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ON THE GROUP OF (1, 1) CONFORMAL MAPPINGS OF AN OPEN RIEMANN SURFACE ONTO ITSELF

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1. The following theorem [4] is well known:

The number of (1, 1) *conformal mappings of a plane region bounded by* p $(\infty > p > 2)$ *Jordan curves onto itself is finite.*

In the present paper we shall consider, instead of a plane region, an open Riemann surface W and shall give two sufficient conditions that W admits only a finite number of (1, 1) conformal mappings onto itself: namely,

THEOREM 1. *If W is an open Riemann surface which has p $(\infty > p > 2)$ boundary elements in the sense of Kerékjártó-Stoilow, then the number of* (1, 1) *conformal mappings of W onto itself is finite.*

THEOREM 2. *If W is an open Riemann surface of genus g $(\infty > g > 0)$, then the number of* (1, 1) *conformal mappings of W onto itself is finite.*

Theorem 1 may be regarded as an extension of the above theorem.

Further we shall consider an open Riemann surface which has precisely two boundary elements. In this case we shall exclude doubly connected planar surfaces from our investigation. There is a non-planar Riemann surface which has two boundary elements and which admits infinitely many (1, 1) conformal mappings onto itself. However we shall prove the following theorem.

THEOREM 3. *If W is an open Riemann surface which has two boundary elements and which is not planar, then the group of* (1, 1) *conformal mappings of W onto itself is finitely generated.*

More generally, let β_1 and β_2 be two boundary elements of an open Riemann surface W which has more than one boundary element and denote by $A(\beta_1, \beta_2)$ the group of (1, 1) conformal mappings φ of W onto itself which have the property that either

(1) $\varphi(p)$ tends to β_1, β_2 for p tends to β_1, β_2 respectively; or else

(2) $\varphi(p)$ tends to β_2, β_1 for p tends to β_1, β_2 respectively.

For such a group we have

THEOREM 4. *If W is an open Riemann surface which has more than one boundary element and which is not a doubly connected planar surface, than $A(\beta_1, \beta_2)$*

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is finitely generated for arbitrary β_1 *and* β_2.

Obviously Theorem 3 is a consequence of this theorem.

In the case that $A(\beta_1, \beta_2)$ contains infinitely many members we shall obtain a detailed information on the structure of $A(\beta_1, \beta_2)$.

In our study the following two theorems play the essential role.

THEOREM A. (Komatu and Mori [5]) *Let W, W* be two Riemann surfaces whose universal covering surfaces are of hyperbolic type, and let* $\{\varphi^{(\nu)}\}_{\nu=1}^{\infty}$ *be a sequence of single valued analytic mappings of W into W*. Then, either*

(1) *there exists a subsequence* $\{\varphi^{(\nu_j)}\}_{j=1}^{\infty}$ *which converges, uniformly in the wider sense in W (with respect to the uniform topology of W* defined by means of Poincaré's hyperbolic metric), to a limit analytic mapping* φ *of W into W*; or else*

(2) *for any point p on W the point sequence* $\{\varphi^{(\nu)}(p)\}_{\nu=1}^{\infty}$ *on W* tends to the ideal boundary of W* uniformly in the wider sense in W.*

The statement (2) *means: if K, K* are compact point sets on W, W* respectively, then* $\varphi^{(\nu)}(K)\cap K^*=\phi$ *for sufficiently large* ν.

THEOREM B. (Heins [2]) *If W is a Riemann surface, which is not simply connected and which has the properties that its universal covering surface is of hyperbolic type and that the fundamental group accociated with W is not cyclic, then the identity mapping of W onto itself can never be expressed as the limit of a sequence* $\{\varphi^{(\nu)}\}_{\nu=1}^{\infty}$ *of single valued analytic mappings of W into itself, where* $\varphi^{(\nu)}(p)\fallingdotseq p$ $(\nu=1, 2, \cdots)$.

These two theorems allow us to infer immediately

LEMMA 1. *If W satisfies the conditions imposed in Theorem* B, *then any sequence* $\{\varphi^{(\nu)}\}_{\nu=1}^{\infty}$ *of* (1, 1) *conformal mappings of W onto itself whose members are distinct tends to the ideal boundary of W in the same sense as Theorem* A.

