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ON CIRCULAR AND RADIAL SLIT DISC MAPPINGS

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§ 1. Introduction.

1. This paper contains ameliorations of some results of Marden and Rodin [7]. Recently Marden and Rodin [7] discussed a circular-radial slit mapping in connection with problems of extremal lengths. For instance they divide boundary components of a plane domain Ω into three sets, α, β and γ, where α is a component and $\alpha\cup\beta$ is closed in the Stoïlow compactification of Ω. They proved that if the α is not so small, a circular-radial slit disc mapping of Ω can be constructed and that the image of Ω under it is bounded by a circle with center at the origin having possible radial incisions, circular slits with possible radial incisions and radial slits under an assumption of β-isolation.

The aims of the present paper are at first to deal with such a mapping without the condition of β-isolation, in the second place to construct a radial-circular slit disc mapping in case where $\alpha\cup\gamma$ is closed, which was treated by them as a dual problem [7] and at last to define a circular and radial slit disc mapping in more general partitions.

We shall also discuss extremal properties of these mappings. Such extremal properties were discussed by Marden and Rodin in connection with the logarithmic area [7]. Our version is more classical. These properties are related to extremal problems treated by Rengel [12] for domains of finite connectivity. One of them was discussed by Reich and Warschawski [11] for circular slit mappings of arbitrary domains and recently by Oikawa [9] for radial slit mappings. The other was due to Grötzsch [6] for radial slit mappings with a restriction which was removed by the author [16].

§ 2. Preliminaries.

2. Let Ω be an open plane domain and let $\hat{\Omega}$ be its Stoïlow ctification [3]. A boundary component σ is defined by a defining sequence $\{\Delta_n\}$ n that the relative boundary of Δ_n is a single Jordan curve, $\Delta_n\supset\bar{\Delta}_{n+1}$ and $\cap\Delta_n=\phi$. Each member of the defining sequence $\{\Delta_n\}$ forms a neighborhood of σ.

The topological representation of $\{\Delta_n\}$ is given by $\cap \mathrm{Cl}(\Delta_n)$ which is denoted by the same letter σ, where $\mathrm{Cl}(*)$ means the closure taken in the Riemann sphere.

Let $T(z)$ be a topological mapping of Ω. $T(z)$ can be extended topologically

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onto its compactification $\hat{\Omega}$. The image of σ defined by $\{\Delta_n\}$ is given by $T(\sigma)$ defined by $\{T(\Delta_n)\}$.

Let C be a closed set of boundary components of σ. Since C is covered by a finite number of members of defining sequences of elements of C, we can construct a defining sequence of C, denoted by $\{D_n\}$, such that D_n consists of a finite number of domain whose relative boundaries are single analytic Jordan curves, $D_n \supset \bar{D}_{n+1}$, and $\cap \mathrm{Cl}(D_n)=C$. $\Omega-\bar{D}_n$ is a domain, denoted by Ω_n. $\{\Omega_n\}$ exhausts Ω which is called an *exhaustion of Ω towards C*.

3. We shall use the method of extremal metrics. Let Γ be a family of curves c running within $\hat{\Omega}$ whose restriction on Ω consists of at most a countable number of locally rectifiable curves in Ω. Let ρ be a measurable metric on Ω which will be used instead of $\rho|dz|$ for short. We mean by $P(\Gamma)$ the admissible class of metrics such that the Lebesgue-Stieltjes integral of ρ along the restriction of c on Ω is defined and satisfies.

$$\int_c \rho|dz| \geqq 1.$$

The module of Γ, denoted by $\mathrm{mod}\,\Gamma$, is defined by

$$\inf_{\rho\in P(\Gamma)} ||\rho||_\Omega^2 = \inf_{\rho\in P(\Gamma)} \iint_\Omega \rho^2 dx\, dy.$$

The extremal length $\lambda(\Gamma)$ is its reciprocal.

Let H be the space of l_2 metrics on Ω. We denote by $P^*(\Gamma)$ the closure of $P(\Gamma)\cap H$ which is called the l_2-admissible class of Γ. Then $\mathrm{mod}\,\Gamma=\infty$, if and only if $P^*(\Gamma)=\phi$. Unless $P^*(\Gamma)=\phi$, there exists a unique metric ρ_0 in H, called the extremal metric, satisfying that $\mathrm{mod}\,\Gamma=||\rho||^2$ [13] and that

$$(1) \qquad ||\rho-\rho_0||^2 \leqq ||\rho||^2-||\rho_0||^2$$

for every $\rho\in P^*(\Gamma)$ [16].

A curve family with vanishing module is called an *exceptional family*. The union of a countable number of exceptional families is exceptional [5]. We say that a proposition holds for almost all (a. a.) $c\in\Gamma$, if it is false only for an exceptional subfamily of Γ. The l_2-admissible class $P^*(\Gamma)$ is equivalent to the class of satisfying

$$\int_c \rho|dz| \geqq 1 \qquad \text{for a.a. } c\in\Gamma$$

[5, 17].

The following lemma will be used frequently.

LEMMA 1. *Let $\{\Gamma_n\}_{n=1}^\infty$ be an increasing sequence of curve families. Put $\Gamma_0 = \cup \Gamma_n$. Then* $\mathrm{mod}\,\Gamma_n$ *tends to* $\mathrm{mod}\,\Gamma_0$. *Furthermore the sequence of the extremal metrics ρ_n tends to the extremal metric ρ_0 of Γ_0 strongly so long as $P^*(\Gamma_0)\neq\phi$.*

The proof of the first half is found in [17] and the convergence of ρ_n is obvious from its proof. Ziemer [19] proved this result for the module of families of complete measures.

4. Let $\Phi(z)$ be a quasiconformal mapping of Ω whose maximal dilatation is K. A curve c in $\hat{\Omega}$ is mapped onto a curve on the Stoïlow compactification of $\Phi(\Omega)$ which is denoted by $\Phi(c)$. The collection of the image curves of Γ is written by $\Phi(\Gamma)$. Then we have

$$\frac{1}{K} \operatorname{mod} \Gamma \leqq \operatorname{mod} \Phi(\Gamma) \leqq K \operatorname{mod} \Gamma. \tag{2}$$

For the proof the readers are referred to [1].

We shall use quasiconformal mappings to modify the conformal structure of Ω and to evaluate modules. We shall need the following lemma later on.

LEMMA 2. *Let Δ and Δ' be subdomains of Ω whose relative boundaries consist of a finite number of analytic closed curves such that $\Delta \supset \bar{\Delta}'$ and $\Delta - \bar{\Delta}'$ is relatively compact. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of univalent functions defined on subdomains Ω_n of Ω such that $\Omega_n \subset \Omega_{n+1}$ and $\cup \Omega_n = \Omega$. Suppose that f_n tends to a univalent function f_0 uniformly on any compact subset of Ω. Then, for a given $\varepsilon > 0$, we can construct a $(1+\varepsilon)$-quasiconformal mapping of a subdomain $\Omega^\varepsilon = (\Omega - \bar{\Delta}) \cup \Omega_n$, denoted by $\Phi^\varepsilon(z)$, such that $\Phi^\varepsilon(z) = f_0(z)$ in $\Omega - \bar{\Delta}$ and $\Phi^\varepsilon(z) = f_n(z)$ in $\Delta' \cap \Omega_n$ for a sufficiently large n.*

Proof. Let D_j $(j=1, 2, \cdots, l)$ be the components of $\Delta - \bar{\Delta}'$. Let C_j be the subset of boundary comsonents of D_j contained in the relative boundary of Δ and let C_j^* be those contained in the relative boundary of Δ'. Denoting by $\omega_j(z)$ the harmonic measure of C_j^* in D_j, we put

$$\Phi^\varepsilon(z) = \begin{cases} f_0(z) & \text{in} \quad \Omega - \Delta, \\ (1-\omega_j(z))f_0(z) + \omega_j(z) f_n(z) & \text{in} \quad D_j, \\ f_n(z) & \text{in} \quad \bar{\Delta}' \cap \Omega_n \end{cases}$$

for so large n that the Ω^ε, defined by $(\Omega - \bar{\Delta}) \cup \Omega_n$, becomes a domain and that

$$\frac{\Phi^\varepsilon(z_1) - \Phi^\varepsilon(z_2)}{z_1 - z_2} \neq 0 \qquad (z_1, z_2 \in D_j, z_1 \neq z_2).$$

Then Φ^ε is univalent in Ω^ε. A simple calculation verifies the statement about the dilatation of Φ^ε for sufficiently large n, which is a desired quasiconformal mapping [1]. Another topological proof of the univalency can be given as in [15].

