$ is a polynomial. Therefore

$$P(z)e^{H(z)}=AQ(z)^n e^{nM(z)}\exp(L(w_1+Q(z)e^{M(z)})).$$

This implies that $P(z)=BQ(z)^n$ and $H(z)=nM(z)+L(w_1+Q(z)e^{M(z)})+C$ with two constants B and C. However the second equation implies that the order of $H(z)$ is not less than 1, which contradicts the assumption on the order of $H(z)$.

THEOREM 2. *Assume that $F(z)$ has the form $P(z)\exp(\exp(z))$ with a non-constant polynomial $P(z)$. Then the functional equation $f\circ g(z)=F(z)$ has no pair of transcendental entire solutions f and g.*

Proof. By the proof of theorem 1 we also have

$$e^z=nM(z)+L(w_1+Q(z)e^{M(z)})+C, \qquad P(z)=BQ(z)^n.$$

If $M(z)$ is a transcendental entire function or a polynomial of degree greater than 1, then the order of $L(w_1+Q(z)e^{M(z)})$ is not less than 2. Further

$$T(r, M)=o(T(r, e^M))=o(T(r, L(w_1+Qe^M)))$$

outside a set of finite measure. Hence the order of $nM(z)+L(w_1+Q(z)e^{M(z)})+C$ is

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not less than 2, which is a contradiction. Hence $M(z)$ must be of the form αz with a non-zero constant α. If L(w) is a transcendental entire function,

$$T(r, M)=o(T(r, e^M)) \quad \text{and} \quad T(r, e^M)=o(T(r, L(w_1+Qe^M)))$$

outside a set of finite measure. This implies that

$$T(r, e^z)\sim T(r, L(w_1+Qe^M)),$$

which is a contradiction. Hence $L(w)$ must be a polynomial. In this case we have the following identity:

$$e^z+Q_pe^{\alpha pz}+Q_{p-1}e^{\alpha(p-1)z}+\cdots+Q_1e^{\alpha z}=n\alpha z+D$$

with suitable polynomials $Q_p, \cdots, Q_1$, among which Q_p is not a constant, and with a suitable constant D. However the above identity leads to a contradiction. This is not trivial. But we can easily modify the reasoning of Nevanlinna's extension [2] of Borel's formulation of Picard's theorem and then we can make use of the fact $Q_p \fallingdotseq$ a constant. See also [1].

As a variant of theorem 1 we have the following theorem.

THEOREM 3. *Assume that $F'(z)$ has the form $P(z)e^{H(z)}$ with a non-constant polynomial $P(z)$ and with an entire function $H(z)$ of order less than* 1. *Then the functional equation $f\circ g(z)=F(z)$ has no pair of transcendental entire solutions f and g.*

Proof. We shall start from the functional equation

$$f'\circ g(z)\cdot g'(z)=P(z)e^{H(z)}.$$

Again by the same reasoning we have the following two possibilities:

Case i) $f'(w)=A(w-w_1)^n e^{L(w)}$, $L(0)=0$, $g(z)=w_1+Q(z)e^{M(z)}$ and $g'(z)=R(z)e^{N(z)}$ with two polynomials $Q(z)$ and $R(z)$, and

Case ii) $f'(w)=e^{L(w)}$ and $g'(z)=Q(z)e^{M(z)}$.

In the first case we have

$$(Q'+QM')e^M=Re^N$$

and

$$AQ^ne^{nM}e^{L(w_1+Qe^M)}Re^N=Pe^H.$$

Hence we have

$$BQ^nR=P$$

and

$$nM+L(w_1+Qe^M)+N=H+C$$

with two constants B and C. Assume that $M(z)$ is not a polynomial. Then $Q'+QM=0$ has only a finite number of solutions and hence $Q'+QM'=Re^K$. Hence

$$K+M=N+D$$

with a constant D. Evidently

$$T(r, N)\leqq T(r, M)+T(r, K)+O(\log r)$$
$$=o(T(r, e^{M}))$$

outside a set of finite measure. Hence we have

$$T(r, H)=T(r, L(w_1+Qe^{M}))+o(T(r, e^{M})),$$

which shows that the order of $H(z)$ is not less than 1. This is a contradiction. Assume that $M(z)$ is a polynomial. Then $M(z)=N(z)+E$ with a constant E. This implies that

$$T(r, H)=T(r, L(w_1+Qe^{M}))+o(T(r, e^{M})),$$

which is again a contradiction.

In the second case we have

$$\exp\left(L\left(B+\int_0^z Q(t)e^{M(t)}dt\right)\right)Q(z)e^{M(z)}=P(z)e^{H(z)}.$$

Hence $Q=CP$ and

$$L\left(B+\int_0^z Q(t)e^{M(t)}dt\right)+M(z)=H(z)+D$$

with two constants C and D. Here we remark that every entire function and its derivatives have the same order. Hence the order of $H(z)$ is not less than that of e^{M}, which is not less than 1. This is absurd.