Proof. Assume that there exists a subsequence $\{\varphi^{(\nu_j)}\}_{j=1}^{\infty}$ which converges to a limit analytic mapping of W into itself in the same sense as Theorem A. Suppose that the limit mapping is not reduced to a single point, then it is a (1, 1) conformal mapping of W onto itself by the aid of Hurwitz's theorem. Theorem B is contradicted. Suppose that the limit mapping is reduced to a single point, then for each cycle c on W which is not homologous to zero $\varphi^{(\nu_j)}(c)$ is homologous to zero if j is sufficiently large. This contradicts that $\varphi^{(\nu_j)}$ is a (1, 1) conformal mapping of W onto itself. Thus we obtain this lemma by applying Theorem A.

There is a theorem of Klein and Poincaré which is a consequence of Theorem B: namely, that under the hypotheses of Theorem B the group of (1, 1) conformal mappings of W onto itself is properly discontinuous. Now we have by virtue of Lemma 1 that under the same hypotheses the group of (1, 1) conformal mappings of W onto itself is countable. Indeed, cover W by a family $\{\varDelta_n\}_{n=1}^{\infty}$ of parametric disks. We enumerate the (1, 1) conformal mappings of W onto itself by counting those mappings φ for which $\varphi(p_0)\in\varDelta_n$, $n=1, 2, \cdots$ with p_0 a point of $\varDelta_0$. Each φ will be counted at least one. But there are only a finite number of such φ for each n by applying Lemma 1. Then altogether there are at most denumerably many φ.

REMARK. The hypotheses of Theorem B implies to exclude the following seven surfaces: (1) a sphere, (2) a once-punctured sphere, (3) a twice-punctured sphere, (4) a torus, (5) a disk, (6) a once-punctured disk, (7) an annulus.

2. Proof of Theorem 1. Let W be an open Riemann surface which has p $(\infty>p>2)$ boundary elements and let $\beta_1, \beta_2, \cdots, \beta_p$ be its boundary elements. We denote by G_i $(i=1,2,\cdots,p)$ a non-compact subregion of W whose relative boundary c_i consists of a Jordan closed curve and which has the properties that β_i is a boundary element of G_i and that $G_i \cap G_j = \phi$ for distinct i, j $(1 \leqq i, j \leqq p)$. Each c_i is not homologous to zero and any two distinct c_i, c_j are not homologous to each other. We set $R = W - \cup_{i=1}^{p} G_i$. Assume that there exist infinitely many distinct (1, 1) conformal mappings of W onto itself, $\{\varphi^{(\nu)}\}_{\nu=1}^{\infty}$. Since R is a campact and connected set on W, applying Lemma 1 there exist integers ν and i such that $G_i \supset \varphi^{(\nu)}(R)$. Thus all $\varphi^{(\nu)}(c_j)$ are contained in G_i. Since each $\varphi^{(\nu)}(c_j)$ is a dividing cycle which is not homologous to zero, $\varphi^{(\nu)}(c_j)$ is homologous to c_i. Hence, for instance, $\varphi^{(\nu)}(c_1)$ is homologous to $\varphi^{(\nu)}(c_2)$ because both are homologous to c_i. By the aid of the fact that $\varphi^{(\nu)}$ is a (1, 1) conformal mapping of W onto itself we have that c_1 is homologous to c_2. It is a contradiction.

3. Proof of Theorem 2. Since W is an open Riemann surface of genus g $(\infty>g>0)$, there exists a relatively compact subregion Ω of W such that each component of $W-\bar{\Omega}$ is planar and there exists a non-dividing cycle c. Assume that there exist infinitely many distinct (1, 1) conformal mappings of W onto itself, $\{\varphi^{(\nu)}\}_{\nu=1}^{\infty}$. Applying Lemma 1 there exists an integer ν such that $\varphi^{(\nu)}(c) \cap \bar{\Omega} = \phi$. Consequently $\varphi^{(\nu)}(c)$ is contained in a component of $W-\bar{\Omega}$. It contradicts that $\varphi^{(\nu)}(c)$ is a non-dividing cycle.