§ 3. Circular-radial slit mapping.

5. We may assume that Ω is a finite domain. Let α be its outer boundary.

Let (α, A, B) denote a partition of $\partial\Omega$ into three sets. Suppose $\alpha \cup A$ is closed. Let $\{\Omega_n\}$ be an exhaustion of towards $\alpha \cup A$. Let α_n denote the outer boundary of Ω_n and let A_n be the subset of its relative boundary other than α_n. Put $B_n = B \cap \hat{\Omega}_n$. Since B_n is closed, we take an exhaustion $\{\Omega_{nj}\}$ of Ω_n towards B_n. Let B_{nj} be the relative boundary of Ω_{nj} in Ω_n, and let α_{nj} and A_{nj} be $\alpha_n \cap \hat{\Omega}_{nj}$ and $A_n \cap \hat{\Omega}_{nj}$ respectively. Let a be a point of Ω. We agree that every member of its exhaustion contains the point a. There exists a circular-radial slit disc mapping such that

i) $f_{nj}(a)=0, \quad f'_{nj}(a)=1$,

ii) $f_n(\alpha_{nj})$ is a circle $|f_{nj}|=R_{nj}$,

iii) $f_{nj}(A_{nj})$ consists of a finite set of circular slits and

iv) $f_{nj}(B_{nj})$ consists of a finite set of radial slits.

The construction of f_{nj} is now classical and the readers are referred to [7].

6. The function f_{nj} induces two extremal metrics of the following module problems. Let Γ^q_{nj} be the family of curves separating the set $|f_{nj}|=q$ from α_{nj} within $\hat{\Omega}_{nj}-A_{nj}$ and let X^q_{nj} be the family of curves joining them within $\hat{\Omega}_{nj}-B_{nj}$. Then the metrics $\rho_{nj}=|f'_{nj}/(2\pi f_{nj})|$ and $\mu_{nj}=|f'_{nj}/((\log R_{nj}/q)f_{nj})|$ are the extremal metrics for Γ^q_{nj} and X^q_{nj} respectively and we get

$$\operatorname{mod} \Gamma^q_{nj} = \frac{1}{2\pi} \log \frac{R_{nj}}{q}$$

and

$$\operatorname{mod} X^q_{nj} = \frac{2\pi}{\log R_{nj}/q}.$$

The quantity R_{nj} is represented in terms of modules. Let $\Gamma_{nj}(q)$ be the family of curves sparating a small circle $|z-a|=q$ from α_{nj} within $\hat{\Omega}_{nj}-A_{nj}$ and let $X_{nj}(q)$ be the joining curve family of them within $\hat{\Omega}_{nj}-B_{nj}$. Then we have

$$\log R_{nj} = \lim_{q\to 0} (2\pi \operatorname{mod} \Gamma_{nj}(q) + \log q)$$

$$= \lim_{q\to 0} (2\pi\lambda(X_{nj}(q)) + \log q),$$

both of which are the limits of monotone increasing sequences. This relation is easily verified from well-known inequalities of extremal lengths as in [2] and [14]. In general the above quantities can be defined for a general domain and an arbitrary partition of $\partial\Omega$ into α, A and B similarly. These two limits may well differ. If these coincide, we denote it by $R(\alpha, A, B)$ and call it the *extremal radius* of α at a with respect to the partition (α, A, B).

7. We first let j tend to infinity. The function f_{nj} converges to a univalent function $f_n(z)$ in such a way that $||f'_n/f_n - f'_{nj}/f_{nj}||^2_{\Omega_{nj}} \to 0$ [7]. It is also verified from the inequality (1). In fact, let M_q be the maximum modulus of f_{nk} on the curve $|f_{nj}|=q$ for $k>j$. Then any curve of $\Gamma^{M_q}_{nk}$ contains a curve of Γ^q_{nj} as a subset, Put $\rho_{nj}=|f'_{nj}/(2\pi f_{nj})|$ which is defined to be zero outside of Ω_{nj}. Let Ω_{nk}^{Mq} denote the set $M_q<|f_{nk}|<R_{nk}$. We have by (1)

$$||\rho_{nj}-\rho_{nk}||^2_{\Omega_{nk}^{Mq}}\leqq\frac{1}{2\pi}\left(\log\frac{R_{nj}}{q}-\log\frac{R_{nk}}{M_q}\right)$$

and letting $q\to 0$ we get

$$\left|\left|\left|\frac{f'_{nj}}{f_{nj}}\right|-\left|\frac{f'_{nk}}{f_{nk}}\right|\right|\right|^2_{\Omega_{nj}}\leqq 2\pi\log\frac{R_{nj}}{R_{nk}}. \tag{3}$$

From the inequality the convergence of f_{nj} is easily verified [18].

The image of Ω_n under f_n is as follows:

i) $f_n(\alpha_n)$ is the circle $|f_n|=R_n$, where $R_n=R(\alpha_n, A_n, B_n)$,

ii) $f_n(A_n)$ consists of a finite set of circular slits and

iii) $f_n(B_n)$ is a minimal set of radial slits.

i) and ii) is obvious, since α_n and A_n are isolated. For the representation of R_n, see the next section no. 8. The property iii) is easily verified by the localization of minimality [10, 15].

8. The f_n again induces two extremal metrics for Γ_n^q and X_n^q, where Γ_n^q is the family of curves separating α_n from the set $|f_n|=q$ within $\hat{\Omega}_n-A_n$ and X_n^q is that of curves joining them within $\hat{\Omega}_n-B_n$ for sufficiently small q. Put $\rho_n=|f_n'/(2\pi f_n)|$. Then from Schwarz's inequality we have

$$||\rho||^2_{\Omega_n^q}\geqq\frac{1}{2\pi}\log\frac{R_n}{q},\qquad \rho\in P(\Gamma_n^q)$$

which implies the extremality of ρ_n, since $\rho_n\in P^*(\Gamma_n^q)$.

Next, we set $\mu_n=|f_n'/(f_n\log R_n/q)|$. Considering the maximum and minimum moduli of f_{nj} on the curve $|f_n|=q$, we can conclude from Lemma 1 that mod $X_n^q=2\pi/\log(R_n/q)$ and μ_n is extremal.