THEOREM 4. *Assume that $F'(z)$ has the form $P(z)\exp(\exp(z))$ with a non-constant polynomial $P(z)$. Then the functional equation $f\circ g(z)=F(z)$ has no pair of transcendental entire solutions f and g.*

Proof. Again we have two possibilities:

Case i). $f'(w)=A(w-w_1)^n e^{L(w)}$, $L(0)=0$, $g(z)=w_1+Q(z)e^{M(z)}$, $M(0)=0$ and $g'(z)$ $R(z)e^{N(z)}$, and

Case ii). $f'(w)=e^{L(w)}$ and $g'(z)=Q(z)e^{M(z)}$, $M(0)=0$.

In the first case we similarly have the following facts: $M(z)=\alpha z$ and L is a polynomial. Then we have the following identity:

$$e^{z}+Q_p e^{\alpha n z}+\cdots+Q_1 e^{\alpha z}=(n+1)\alpha z+E, \qquad \alpha\neq 0$$

with a constant E and polynomials $Q_1, \cdots, Q_p$ and Q_p is not a constant. Again by Nevanlinna's extension of Borel's formulation of Picard's theorem we have a contradiction.

In the second case we have

$$\exp\left(L\left(B+\int_0^z Q(t)e^{M(t)}dt\right)\right)Q(z)e^{M(z)}=P(z)e^{e^{z}}$$

and hence $Q(z)=CP(z)$ and

$$L\left(B+\int_0^z Q(t)e^{M(t)}dt\right)+M(z)=e^{z}+D.$$

By this identity and by the fact that every entire function and its derivatives have the same order, $M(z)$ must be of the form αz and L must be a polynomial. By a simple calculation we have the following identity:

$$e^{z}+Q_{p}e^{p\alpha z}+\cdots+Q_{1}e^{\alpha z}=\alpha z+E$$

with a constant E and with a non-constant polynomial Q_p and polynomials $Q_{p-1},\cdots,Q_1$. Again we have a contradiction.

In theorems 3 and 4 the non-constancy of $P(z)$ does not play an essential role. By a little more precise examination of Nevanlinna's extension of Borel's formulation of Picard's theorem we can drop the non-constancy assumption in theorems 3 and 4. Hence the functional equation $f\circ g(z)=F(z)$ for

$$F=\int_0^z e^{H(z)}dz \qquad \left(\text{or } \int_0^z e^{e^z}dz\right)$$

has no pair of entire transcendental solutions f and g, if $H(z)$ is an entire function of order less than 1.

We can prove several other results. For example the functional equations

$$f\circ g(z)=P(z)e^{\sin z} \qquad \text{and} \qquad f\circ g(z)=\int_0^z P(z)e^{\sin z}dz$$

have no pair of transcendental entire solutions f and g, respectively, when $P(z)$ is a non-constant polynomial.

We shall extend theorem 2 to the following theorem 5. In order to prove it we need the following two lemmas.

LEMMA 1. *Let f and g be two transcendental entire functions of finite order. Then*

$$\overline{\lim_{r\to\infty}}\frac{\log\log\log M_{f\circ g}(r)}{\log r}\leqq\overline{\lim_{r\to\infty}}\frac{\log\log M_g(r)}{\log r}.$$

Proof. By the maximum principle

$$M_{f\circ g}(r)\leqq M_f\circ M_g(r).$$

Let ρ_f and ρ_g denote the orders of f and g, respectively. For an arbitrary given positive number ε there exists an r_0 such that for any $r\geqq r_0$

$$M_f(r)\leqq e^{r^{\rho_f+\varepsilon}}, \qquad M_g(r)\leqq e^{r^{\rho_g+\varepsilon}}.$$

Hence

$$M_{f\circ g}(r)\leqq\exp\left(e^{r^{\rho_g+\varepsilon}}\right)^{\rho_f+\varepsilon}=\exp\left(e^{(\rho_f+\varepsilon)r^{\rho_g+\varepsilon}}\right),$$

which implies that

$$\overline{\lim_{r\to\infty}}\frac{\log\log\log M_{f\circ g}(r)}{\log r}\leqq\rho_g+\varepsilon.$$

Here ε is arbitrary. We have the desired result.

LEMMA 2. *Let $F(z)$ be an entire function satisfying*

$$\hat{\rho}_F \equiv \overline{\lim_{r\to\infty}} \frac{\log\log\log M_F(r)}{\log r} < \infty,$$

and

$$\rho_{N(r;0,F)} \equiv \overline{\lim_{r\to\infty}} \frac{\log N(r;0,F)}{\log r} < \hat{\rho}_F.$$

If there is a pair of transcendental entire solutions f and g of the functional equation $f\circ g(z)=F(z)$, then $f(w)$ is of the form

$$f(w)=A(w-w_1)^n e^{M(w)},$$

unless it is of infinite order.

Proof. Assume that $f(w)$ is of finite order and $f(w)=0$ has at least two roots w_1 and w_2. Then

$$N(r;0,F)=N(r;0,f\circ g)\geqq N(r;w_1,g)+N(r;w_2,g)$$
$$\geqq m(r,g)-O(\log rm(r,g))$$

outside a set of finite measure firstly by the second fundamental theorem for g and then without any exceptional set. Hence

$$\rho_g \leqq \rho_{N(r;0,F)} < \hat{\rho}_F.$$

On the other hand we have $\hat{\rho}_F \leqq \rho_g$ by Lemma 1. This is a contradiction. Therefore $f(w)$ has only one zero w_1, which implies the desired result.