4. Proof of Theorem 4. In order to prove Theorem 4 we shall prove several lemmas. We begin with introducing some notations.

Let W be an open Riemann surface and let β be a boundary element of W. We denote by W_β a non-compact subregion of W which has the property that β is a boundary element of W_β and whose relative boundary ∂W_β consists of at most an enumerably infinite number of analytic curves clustering nowhere in W. Let $\{G_n\}_{n=1}^{\infty}$ be a defining sequence of β whose relative boundary ∂G_n consists of an analytic Jordan closed curve. Further let $\{\Omega_k\}_{k=1}^{\infty}$ be an exhaustion of W whose relative boundary $\partial\Omega_k$ consists of analytic Jordan closed curves and whose closure $\bar{\Omega}_k$ is compact, and which has the property that $\bar{\Omega}_k \subset \Omega_{k+1}$ for all k.

Consider the harmonic function $\omega_{n,k}$ in $(W_\beta - \bar{G}_n) \cap \Omega_k$ which has the boundary values 1 on ∂G_n and 0 on $\partial W_\beta \cap \Omega_k$ and whose normal derivative vanishes on $\partial\Omega_k \cap (W_\beta - \bar{G}_n)$. The sequence $\{\omega_{n,k}\}$ converges to a harmonic function ω_β uniformly on every compact subset of W_β:

$$\omega_\beta = \lim_{n\to\infty} \lim_{k\to\infty} \omega_{n,k} \qquad \text{and} \qquad \|d\omega_\beta\|_{W_\beta} = \lim_{n\to\infty} \lim_{k\to\infty} \|d\omega_{n,k}\|_{(W_\beta - \bar{G}_n)\cap\Omega_k}.$$

ω_β is independent of the particular $\{G_n\}_{n=1}^\infty$ and $\{\Omega_k\}_{k=1}^\infty$, and ω_β has the following properties:

(1) $\sup_{W_\beta \ni p} \omega_\beta(p)=1$ if ω_β is not reduced to the constant zero,

(2) let W_β, W'_β be two admitted non-compact subregions of W associated with β, then $\omega_\beta \equiv 0$ is equivalent to $\omega'_\beta \equiv 0$, where ω_β, ω'_β are the above harmonic functions associated with W_β, W'_β respectively.

We call ω_β the harmonic measure of β.

Let $g_{n,k}(p, q)$ be the function which is harmonic in $(W_\beta-\bar{G}_n)\cap\Omega_k$ except for the singularity $-\log|z|$ at a point $q\in(W_\beta-\bar{G}_n)\cap\Omega_k$, $=0$ on $(\partial W_\beta\cap\Omega_k)\cup\partial G_n$ and whose normal derivative vanishes on $\partial\Omega_k\cap(W_\beta-\bar{G}_n)$. Then $\lim_{n\to\infty}\lim_{k\to\infty} g_{n,k}(p, q)$ exists and the limit function $g(p, q)$ is harmonic in W_β except for the singularity $-\log|z|$ at q, $=0$ on ∂W_β and has the symmetry property: $g(p, q)=g(q, p)$.

In the case that the relative boundary of W_β is compact for a given continuous function f on ∂W_β we can construct the harmonic function in W_β denoted by H_f^c as follows. Let $u_{n,k}^c$ be the harmonic function in $(W_\beta-\bar{G}_n)\cap\Omega_k$ which has the boundary values f on ∂W_β and a constant c on ∂G_n and whose normal derivative vanishes on $\partial\Omega_k\cap(W_\beta-\bar{G}_n)$. Here we assume that Ω_1 contains ∂W_β without loss of generality. The sequence $\{u_{n,k}^c\}$ converges to a harmonic function H_f^c uniformly on every compact subset of W_β: $H_f^c=\lim_{n\to\infty}\lim_{k\to\infty} u_{n,k}^c$. If the harmonic measure of β vanishes, H_f^c has the following properties:

(1) $H_f^c=H_f^0$ for every c,

(2) $\min_{\partial W_\beta} f \leqq H_f^c \leqq \max_{\partial W_\beta} f$ on W_β,

(3) $\int_\gamma \frac{\partial H_f^c}{\partial n}\, ds=0$

where γ is an analytic Jordan closed curve on W_β separating β from ∂W_β.