9. Since the family Γ_n of curves separating α_n from the point a within $\hat{\Omega}_n-A_n$ is increasing, so is R_n (cf. no. 6). Suppose $R_0=\lim R_n<\infty$. Then letting $n\to\infty$, we obtain a univalent function $f_0(z)$ such that $||f_n'/f_n-f_0'/f_0||^2_{\Omega_n}\to 0$ [7]. This is a direct result from an inequality similar to (3)

$$\left|\left|\left|\frac{f_m'}{f_m}\right|-\left|\frac{f_n'}{f_n}\right|\right|\right|^2_{\Omega_n}\leqq 2\pi\log\frac{R_m}{R_n},\qquad \text{for}\quad m>n$$

since $R_n\leqq R_0$. We now state

THEOREM 1. *Under the assumption that $R_0<\infty$, the function f_0 constructed above possesses the following properties:*

i) *$f_0(\alpha)$ is the circle $|f_0|=R_0$ with possible radial incisions of angular measure zero emanating from it.*

ii) *$f_0(\sigma)$, $\sigma \in A$, is a circular slit (possibly a point) with possible radial incisions of angular measure zero,*

iii) *$f_0(B)$ is a minimal set of radial slits,*

vi) *the total area of the image of the boundary of Ω under f_0 vanishes,*

v) *the metric $\rho_0 = |f_0'/(2\pi f_0)|$ is extremal for the curve family Γ_0^q of curves separating α from the set $|f_0| = q$ within $\hat{\Omega} - A$ for sufficiently small q and* $\operatorname{mod} \Gamma_0^q = (2\pi)^{-1} \log R_0/q$, *and*

vi) *the metric $\mu_0 = |f_0'/(f_0 \log R_0/q)|$ is extremal for the family X_0^q of curves joining them within $\hat{\Omega} - B$ and* $\operatorname{mod} X_0^q = 2\pi/\log R_0/q$.

The properties i) and ii) are discussed by Marden and Rodin [7] under an additional assumption "β-isolation." They showed that $f_0(\sigma)$, $\sigma \in B$, is a radial slit. Here a minimal set is a quasiminimal set in [15]. The property iv) is a common property of canonical slit mappings stated in [15]. The module problems were discussed by them [7]. A special module problem for the family of collections of curves was dealt with by Andreian-Casacu [4].

10. Before proving Theorem 1, we prepare the following

LEMMA 3. *Let ρ_n be a sequence of metrics such that $||\rho_n||^2 \to 0$. Let Γ be a family of curves on which ρ_n is defined and measurable. Then we have*

$$\lim_{n\to\infty} \int_c \rho_n |dz| = 0 \qquad \textit{for a.a. } c \in \Gamma.$$

This is due to Fuglede [5] (cf. [7]).

11. *The proof of Theorem* 1. The property iii) is a direct result of the localization of minimality [15] which is also proved by a characterization due to Oikawa [10].

We first show the property ii). The proof of i) is its analogue. Let σ be an element of A. We can select a defining sequence of σ, denoted by $\{\Delta_n\}$, from the components of $\Omega - \bar{\Omega}_n$, where $\{\Omega_n\}$ is an exhaustion of Ω towards $\alpha \cup A$ to define f_0 as before. Let σ_n be the relative boundary of Δ_n. The image of σ_n under f_n is a circular slit with radius r_n. Selecting a subsequence, we may assume that $\lim r_n = r_0$, since r_n is bounded by R_0. Put $u_0 = \log |f_0|$, $u_n = \log |f_n|$ in $\Delta_1 \cap \Omega_n$, and extend u_n on $\Delta_1 - \bar{\Omega}_n$ by the constants taken by it on each component of the relative boundary of $\Delta_1 - \bar{\Omega}_n$. Let $X(\sigma)$ be the family of curves joining σ and σ_1 within $\hat{\Delta}_1 - B$. The function u_n is continuous on $c \in X(\sigma)$. We set $\rho_n = |\operatorname{grad}(u_0 - u_n)|$. Then the convergence of f_n'/f_n in no. **9** and Lemma 3 shows that

$$\lim_{n\to\infty} \int_c d(u_0 - u_n) = 0, \qquad \text{for a.a. } c \in X(\sigma).$$

Using the uniform convergence of u_n on σ_1, we have

$$(4) \qquad \int_c du_0 = \log r_0 - u_0(z_c) \qquad \text{for a.a. } c \in X(\sigma),$$

where z_c is the initial point of c on σ_1.

Next we evaluate the module of the curve family, denoted by Λ, consisting of the curves on which (4) is false. From the construction of f_0 in nos. **7** and **9** we can select a subsequence $\{f_{nj(n)}\}$ from the sequence $\{f_{nj}\}$ such that $\|f_0'/f_0 - f'_{nj(n)}/f_{nj(n)}\|^2_{\Omega_{nj(n)}} \to 0$. Let Ω_0 be a relatively compact open set $|f_0|<q$ and let D_k be $\Omega_k - \bar{\Omega}_{k-1}$ $(k \geqq 1)$ which consists of a finite number of domains, say D_{kl} $(l=1, 2, \cdots, N_k)$. Let ε be a given positive number. Then using Lemma 2, we can construct a subdomain D^{ε}_{kl} of each D_{kl}, given by $D_{kl} \cap \Omega_{nj(n)}$, and its $(1+\varepsilon)$-quasiconformal mapping $\Phi^{\varepsilon}_{kl}(z)$ such that $\Phi^{\varepsilon}_{kl} = f_0$ in $D_{kl} - \bar{\Delta}_{kl}$ and $\Phi^{\varepsilon}_{kl} = f_{nj(n)}$ in $\Delta'_{kl} \cap \Omega_{nj(n)}$, where Δ_{kl} and Δ'_{kl} are suitably chosen ends of D_{kl} containing its ideal boundary components which correspond to Δ and Δ' in Lemma 2 respectively. Set $\Omega^{\varepsilon} = \cup_{kl} D^{\varepsilon}_{kl} \cup \Omega_0$ and put

$$\Phi^{\varepsilon} = \begin{cases} \Phi^{\varepsilon}_{kl} & \text{in } D^{\varepsilon}_{kl}, \\ f_0 & \text{in } \Omega_0. \end{cases}$$

Then Ω^{ε} has at most a countable number of relative boundary components whose images under Φ^{ε} are radial slits. Furthermore the image of $\tau \in A \cup \alpha$ under Φ^{ε} coincides with $f_0(\tau)$.

In the image domain $\Phi^{\varepsilon}(\Delta_1 \cap \Omega^{\varepsilon})$ a ray $\arg w = \text{const}$ emanating from $\Phi^{\varepsilon}(\sigma_1)$ $(=f_0(\sigma_1))$ contains the image of a curve joining σ_1 and σ within $\hat{\Delta}_1 - B$, if it intersects $\Phi^{\varepsilon}(\sigma)$ and if it is disjoint from the radial slits which is the image of the relative boundary of Ω^{ε}. Let W be the doubly connected domain bounded by $f_0(\sigma_1)$ and $f_0(\sigma)$. Since $f_0(\sigma_1)$ encloses $f_0(\sigma)$, a ray $\arg w = \text{const}$ contains two radial segments joining $f_0(\sigma_1)$ and $f_0(\sigma)$ within W. The set of the arguments of these segments makes two intervals $[a, b]$, where a and b are the minimum and maximum values of the arguments of the rays. One is the set of segments on which $|w|$ increase from $f_0(\sigma_1)$ to $f_0(\sigma)$ and on the other set the contrary holds. The subset Ξ of the arguments of the segments along which the relation

$$\lim \log |w| = \log r_0$$

does not hold is a set of F_σ, where r_0 is the quantity in (4). Thus the set Ξ^{ε} of the arguments of the rays in the domain $\Phi^{\varepsilon}(\Delta_1)$ mentioned above is a measurable subset of Ξ with the same measure. Let $l^{\varepsilon}(\theta)$ denote the logarithmic length of the curve on the ray in the $\Phi^{\varepsilon}(\Delta_1 \cap \Omega^{\varepsilon})$ for $\theta \in \Xi^{\varepsilon}$ and let $l(\theta)$ be the length of the segment for $\theta \in \Xi$ which satisfies $l(\theta) \geqq l^{\varepsilon}(\theta)$. Let ρ be an admissible metric for the image curve on the ray with argument $\theta \in \Xi^{\varepsilon}$. From the Schwarz inequality we have

$$\int_{\arg w = \theta} \rho^2 r dr \geqq \frac{1}{l^{\varepsilon}(\theta)}, \qquad \theta \in \Xi^{\varepsilon}$$

and since the inverse image of the curve for $\theta \in \Xi^{\varepsilon}$ belongs to Λ

$$\operatorname{mod} \Lambda \geqq \frac{1}{1+\varepsilon} \int_{\Xi^\varepsilon} \frac{d\theta}{l^\varepsilon(\theta)}.$$

Letting $\varepsilon \to 0$, we get

$$(5) \qquad \operatorname{mod} \Lambda \geqq \int_{\Xi} \frac{d\theta}{l(\theta)}.$$

The above inequality was first obtained by Strebel for the radial slit mapping [14].