THEOREM 5. *Let $L(z)$ be a transcendental entire function of order less than 1 and P a polynomial. Then the functional equation $f\circ g(z)=L(z)\exp(Pe^z)$ has no pair of transcendental entire solutions f and g of finite order.*

Proof. By a simple calculation

$$\hat{\rho}_F=1, \qquad F=L\exp(Pe^z)$$

and $\rho_{N(r;0,F)} \leqq \rho_L < 1$. Hence we can make use of Lemma 2. Therefore $f(w)$ has only one zero w_1 and it has the following form:

$$f(w)=A(w-w_1)^n e^{K(w)}.$$

By the given functional equation $g(z)$ must have the following form

$$g(z)=w_1+Q(z)e^{N(z)}, \qquad N(0)=0.$$

Therefore we have the following functional equations

$$L(z)=DQ(z)^n \qquad \text{and} \qquad P(z)e^z+C=nN(z)+K(w_1+Q(z)e^{N(z)})$$

with two constants C and D. By $\rho_g \leqq \rho_L < 1$ $N(z)$ vanishes identically and $Q(z)$ is an entire function of order less than 1. Then we have

$$P(z)e^z+C=K(w_1+Q(z)).$$

We already proved that there is no pair of transcendental entire solutions K and Q for the above identity [3]. Hence K must be a polynomial. In this case $K(w_1+Q(z))$ is of order less than 1, which is a contradiction.

THEOREM 6. *Let F be a transcendental entire function of finite order having p non-contiguous asymptotic paths, along which F tends to a finite value, respectively. Further assume that the order of $N(r; A, F)$ for an A is less than $p/2$. Then there is no pair of transcendental entire functions f and g satisfying the functional equation $f\circ g(z)=F(z)$.*

Proof. In this case f must be a transcendental entire function of order zero by Pólya's result [4]. If A is a Picard exceptional value of F, then the desired result holds by our earlier result in [3]. Hence we may assume that $F(z)=A$ has an infinite number of roots. Since f is of order zero, there is an infinite number of solutions of $f(w)=A$. Take two roots w_1 and w_2. Then

$$N(r; A, F)=N(r; A, f\circ g)\geqq(r; w_1, g)+N(r; w_2, g)$$
$$\geqq m(r, g)-O(\log rm(r, g))$$

outside a set of finite measure firstly by the second fundamental theorem for g and then without any exceptional set. Hence

$$\rho_g\leqq\rho_{N(r;A,F)}<\frac{p}{2}.$$

Now consider p non-contiguous asymptotic paths $\Gamma_1, \cdots, \Gamma_p$ of $F(z)$. If $g(\Gamma_1)$ is unbounded, $f\circ g(\Gamma_1)$ is unbounded by Wiman's theorem. This is a contradiction, since $f\circ g(z)=F(z)$ tends to a finite value along Γ_1. If $g(\Gamma_1)$ is bounded but $g(z)$ does not tend to a finite value along Γ_1, then $f\circ g(z)$ does not tend to a single point along Γ_1, which is again untenable. Hence $g(z)$ tends to a finite value along Γ_1. The same holds for each Γ_j. Assume that Γ_1 and Γ_2 are contiguous for $g(z)$. Then there is an unbounded domain D, which is bounded by Γ_1 and Γ_2 and in which $g(z)$ is bounded. In this case $f\circ g(z)$ is bounded in D, that is, Γ_1 and Γ_2 are contiguous for $f\circ g(z)=F(z)$. This is a contradiction. The same holds for $\Gamma_1, \cdots, \Gamma_p$. We have, therefore, that $\Gamma_1, \cdots, \Gamma_p$ are p non-contiguous asymptotic paths of $g(z)$, along which $g(z)$ tends to a finite value, respectively.

Now we can apply the Denjoy-Carleman-Ahlfors theorem and then we have

$$p\leqq 2\rho_g.$$

This contradicts $\rho_g<p/2$. Thus we have the desired result.

THEOREM 7. *Let F be the same as in theorem 6. Assume that the order of $N(r; 0, F')$ is less than $p/2$. Then there is no pair of transcendental entire functions f and g satisfying the functional equation $f\circ g(z)=F(z)$.*

Proof. By our earlier result in [3] we may assume that $F'(z)=0$ has an infinite number of roots. Then by the derived functional equation

$$f'\circ g(z)\cdot g'(z)=F'(z)$$

we have

$$N(r;0,F')=N(r;0,f'\circ g)+N(r;0,g')$$
$$\geqq N(r;w_1,g)+N(r;w_2,g)+N(r;0,g')$$

for two roots w_1 and w_2 of $f'(w)=0$. By the second fundamental theorem

$$N(r;0,F')\geqq m(r,g)-O(\log rm(r,g)),$$

which implies $2\rho_g<p$.

The remaining reasoning is the same as in theorem 6. Hence we have the desired result.

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