In the case that the harmonic measure of β vanishes we denote by $H_f^{W_\beta}$ the above function associated with W_β. Then we can verify easily that $g(p, q)=H_{g(r,q)}^{G_n}(p)$ on G_n provided that G_n does not contain q.

LEMMA 2. *If the harmonic measure of β vanishes, there exists a positive harmonic function v in W_β which has the following properties*:

(1) $v=0$ *on* ∂W_β,

(2) $v=\lim_{n\to\infty}\lim_{k\to\infty} v_{n,k}$,

where $v_{n,k}$ denotes the harmonic function in $(W_\beta-\bar{G}_n)\cap\Omega_k$ which has the boundary values v on $(\partial W_\beta\cap\Omega_k)\cup\partial G_n$ and whose normal derivative vanishes on $\partial\Omega_k\cap(W_\beta-\bar{G}_n)$.

Proof. Since $g(p, q)=H_{g(r,q)}^{G_n}(p)$ on G_n provided that G_n does not contain q, we have that

$$g(p, q)\geqq\min_{\partial G_n} g(r, q) \quad \text{on } G_n, \qquad q\notin G_n,$$

and so that

$$\varliminf_{p\to\beta} g(p, q)\geqq\min_{\partial G_n} g(r, q)>0, \qquad q\notin G_n.$$

Hence we can select a sequence $\{q_j\}_{j=1}^\infty$ tending to β such that $\{g(p, q_j)\}_{j=1}^\infty$ converges to a positive harmonic function v in W_β: $v(p)=\lim_{j\to\infty} g(p, q_j)$.

It is evident that v has the property (1). There remains to be shown that v has the property (2). Let $\tilde{g}_{n,k}(p,q)$ be the harmonic function in $(W_\beta-\bar{G}_n)\cap\Omega_k$ which has the boundary values $g(r,q)$ on $(\partial W_\beta\cap\Omega_k)\cup\partial G_n$ and whose normal derivative vanishes on $\partial\Omega_k\cap(W_\beta-\bar{G}_n)$. $\lim_{k\to\infty}\tilde{g}_{n,k}(p,q)$ exists and is harmonic in $W_\beta-\bar{G}_n$. Since for $m>n$

$$|\tilde{g}_{n,k}(p,q_j)-g_{m,k}(p,q_j)|\leqq\max_{\partial G_n}|g(p,q_j)-g_{m,k}(p,q_j)| \qquad \text{on} \quad (W_\beta-\bar{G}_n)\cap\Omega_k$$

provided that G_n contains q_j, we have that

$$g(p,q_j)=\lim_{k\to\infty}\tilde{g}_{n,k}(p,q_j) \qquad \text{on} \quad W_\beta-\bar{G}_n \qquad \text{for} \quad q_j\in G_n.$$

Hence by the inequality

$$|v(p)-v_{n,k}(p)|\leqq|v(p)-g(p,q_j)|+|g(p,q_j)-\tilde{g}_{n,k}(p,q_j)|+\max_{\partial G_n}|g(p,q_j)-v(p)|$$
$$\text{on} \quad (W_\beta-\bar{G}_n)\cap\Omega_k$$

we obtain that

$$v=\lim_{k\to\infty}v_{n,k} \qquad \text{on} \quad W_\beta-\bar{G}_n$$

and that hence

$$v=\lim_{n\to\infty}\lim_{k\to\infty}v_{n,k} \qquad \text{on} \quad W_\beta.$$

If the harmonic measure of β vanishes, v is unbounded. Indeed, if v were bounded: $v\leqq M$ on W_β, then $v_{n,k}\leqq M\omega_{n,k}$ on $(W_\beta-\bar{G}_n)\cap\Omega_k$ and therefore

$$0\leqq v=\lim_{n\to\infty}\lim_{k\to\infty}v_{n,k}\leqq M\lim_{n\to\infty}\lim_{k\to\infty}\omega_{n,k}=0.$$