From (5) we conclude that the subset of $f_0(\sigma)$ not lying on the circle $|f_0|=r_0$ is possibly a set of radial incisions of angular measure zero.

12. *Continued.* We now prove the properties iv), v) and vi). As is seen in no. 8, $\rho_n=|f_n'/(2\pi f_n)|$ is extremal for the dividing curve family Γ_n^q. Then selecting a subsequence of $\{f_n\}$, if necessary, we can construct such a sequence $\{q_n\}$ with limit q that $\{\Gamma_n^{q_n}\}$ is increasing and that $\cup \Gamma_n^{q_n}=\Gamma_0^q$, since f_n converges to f_0 uniformly on a neighborhood of the set $|f_0|=q$ for a sufficiently small q. Thus Lemma 1 shows that $\operatorname{mod} \Gamma_0^q=(2\pi)^{-1} \log R_0/q$ and $\rho_0=|f_0'/(2\pi f_0)|$ is extremal.

The property iv) is obvious from the equality $||\rho_0||^2=(2\pi)^{-1}\log(R_0/q)$.

Finally the metric $\mu_0=|f_0'/(f_0 \log R_0/q)|$ is l_2-admissible and hence we have $\operatorname{mod} X_0^q \leqq 2\pi/\log(R_0/q)$. We apply the inequality (5) to X_0^q and have

$$(6) \qquad \operatorname{mod} X_0^q \geqq \int_0^{2\pi} \frac{d\theta}{l(\theta)},$$

Since $l(\theta) \leqq \log(R_0/q)$, we get $\operatorname{mod} X_0^q \geqq 2\pi/\log(R_0/q)$ which implies vi). This inequality was obtained by Strebel [14] in case where $A=\phi$.

From (6) we also see that $f_0(\alpha)$ is the circle $|f_0|=R_0$ with possible radial incisions of angular measure zero.

REMARK. We conclude that $R_0=R(\alpha, A, B)$ from v) and vi).

We call the function f_0 an (*extremal*) *circular-radial slit* disc mapping of Ω with respect to the partition (α, A, B). Here the closedness of $\alpha \cup A$ and finiteness of R_0 are assumed.

§ 4. Extremal Properties.

13. We discuss some extremal properties of the circular-radial slit disc mapping which characterize itself. Marden and Rodin dealt with extremal properties intimately related to the extremal length [7]. We shall show these extremal properties as extensions of classical theorems.

Let Ω be a finite domain and let α, A and B be a partition of $\partial\Omega$ such that α is its outer boundary and $\alpha \cup A$ is closed. We denote by $\mathfrak{F}(\alpha, A, B)$ the family of univalent functions satisfying

i) $f(a)=0, \quad f'(a)=1, \quad a\in\Omega,$

ii) $f(\alpha)$ is the outer boundary of $f(\Omega)$ and

iii) $\left|\int_c d\arg f\right|\geqq 2\pi$ for a. a. $c\in\Gamma(q)$,

where $\Gamma(q)$ is the family of curves separating α from a compact disc $|z-a|\leqq q$ within $\hat{\Omega}-A$. Put

$$M(f)=\sup_{z\in\Omega}|f(z)|.$$

Then we have

THEOREM 2. *Suppose $R(\alpha, A, B)<\infty$. Then the circular-radial slit disc mapping f_0 is the unique function minmizing the quantity $M(f)$ within $\mathfrak{F}(\alpha, A, B)$.*

Proof. We first show that $f_0\in\mathfrak{F}(\alpha, A, B)$. In fact, as is seen in no. 11 we have $||f_{nm(n)}'/f_{nm(n)}-f_0'/f_0||^2_{\Omega_{nm(n)}}\to 0$ for a subsequence $\{f_{nm(n)}\}$ of $\{f_{nm}\}$. Applying Lemma 3 to the metric

$$\rho_n=\begin{cases}\dfrac{1}{2\pi}\left|\operatorname{grad}\log\left|\dfrac{f_{nm(n)}}{f_0}\right|\right| & \text{in } \Omega_{nm(n)},\\[2ex] \dfrac{1}{2\pi}|\operatorname{grad}\log|f_0|| & \text{in } \Omega-\Omega_{nm(n)},\end{cases}$$

we get

$$\int_c d\arg f_0-\int_{c\cap\Omega_{nm(n)}} d\arg f_{nm(n)}\to 0.$$

for a. a. $c\in\Gamma(q)$. Thus we have

$$\left|\int_c d\arg f_0\right|\geqq 2\pi \qquad \text{for a. a. } c\in\Gamma(q).$$

Next we remark that the condition iii) is independent of the choice of neighborhoods. Indeed, for $q'<q$, we take an $r>q$ such that $q'\leqq|z-a|\leqq r$ is contained in Ω. Then

$$\Phi(z)=\begin{cases} r\left|\dfrac{z}{r}\right|^{\frac{\log q'-\log r}{\log q-\log r}} e^{i\arg z} & \text{in } q<|z|\leqq r\\[2ex] z & \text{in } \Omega^r\end{cases}$$

is a quasiconformal mapping of Ω^q onto $\Omega^{q'}$. Φ maps $\Gamma(q)$ onto $\Gamma(q')$ and a curve satisfying the inequality in iii) corresponds to a curve with the same property since the condition

$$\left|\int_{f(c)} d\arg w\right|\geqq 2\pi$$

is due to the behavior of the curve near the boundary of Ω. Thus from (2) we conclude the independence.

Put $\rho_0=|f_0'/(2\pi f_0)|$ and $\rho=|f'/(2\pi f)|$ for $f\in\mathfrak{F}(\alpha, A, B)$. From Theorem 1 and (1) we have

$$\left\|\left|\frac{f'}{f}\right|-\left|\frac{f_0'}{f_0}\right|\right\|^2\leqq 2\pi\log\frac{M(f)}{R_0}.$$

Thus we have the assertion.

This extremal property can be deduced from Marden and Rodin [7].

14. From Theorem 2 in case where $A=\phi$, we obtain a characterization of the minimality of radial slits which will be needed the next corollary.

COROLLARY 1. *Let E be a compact set contained in an annulus $G: q<|z|<Q$. Then E is a minimal set of radial slits if and only if*

$$\left|\int_c d\arg z\right|\geqq 2\pi \tag{7}$$

for a.a. c of the family of curves separating the circle $|z|=Q$ from $|z|=q$ within the compactification of $G-E$.

Proof. Let ρ be an admissible metric for the above separating curve family. Then the Schwarz inequality shows $\|\rho\|^2\geqq(2\pi)^{-1}\log(Q/q)$. From (7) $\rho_0=|2\pi z|^{-1}$ is l_2-admissible and $\|\rho_0\|^2\geqq(2\pi)^{-1}\log(Q/q)$. Hence ρ_0 is extremal and we see that the radial slit disc mapping of the disc $|z|<Q$ less E is the function z, which implies the minimality of E from Theorem 1.