Further $\|dv\|_{W_\beta}=\infty$. Suppose that $\|dv\|_{W_\beta}<\infty$. We construct the harmonic function $\tilde{v}_{n,k}$ in $(G_1-\bar{G}_n)\cap\Omega_k$ which has the boundary values v on ∂G_n and 0 on ∂G_1 and whose normal derivative vanishes on $\partial\Omega_k\cap(G_1-\bar{G}_n)$. Then the sequence $\{\tilde{v}_{n,k}\}$ converges to a harmonic function $\tilde{v}$ uniformly on every compact subset of G_1: $\tilde{v}=\lim_{n\to\infty}\lim_{k\to\infty}\tilde{v}_{n,k}$ and $\|d\tilde{v}\|_{G_1}$ is finite, and moreover $\|d\tilde{v}\|_{G_1}=\lim_{n\to\infty}\lim_{k\to\infty}\|d\tilde{v}_{n,k}\|_{(G_1-\bar{G}_n)\cap\Omega_k}$. By the inequality

$$\|d\tilde{v}_{n,k}\|_{(G_1-\bar{G}_n)\cap\Omega_k}\cdot\|d\omega_{n,k}\|_{(G_1-\bar{G}_n)\cap\Omega_k}\geqq|(d\tilde{v}_{n,k},d\omega_{n,k})_{(G_1-G_n)\cap\Omega_k}|$$
$$=\left|\int_{\partial G_n}\frac{\partial\tilde{v}_{n,k}}{\partial n}ds\right|=\left|\int_{\partial G_1}\frac{\partial\tilde{v}_{n,k}}{\partial n}ds\right|$$

where $\omega_{n,k}$ denotes the harmonic function in $(G_1-\bar{G}_n)\cap\Omega_k$ which has the boundary values 1 on ∂G_n and 0 on ∂G_1 and whose normal derivative vanishes on $\partial\Omega_k\cap(G_1-\bar{G}_n)$, we obtain that

$$\int_{\partial G_1}\frac{\partial\tilde{v}}{\partial n}ds=0$$

because $\lim_{n\to\infty}\lim_{k\to\infty}\|d\omega_{n,k}\|_{(G_1-\bar{G}_n)\cap\Omega_k}=0$ and that hence $\tilde{v}\equiv0$. Consequently $v=v-\tilde{v}$ is bounded on G_1 and therefore v is bounded on W_β. We conclude that v is identically equal to zero as before.

From now on we assume that W is an open Riemann surface which has more than one boundary element and which is not a doubly connected planar surface. Further we assume that $A(\beta_1, \beta_2)$ contains infinitely many distinct members.

LEMMA 3. *There exists a member φ of $A(\beta_1, \beta_2)$ such that φ_n is not the identity mapping for any integer $n(\neq 0)$, where $\{\varphi_n\}_{n=-\infty}^{\infty}$ denotes the sequence of iterates of φ: $\varphi_1=\varphi$, $\varphi_n=\varphi\circ\varphi_{n-1}$ for positive integers n, and φ_{-1} denotes the inverse mapping of φ, $\varphi_n=\varphi_{-1}\circ\varphi_{n+1}$ for negative integers n.*