Conversely if E is minimal, $G-E$ is a minimal radial slit annulus [15]. Let $\{G_n\}$ be its exhaustion towards E. Then we have $\|1/z-g_n'/g_n\|^2_{G_n}\to 0$, where g_n is the radial slit annulus mapping of G_n with the normalizations $g_n(Q)=Q$ and preserving the outer boundary. Put

$$\rho_n=\begin{cases}\dfrac{1}{2\pi}\left|\operatorname{grad}\log\left|\dfrac{g_n}{z}\right|\right| & \text{in } G_n,\\[2ex] |2\pi z|^{-1} & \text{in } G-G_n.\end{cases}$$

Applying Lemma 3 to ρ_n, we have

$$\int_c d\arg z-\int_{c\cap G_n} d\arg g_n\to 0,\qquad \text{for a.a. } c,$$

which implies (7).

We call a univalent function f a *radial slit mapping* (with respect to B) if $f(B)$ is a minimal set of radial slits. We get

COROLLARY 2. *f_0 is the unique function minimizing $M(f)$ among the radial slit mappings f satisfying* i) *and* ii).

This extremal property was found by Oikawa [9] in case where $A=\phi$. The case where $B=\phi$ is classical [11, 12].

Proof. It is sufficient to prove that the condition iii) is equivalent to the minimality of $f(B)$. Suppose iii). In the image plane, we take a compact subset E of $f(B)$. Then we can take a disc $|w|\leqq q\subset f(\Omega)$ and an analytic closed curve κ which separates the image of $\alpha\cup A$ under f from the disc and E in $f(\Omega)$. Let W be the domain whose boundary consists of the circle $|w|=q$, and the subset of boundary components of $f(B)$ contained in the interior of κ, say E', which is closed and contains E. Then we have (7) for a.a. c of the family of curves separating κ from the circle $|w|=q$ within W. Let G be an annulus $q<|w|<Q$ less E' containing W. Then similarly as in the proof of Theorem 2 we can construct a quasiconformal mapping Φ of W onto G such that $\Phi(w)=w$ in a neighborhood of E' (cf. [16] pp. 224–225). Then we have the validity of (7) for G, which implies the minimality of E' and hence of E from Corollary 1.

Next suppose $f(B)$ is minimal. Let $\{\Omega_n\}$ be an exhaustion of Ω towards $\alpha\cup A$. Let α_n be the outer boundary of Ω_n and let A_n be the relative boundary of Ω_n other than α_n. Put $B_n=\hat{\Omega}_n\cap B$. Denoting by $\Gamma_n(q)$ the family of curves separating the circle $|z-a|=q$ and α_n in Ω_n-A_n, we have (7) for $\Gamma_n(q)$ from Corollary 1, since $f(B_n)$ is a compact minimal set of radial slits and $\Gamma_n(q)$ is a subfamily of the corresponding family of a large annulus less $f(B_n)$. $\Gamma_0(q)=\cup\Gamma_n(q)$ and a countable union of exceptional families is also exceptional. So we get (7) for $\Gamma_0(q)$.

15. We now deal with another extremal problem. Let $\mathfrak{F}$ be the family of univalent functions f in Ω satisfying i) and ii). Let X be the family of curves joining α and a within $\hat{\Omega}-B$.

Then the limit

$$\lim_{t\to1}\left|\int_{c_t} df\right|=M_c(f)$$

exists for a.a. $c\in X$, where c_t is a subarc of c with its representation $z(s)$, ($0\leqq s\leqq t$, $z(0)=a$) and tending to α as $t\to1$ [8]. Here the module of the above exceptional family is measured by the set of subarcs starting from a simply connected compact neighborhood of a whose exceptionality does not depend on the choice of neighborhood [16]. We define by $m^*(f)$ the least upper bound of m satisfying $M_c(f)\geqq m$ for a.a. $c\in X$. Then we state

THEOREM 3. *Under the same assumption in Theorem 2, the circular-radial slit disc mapping f_0 is the unique function maximizing the quantity $m^*(f)$ within the* *family* $\mathfrak{F}$.

The Proof is analogous to that of Theorem 4 in [16] and omitted.

§5. Radial-circular slit mapping.

16. Throughout this section we assume that Ω is a finite domain containing a, α is its outer boundary and $\alpha \cup B$ is closed. Let $\{\Omega_n\}$ be an exhaustion of Ω towards α and let α_n be the outer boundary of Ω_n. Put $A_n = A \cap \hat{\Omega}_n$ and $B_n = B \cap \hat{\Omega}_n$. We take an exhaustion of Ω_n towards B_n, denoted by $\{\Omega_{nj}\}$. Let α_{nj} denote the outer boundary of Ω_{nj} and let B_{nj} be the set of its relative boundary components other than α_{nj}. Put $A_{nj} = A_n \cap \hat{\Omega}_{nj}$. Since $\alpha_{nj} \cup A_{nj}$ is closed in $\hat{\Omega}_{nj}$, there exists the circular-radial slit disc mapping f_{nj}. The image $f_{nj}(\alpha_{nj})$ is a circle $|f_{nj}| = R_{nj}$, where $R_{nj} = R(\alpha_{nj}, A_{nj}, B_{nj})$, $f_{nj}(B_{nj})$ is a finite set of radial slits and $f_{nj}(A_{nj})$ is a minimal set of circular slits. The incisions do not appear because B_{nj} is a finite set. Set $\rho_{nj} = |f_{nj}'/(2\pi f_{nj})|$. Since ρ_{nj} is extremal for the family Γ^q_{nj} of curves separating α_{nj} from the set $|f_{nj}| = q$ within $\hat{\Omega}_{nj} - A_{nj}$ and $\rho_{nk} \in P^*(\Gamma^q_{nj})$ for $k<j$, we have from (1)

$$\left\| \left| \frac{f_{nk}'}{f_{nk}} \right| - \left| \frac{f_{nj}'}{f_{nj}} \right| \right\|^2_{\Omega_{nj}} \leqq 2\pi \log \frac{R_{nk}}{R_{nj}}, \tag{8}$$

letting $q \to 0$. The sequence R_{nj} is monotone decreasing which tends to a limit R_n. From (8) there exists a univalent function g_n such that

$$\left\| \frac{g_n'}{g_n} - \frac{f_{nj}'}{f_{nj}} \right\|^2_{\Omega_{nj}} \to 0 \quad \text{as} \quad j \to \infty. \tag{9}$$

17. We prepare

LEMMA 4. *The function g_n possesses the following properties*:

i) *$g_n(\alpha_n)$ is the circle $|g_n| = R_n$*,

ii) *$g_n(A_n)$ is a minimal set of circular slits*,

iii) *$g_n(\sigma)$, $\sigma \in B_n$, is a radial slit (possibly a point) with possible circular incisions emanating from it*,

iv) *the area of $g_n(\partial \Omega_n)$ is equal to zero*,

v) *the metric $\rho_n = |g_n'/(2\pi g_n)|$ is extremal for the family Γ^q_n of curves separating α_n from the set $|g_n| = q$ within $\hat{\Omega}_n - A_n$ for sufficiently small q and* mod $\Gamma^q_n = (2\pi)^{-1} \log R_n/q$ *and*

vi) *the metric $\mu_n = |g_n'/(g_n \log(R_n/q))|$ is extremal for the family X^q_n of curves joining them within $\hat{\Omega}_n - B_n$ and* mod $X^q_n = 2\pi/\log(R_n/q)$.

Proof. i) is obvious, since α_n is isolated. ii) is the property of minimal sets which is shown in no. 8. In order to prove iii) we return to the definition of f_{nj}. Let $\{\Omega^k_{nj}\}$ be an exhaustion of Ω_{nj} towards $\alpha_{nj} \cup A_{nj}$. Let α^k_{nj} be the outer boundary of Ω_{nj}^k and let A_{nj}^k be the subset of the relative boundary of Ω_{nj}^k in Ω_{nj} other than α_{nj}^k. Put $B_{nj}^k = B_{nj} \cap \hat{\Omega}_{nj}^k$. Let f_{nj}^k be the circular-radial slit disc mapping with respect to $(\alpha_n^k, A_{nj}^k, B_{nj}^k)$. Then we can select a subsequence $\{f_{nj}^{k(j)}\}$ of $\{f_{nj}^k\}$ such that $\|f_{nj}^{k(j)\prime}/f_{nj}^{k(j)} - g_n'/g_n\|_{\Omega_{nj}^{k(j)}} \to 0$ as $j \to \infty$. Thus the same proof as in iii) of Theorem 1 is applicable. The details are omitted.

The properties v) and vi) is proved similarly as in no. 12 and iv) follows from v).

REMARK. From the properties v) and vi) we see that

$$R_n = R(\alpha_n, A_n, B_n).$$

18. We have $\| |g_n'/g_n| - |g_m'/g_m| \|_{\Omega_m}^2 \leqq 2\pi \log R_n/R_m$ $(n > m)$ as before. R_n is increasing and we put $R_0 = \lim R_n$. Suppose the sequence R_n is bounded. Then there exists a univalent function g_0 such that

$$\left\| \frac{g_n'}{g_n} - \frac{g_0'}{g_0} \right\|_{\Omega_n}^2 \to 0. \tag{10}$$

Now we state

THEOREM 4. *Under the assumption that $R_0 < \infty$, the function g_0 has the following properties.*

i) *$g_0(\alpha)$ is the circle $|g_0| = R_0$ with possible radial incisions emanating from it,*

ii) *$g_0(A)$ is a minimal set of circular slits,*

iii) *$g_0(\sigma)$, $\sigma \in B$, is a radial slit (possibly a point) with possible circular incisions emanating from it,*

iv) *the area of $g_0(\partial\Omega)$ vanishes,*

v) *$\rho_0 = |g_0'/(2\pi g_0)|$ is extremal for the Γ_0^q of curves separating α from the set $|g_0| = q$ within $\hat{\Omega} - A$ and* $\operatorname{mod} \Gamma_0^q = (2\pi)^{-1} \log (R_0/q)$ *and*

vi) *$\mu_0 = |g_0'/(g_0 \log (R_0/q))|$ is extremal for the family X_0^q joining them within $\hat{\Omega} - B$ and* $\operatorname{mod} X_0^q = 2\pi/\log (R_0/q)$.

Most of the proof of the theorem is analogous to that of Theorem 1. We shall prove the properties i), iii) and vi).

Proof. To prove iii), similarly as in the proof of iii) of Lemma 4 we select a subsequence $\{f_{nj(n)}^{k(n)}\}$ of $\{f_{nj}^k\}$ such that

$$\left\| \frac{g_0'}{g_0} - \frac{f_{nj(n)}^{k(n)\prime}}{f_{nj(n)}^{k(n)}} \right\|_{\Omega_{nj(n)}^{k(n)}}^2 \to 0.$$

Using this sequence to establish a similar inequality to (5), we can prove iii) analogously as in the proof of ii) of Theorem 1.

Next we show vi). The metric $\mu_0 = |g_0'/(g_0 \log (R_0/q))|$ is l_2-admissible for X_0^q and we have $\operatorname{mod} X_0^q \leqq 2\pi/\log (R_0/q)$. It is a direct result from the fact that

$$\left\| \frac{g_0'}{g_0 \log (R_0/q)} - \frac{g_n'}{g_n \log (R_n/M_q(g_n))} \right\|^2_{\Omega_q^{M_q(g_n)}} \to 0,$$

where $M_q(g_n)$ is the maximum modulus of g_n on the set $|g_0|=q$, tending to q. In order to prove Strebel's inequality we take a subsequence $\{f_{nj(n)}\}$ of $\{f_{nj}\}$ such that

$$\left\| \frac{g_0'}{g_0} - \frac{f'_{nj(n)}}{f_{nj(n)}} \right\|^2_{\Omega_{nj(n)}} \to 0.$$

Let Ω_0 be a relatively compact set $|g_0|<q$, put $\Delta_k=\Omega_k-\bar{\Omega}_{k-1}$ and $\Delta_0=\Omega_0$. Then applying Lemma 2 to Δ_k and the sequence $\{f_{nj(n)}\}$ we can construct a subdomain Ω^ε of Ω whose relative boundary consists of a countable number of closed analytic curves enclosing all the boundary components of B and its $(1+\varepsilon)$-quasiconformal mapping $\Phi^\varepsilon(z)$ such that $\Phi^\varepsilon(\alpha)$ is equal to $g_0(\alpha)$. Since the radial slits of $\Phi^\varepsilon(\Omega^\varepsilon)$ is countable we get Strebel's inequality (6). Thus we have $\bmod X_0^q \geqq 2\pi/\log (R_0/q)$ which implies vi). The property i) follows from (6) because $l(\theta)=\log R_0/q$ except for a set of angular measure zero.

Remark. In this case, R_0 is equal to $R(\alpha, A, B)$.

19. We call the function g_0 a *radial-circular slit disc mapping* of Ω. We can show the same extremal properties of g_0 as f_0 stated in Theorems 2 and 3.

§ 6. Circular and radial slit mapping.

20. Let (α, A, B) be an arbitrary partition of $\partial\Omega$, where Ω and α are as before. Let A_1 be a subset of A such that $\alpha \cup A_1$ is closed. Put $\partial\Omega-\alpha\cup A_1=B^1$. If $R(\alpha, A_1, B^1)<\infty$, from Theorem 1 there exists the circular-radial slit disc mapping of Ω, denoted by $f_{A_1}(z)$. Let $\Gamma(A_1)$ be the family of curves separating α from the point a within $\hat{\Omega}-A_1$ and let $X(B^1)$ be that joining them within $\Omega-B^1$. If $A_1\subset A_2$, $\Gamma(A_1)\supset\Gamma(A_2)$ and $X(B^1)\subset X(B^2)$. Thus we have $R(\alpha, A_1, B^1)\geqq R(\alpha, A_2, B^2)$.

Put

$$\bar{R}(A)= \inf_{A_1\subset A} R(\alpha, A_1, B^1)$$

for every compact $\alpha\cup A_1$. Let $\{A_n\}$ be a minimal sequence satisfying that $A_n\subset A$, $\alpha\cup A_n$ is compact and $\lim R(\alpha, A_n, B^n)=\bar{R}(A)$. Then we have

Lemma 5. *Let $f_{A_n}(z)$ be the circular-radial slit disc mapping of Ω with respect to the partition (α, A_n, B^n). Then the sequence $f_{A_n}(z)$ tends to a univalent function $f_A(z)$ such that*

$$\left\| \frac{f'_{A_n}}{f_{A_n}} - \frac{f'_A}{f_A} \right\|^2_{\Omega} \to 0.$$

The function $f_A(z)$ is independent of the choice of minimal sequences.

Proof. Taking a new sequence $\{\cup_{j=1}^{n} A_j\}$, we may assume that A_n is increasing. Then from the monotonity of $\Gamma(A_n)$ the same reason as the proof of (3) shows

$$\left\|\left|\frac{f'_{A_m}}{f_{A_m}}\right|-\left|\frac{f'_{A_n}}{f_{A_n}}\right|\right\|_{\Omega}^{2} \leq 2\pi \log \frac{R(\alpha, A_m, B^m)}{R(\alpha, A_n, B^n)} \tag{11}$$

for $n>m$ which implies the existence of f_A such that $\|f'_{A_n}/f_{A_n}-f'_A/f_A\|^2 \to 0$.
The independence of f_A follows from (11).