Proof. Let γ be a Jordan closed curve dividing W into two parts W_1, W_2 such that β_i is a boundary element of W_i $(i=1,2)$. We denote by $\{\varphi^{(\nu)}\}_{\nu=1}^{\infty}$ a sequence of distinct members of $A(\beta_1, \beta_2)$. By Lemma 1 we can assume without loss of generality that $\varphi^{(\nu)}(\gamma)$ are contained in W_1 for all ν. This implies that either $\varphi^{(\nu)}(W_1)\subsetneqq W_1$ or else $\varphi^{(\nu)}(W_2)\subsetneqq W_1$. If $\varphi^{(1)}(W_1)\subsetneqq W_1$, $\varphi^{(1)}$ is a desired mapping. In the case that $\varphi^{(1)}(W_2)\subsetneqq W_1$ we take a relatively compact subregion K of $W_1-\varphi^{(1)}(W_2)$ whose relative boundary contains γ and $\varphi^{(1)}(\gamma)$. Applying Lemma 1 there exists a mapping $\varphi^{(\nu)}$ such that $\varphi^{(\nu)}(\bar{K})\cap\bar{K}=\phi$. Since $\varphi^{(\nu)}(\gamma)$ is contained in W_1 and separates β_1 from β_2, $\varphi^{(\nu)}(K)$ must be contained in $\varphi^{(1)}(W_2)$. Hence $\varphi^{(\nu)}\circ\varphi^{(1)}(\gamma)$ is contained in $\varphi^{(1)}(W_2)$. This implies that either $\varphi^{(\nu)}\circ\varphi^{(1)}(W_1)\subsetneqq\varphi^{(1)}(W_2)$ or else $\varphi^{(\nu)}\circ\varphi^{(1)}(W_2)\subsetneqq\varphi^{(1)}(W_2)$. In the former case we have that $\varphi^{(\nu)}\circ\varphi^{(1)}(W_1)\subsetneqq W_1$ and hence $\varphi^{(\nu)}\circ\varphi^{(1)}$ is a desired mapping. In the latter case $\varphi^{(\nu)}$ is a desired mapping.

This mapping has the properties that $\lim_{n\to\infty}\varphi_n(p)=\beta_1$ and that $\lim_{n\to\infty}\varphi_{-n}(p)=\beta_2$. Then we have

LEMMA 4. *The harmonic measure of β_i vanishes $(i=1,2)$.*

Proof. Let R_0 be a closed annulus separating β_1 from β_2 with Jordan boundary on W and we denote by W_i $(i=1,2)$ the component of $W-R_0$ such that β_i is a boundary element of W_i. We set $R_n=\varphi_n(R_0)$. We can assume that $R_m\cap R_n=\phi$ for distinct integers m, n. We denote by F_n the family of arcs in R_n which join the opposite contours of R_n and denote by $\lambda(F_n)$ the extremal length of F_n. We have that $\lambda(F_n)=\lambda(F_0)>0$. Further we denote by $\mathfrak{F}_i$ $(i=1,2)$ the family of arcs in W_i with the initial point on ∂R_0 and extending to β_i, and denote by $\lambda(\mathfrak{F}_i)$ the extremal length of $\mathfrak{F}_i$. Then we have that $\lambda(\mathfrak{F}_1)>\sum_{k=1}^{n}\lambda(F_k)$ for all positive integers n and that $\lambda(\mathfrak{F}_2)>\sum_{k=1}^{n}\lambda(F_{-k})$ for all positive integers n. This implies that $\lambda(\mathfrak{F}_i)=\infty$ $(i=1,2)$. On the other hand we have that $\lambda(\mathfrak{F}_i)=\|d\omega_{\beta_i}\|_{W_i}^{-2}$ where ω_{β_i} denotes the harmonic measure of β_i associated with W_i [7]. Hence we obtain that $\omega_{\beta_i}\equiv 0$.

Let γ be an analytic Jordan closed curve dividing W into two parts W_1, W_2 such that β_i is a boundary element of W_i $(i=1,2)$. We orient γ positively with respect to W_1. By Lemma 2 and Lemma 4 there exists a positive harmonic function v in W_i $(i=1,2)$ which has the properties imposed in Lemma 2. We denote by $\{G_n^{(i)}\}_{n=1}^{\infty}$ a defining sequence of β_i $(i=1,2)$. Since $\min_{\partial G_n^{(i)}} H_f^{W_i}\leqq H_f^{W_i}\leqq\max_{\partial G_n^{(i)}} H_f^{W_i}$ on $G_n^{(i)}$ and $\|dH_f^{W_i}\|_{W_i}<\infty$, we can prove the following lemma by applying the same method as in the proof of Theorem 13.1 and Lemma 11.1 in Heins' paper [3].