21. Next we take a subset B_1 of B such that $\alpha \cup B_1$ is compact. Let $\Gamma(A^1)$ and $X(B_1)$ be the families defined as before. Considering the family $\Gamma(A_1)$, we see that $R(\alpha, A^1, B_1)$ is increasing with respect to B_1. Put

$$\underline{R}(B)=\sup_{B_1 \subset B} R(\alpha, A^1, B_1)$$

for B_1 such that $\alpha \cup B_1$ is compact. Let $\{B_n\}$ be a maximal sequence such that $\lim R(\alpha, A^n, B_n)=\underline{R}(B)$. Then we have

LEMMA 6. *Suppose $\underline{R}(B)<\infty$. Let $g_{B_n}(z)$ be the radial-circular slit disc mapping of Ω with respect to the partition (α, A^n, B_n). Then there exists a univalent function $g_B(z)$ such that*

$$\left\|\frac{g'_B}{g_B}-\frac{g'_{B_n}}{g_{B_n}}\right\|_{\Omega}^{2} \to 0.$$

The function g_B is independent of the choice of maximal sequences.

The proof is similar to Lemma 5, which may be omitted.

22. We now state

THEOREM 5. *Suppose $\overline{R}(A)=\underline{R}(B)<\infty$. Then the function f_A defined in Lemma 5 coincides with the function g_B in Lemma 6.*

Put $\overline{R}(A)=R(\alpha, A, B)$ and $f_A=\varphi_{A,B}$. Then the function $\varphi_{A,B}$ possesses the following properties:

i) *$\varphi_{A,B}(\alpha)$ is the circle $|\varphi_{A,B}|=R(\alpha, A, B)$ with possible radial incisions emanating from it,*

ii) *$\varphi_{A,B}(\sigma)$, $\sigma \in A$, is a circular slit (possibly a point) with possible radial incisions emanating from it,*

iii) *$\varphi_{A,B}(\sigma)$, $\sigma \in B$, is a radial slit (possibly a point) with possible circular incisions emanating from it,*

iv) *the area of $\varphi_{A,B}(\partial\Omega)$ vanishes,*

v) *the metric $\rho_0=|\varphi'_{A,B}/(2\pi\varphi_{A,B})|$ is extremal for the family $\Gamma^q(A)$ of curves separating α from the set $|\varphi_{A,B}|=q$ within $\hat{\Omega}-A$ for sufficiently small q and* mod $\Gamma^q(A)=(2\pi)^{-1}\log R(\alpha, A, B)/q$ *and*

vi) *the metric $\mu_0=|\varphi'_{A,B}/(\varphi_{A,B}\log(R(\alpha, A, B)/q))|$ is extremal for the family $X^q(B)$ of curves joining them within $\hat{\Omega}-B$ and* mod $X^q(B)=2\pi/\log(R(\alpha, A, B)/q)$.

Proof. Let $\{A_n\}$ be a minimal sequence in Lemma 5 and let $\{B_n\}$ be a maximal sequence in Lemma 6. Then we get similarly as in (11)

$$\left\|\left|\frac{f'_{A_n}}{f_{A_n}}\right|-\left|\frac{g'_{B_n}}{g_{B_n}}\right|\right\|_{\Omega}^2 \leqq 2\pi \log \frac{R(\alpha, A_n, B^n)}{R(\alpha, A^n, B_n)}$$

since $\Gamma(A_n)\supset\Gamma(A^n)$, which implies the coincidence of f_A and g_B.

Next we show the properties v) and vi). Taking a subsequence of $\{f_{A_n}\}$, if necessary, we can construct such a sequence $\{q_n\}$ with limit q that $\{\Gamma^{q_n}(A_n)\}$ is decreasing and $\Gamma^{q_n}(A_n)\supset\Gamma^q(A)$. Here $\Gamma^{q_n}(A_n)$ is the family of curves separating α from the set $|f_{A_n}|=q_n$ within $\Omega-A_n$. Let $\{q^n\}$ be a sequence with the same limit such that $\{\Gamma^{q_n}(A_n)\}$ is increasing and $\Gamma^{q_n}(A^n)\subset\Gamma^q(A)$, where $\Gamma^{q_n}(A^n)$ is the similar curve family for A^n and g_{B_n}. Then we have $\cap\Gamma^{q_n}(A_n)\supset\Gamma^q(A)\supset\cup\Gamma^{q_n}(A^n)$.

From Lemma 1 the metric $\rho_0=|\varphi'_{A,B}/(2\pi\varphi_{A,B})|$ is extremal for $\cup\Gamma^{q_n}(A^n)$. On the other hand, $\rho_0\in P^*(\cap\Gamma^{q_n}(A_n))$ from the strong convergence of $\rho_n=|f'_{A_n}/(2\pi f_{A_n})|$. Thus ρ_0 is extremal for $\cap\Gamma^{q_n}(A_n)$ and so is for $\Gamma^q(A)$. The module is calculated from the convergence of ρ_n. The extremality of μ_0 is proved analogously.

The property iv) follows from e.g. the fact the mod $\Gamma^q(A)=(2\pi)^{-1}\log(R(\alpha, A, B)/q)$.

23. *Continued.* Finally we prove i), ii) and iii). In order to show the property ii), we may assume that $\sigma\in A_1$ of $\{A_n\}$, where $\{A_n\}$ is an increasing sequence such that f_{A_n} tends to f_A. Let $\{\Omega_m\}$ be an exhaustion of Ω towards σ such that $\Omega_1\ni a$ and let σ_m be the relative boundary of Ω_m. Put $u_n=\log|f_{A_n}|$ and $u_0=\log|f_A|$. Let $X(\sigma, B^n)$ denote the family of curves joining σ_1 and σ within $\hat{\Omega}-B^n$. Then as in the proof of Theorem 1, we have for a constant $r_n(\sigma)$

$$\int_c du_n=\log r_n(\sigma)-u_n(z_c) \qquad \text{for a.a. } c\in X(\sigma, B^n),$$

where z_c is the initial point of c on σ_1. Since $\|\operatorname{grad}(u_n-u_0)\|^2\to 0$, there exists an $r_0(\sigma)$ such that

$$\int_c du_0=\log r_0(\sigma)-u_0(z_c) \qquad \text{for a.a. } c\in\cup X(\sigma, B^n). \tag{12}$$

This is easily seen from Lemma 3 and the uniform convergence of u_n on σ_1.

Set $\varDelta_m=\Omega_m-\bar{\Omega}_{m-1}$ $(m\geqq 1, \Omega_0=\phi)$. Considering the sequence $\{f_{A_n}\}$ in each $\varDelta_m$, from Lemma 2 we can construct a $(1+\varepsilon_1)$-quasiconformal mapping $\Phi^{\varepsilon_1}(z)$ of Ω such that $\Phi^{\varepsilon_1}=f_{A_{n(m)}}$ in a subdomain $\varDelta'_m$ of $\varDelta_m$ whose complement with respect to $\varDelta_m$ is relatively compact in Ω, $\Phi^{\varepsilon_1}=f_A$ in a neighborhood of the relative boundary of $\varDelta_m$ and $f_A(\sigma)=\Phi^{\varepsilon_1}(\sigma)$. Let $\tilde{B}^{n(m)}$ denote $B^{n(m)}\cap\varDelta_m$ which is open in $\partial\Omega$. Put $B^{\varepsilon_1}=\cup_m\tilde{B}^{n(m)}$ and $A_{\varepsilon_1}=\partial\Omega-\alpha-B^{\varepsilon_1}$. Then B^{ε_1} is open. We see that $\Phi^{\varepsilon_1}(\tau)=f_{A_{n(m)}}(\tau)$, $\tau\in A_{n(m)}\cap\hat{\varDelta}_m$ and $\Phi^{\varepsilon_1}(\alpha)=f_{A_{n(1)}}(\alpha)$. Furthermore we show that $\Phi^{\varepsilon_1}(B^{\varepsilon_1})$ is a minimal set of radial slits. In fact, any compact subset of $\Phi^{\varepsilon_1}(B^{\varepsilon_1})$ is covered by a finite number of mutually disjoint open sets $\Phi^{\varepsilon_1}(\tilde{B}^{n(m)})$'s. The intersection of the subset with each member of the covering is a compact minimal set and hence the union of these intersections is also minimal [15], which implies the minimality of $\Phi^{\varepsilon_1}(B^{\varepsilon_1})$.

Put $W=\Phi^{\varepsilon_1}(\Omega)$. Consider an exhaustion of W towards $\Phi^{\varepsilon_1}(\alpha\cup A_{\varepsilon_1})$, denoted by $\{W_k\}$ and set $V_k=W_k-\overline{W}_{k-1}$ ($k\geqq 1$, $W_0=\phi$). Each V_k consists of a finite number of domains, say V_{kj} ($j=1, 2, \cdots, N_k$). The set $\Phi^{\varepsilon_1}(B^{\varepsilon_1})\cap\hat{V}_{kj}$, denoted by B_{kj}, is a compact minimal set of radial slits and we put $D_{kj}=(B_{kj})^c$, where the complement is taken in the extended w-plane. Then, for an exhaustion $\{D_{kj}^l\}_{l=1}^{\infty}$ of D_{kj}, the radial slit mapping h_{kj}^l with the normalizations that $h_{kj}^l(w)=w+\cdots$ near the point at infinity and that $h_{kj}^l(0)=0$ tends to the function w uniformly on any compact subset (e.g. [15]). Again using Lemma 2 we can construct a subdomain W^{ε_2} of W and its $(1+\varepsilon_2)$-quasiconformal mapping with the following properties: the relative boundary of W^{ε_2} consists of at most a countable number of analytic curves enclosing the elements of $\Phi^{\varepsilon_1}(B^{\varepsilon_1})$ only, its image under Φ^{ε_2} is a set of radial slits and $\Phi^{\varepsilon_2}\circ\Phi^{\varepsilon_1}(\tau)=\Phi^{\varepsilon_1}(\tau)$ for $\Phi^{\varepsilon_1}(\tau)\in\hat{W}^{\varepsilon_2}\cap\Phi^{\varepsilon_1}(\alpha\cup A_{\varepsilon_1})$. Let Ω^* be the inverse image of W^{ε_2}under Φ^{ε_1}. We denote by $\Lambda^*(\sigma)$ the subfamily of $\cup_n X(\sigma, B_n)$ consisting of the curves contained in $\hat{\Omega}^*$ less its relative boundary along which (12) is false. Clearly $\Phi^{\varepsilon_2}\circ\Phi^{\varepsilon_1}$ is a $(1+\varepsilon_1)(1+\varepsilon_2)$-quasiconformal mapping. Then each radial ray joining the images of of σ and σ_1 under $\Phi^{\varepsilon_2}\circ\Phi^{\varepsilon_1}$ within $\Phi^{\varepsilon_2}(\hat{W}^{\varepsilon_2})$ less the image of the relative boundary of Ω^* contain an image curve of $\cup X(\sigma, B)$. Since the number of relative boundary components of $\Phi^{\varepsilon_2}(W^{\varepsilon_2})$ is at most countable, the inequality (5) is applicable to $\Phi^{\varepsilon_1}\circ\Phi^{\varepsilon_2}(\Lambda^*(\sigma))$ and ii) follows. The proof of i) is similar, because we can establish Strebel's inequality (6). The proof of iii) is analogous under use of $\{g_{B_n}\}$. We complete the proof of Theorem 5.

24. We call the function $\varphi_{A,B}$ in Theorem 5 a *circular and radial slit disc mapping* of Ω with respect to the partition (α, A, B). The same extremal properties stated in Theorems 2 and 3 are valid for this function.

We can see from these theorems that both the circular-radial and radial-circular slit mappings are indeed circular and radial slit disc mappings. It follows from the following.

THEOREM 6. *If A or B is closed in $\partial\Omega-\alpha$, the quantities $\overline{R}(A)$ and $\underline{R}(B)$ coincide with $R(\alpha, A, B)$. Here $R(\alpha, A, B)$ is the extremal radius in Theorems* 1 *and* 4, *if it is finite and $R(\alpha, A, B)=\infty$ otherwise.*

Proof. Suppose, at first, that $\alpha\cup A$ is compact. Then clearly $\overline{R}(A)=\underline{R}(\alpha, A, B)$. To show that $\underline{R}(B)=R(\alpha, A, B)$, consider an exhaustion $\{\Omega_n\}$ of Ω towards $\alpha\cup A$. Put $B_n=\hat{\Omega}_n\cap B$ and $A^n=\partial\Omega-\alpha-B_n$. Then we have under the same notations in no. **20** $\Gamma(A)=\cup\Gamma(A^n)$. By Lemma 1, we have $R(\alpha, A^n, B_n)\to R(\alpha, A, B)$, which implies $\underline{R}(B)=R(\alpha, A, B)$, since $\alpha\cup B_n$ is compact.

Next, if $\alpha\cup B$ is compact, $\underline{R}(B)=R(\alpha, A, B)$. When $R(\alpha, A, B)=\infty$, we get $\overline{R}(A)=\infty$ from the monotonity of module. In case $R(\alpha, A, B)<\infty$, considering an exhaustion $\{\Omega_m\}$ of Ω towards α, set $V_m=\Omega_m-\bar{\Omega}_{m-1}$ ($m\geqq 1$, $\Omega_0=\phi$), $A^{(m)}=\hat{V}_m\cap A$. Let $\{V_{mn}\}$ denote an exhaustion of V_m towards $\partial V_m-A^{(m)}$ and let $A_n^{(m)}$ be the open and closed set $\hat{V}_{mn}\cap A^{(m)}$. We set $A_{mn}=A_n^{(m)}\cup(A-A^{(m)})$ and $B_{mn}=\partial\Omega-\alpha-A_{mn}$ which is closed. We have $A=\cup_n A_{mn}$ and $X(B)=\cup_n X(B_{mn})$ as above. Putting $m=1$, from Lemma 1 and (1) we have $||g'_B/g_B-g'_{B_{1n}}/g_{B_{1n}}||^2\to 0$. Thus there exists an n_1 such that

$$R(\alpha, A_{1n_1}, B_{1n_1}) < R(\alpha, A, B) + \frac{\varepsilon}{2}.$$

For the partition $(\alpha, A_{1n_1}, B_{1n_1})$, applying the same argument to V_2, we have an open subset A_{2n_2} of A_{1n_1} such that $\hat{\Omega}_m \cap A_{2n_2}$ is compact for $m \leqq 2$ and that

$$R(\alpha, A_{2n_2}, B_{2n_2}) < R(\alpha, A_{1n_1}, B_{1n_1}) + \frac{\varepsilon}{2^2}$$

and so on. Summing up these inequalities, we have

$$R(\alpha, A_{kn_k}, B_{kn_k}) < R(\alpha, A, B) + \varepsilon.$$

We now prove that $A_\varepsilon = \cap_k A_{kn_k}$ is a closed subset of A in $\partial\Omega - \alpha$ and that $R(\alpha, A_{kn_k}, B_{kn_k}) \to R(\alpha, A_\varepsilon, B^\varepsilon)$ as $k \to \infty$, where $(\alpha, A_\varepsilon, B^\varepsilon)$ is determined by A_ε, which implies the assertion. In fact $\hat{\Omega}_m \cap A_\varepsilon$ is compact for all m, whence $\alpha \cup A_\varepsilon$ is compact. The convergence of the extremal radii follows from the fact that $\Gamma(A_\varepsilon) = \cup_k \Gamma(A_{km_k})$.

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