LEMMA 5. *$H_f^{W_i}$ possesses a limit at β_i and*

$$\lim_{p\to\beta_i} H_f^{W_i}(p)=\int_\gamma H_f^{W_i}\frac{\partial v}{\partial n}\,ds$$

where v denotes a positive harmonic function in W_1 (or W_2) which has the properties imposed in Lemma 2 and

$$\int_\gamma \frac{\partial v}{\partial n}\,ds=1 \qquad (or\ -1).$$

Accordingly we obtain the following lemma by applying the same method as in the proof of Theorem 11. 2 in Heins' paper [3].

LEMMA 6. *There is only one positive harmonic function v in W_1 (or W_2) which has the properties imposed in Lemma 2 and*

$$\int_\gamma \frac{\partial v}{\partial n}\,ds=1 \qquad (or\ -1).$$

Since $\|dv\|_{W_i}=\infty$ as stated above, we obtain the following lemma by applying the same method as in the proof of Theorem 12. 1 in Heins' paper [3].

LEMMA 7. *v has limit ∞ at β_i $(i=1,2)$.*

Now we can prove the following lemma by the aid of above those lemmas.

LEMMA 8. *There exists a harmonic function h_0 in W satisfying the following conditions:*

(1) $$\lim_{p\to\beta_1} h_0(p)=+\infty, \qquad \lim_{p\to\beta_2} h_0(p)=-\infty,$$

(2) *for all analytic Jordan closed curves c separating β_1 from β_2,*

$$\int_c \frac{\partial h_0}{\partial n}\,ds=1,$$

where c are oriented positively with respect to the component of $W-c$ which has β_1 as its boundary element.

(3) *$h_0=\lim_{n\to\infty}\lim_{k\to\infty} h_{n,k}$, where $h_{n,k}$ denotes the harmonic function in $(W-\bar{G}_n^{(1)}-\bar{G}_n^{(2)})\cap\Omega_k$ which has the boundary values h_0 on $\partial G_n^{(1)}\cup\partial G_n^{(2)}$ and whose normal derivative vanishes on $\partial\Omega_k\cap(W-\bar{G}_n^{(1)}-\bar{G}_n^{(2)})$.*

The imposed conditions determine h_0 up to an additive constant.

Proof. Let γ be an analytic Jordan closed curve dividing W into two parts W_1, W_2 such that β_i is a boundary element of W_i $(i=1,2)$ and let v_0 be a positive harmonic function in W_2 which has the properties imposed in Lemma 2 and

$$\int_\gamma \frac{\partial v_0}{\partial n}\,ds=-1.$$

We orient γ positively with respect to W_1. By Lemma 7 we have that

$$\lim_{p\to\beta_2} v_0(p)=+\infty.$$

For each positive number r we denote by W_r^* the region $\{p \mid v_0(p)<r\} \cup W_1$. We note that the set $\{p \mid v_0(p)<r\}$ is connected. Let ρ_1, ρ_2 $(\rho_2>\rho_1)$ be two sufficiently small positive numbers such that the niveau curve $\{p \mid v_0(p)=\rho_k\}$ is an analytic Jordan closed curve $(k=1, 2)$. By Lemma 2 there exists a positive harmonic function v_r in W_r^* which has the properties imposed in Lemma 2 and

$$\int_r \frac{\partial v_r}{\partial n} ds=1.$$

We introduce $\tilde{h}_r$ defined in W_r^* as $v_r-\min_{\partial W_{\rho_2}^*} v_r$ and show that the family $\{\tilde{h}_r\}$ is normal. Consider the harmonic function

$$V_r=\tilde{h}_r-\left(-\min_{\partial W_{\rho_2}^*} v_r\right)\frac{v_0-\rho_2}{r-\rho_2}.$$

This function vanishes on ∂W_r^* and a point of $\partial W_{\rho_2}^*$. Let V_r^k be the harmonic function in $(W_r^*-\overline{W}_{\rho_1}^*)\cap \Omega_k$ which has the boundary values V_r on $(\partial W_r^* \cap \Omega_k)\cup \partial W_{\rho_1}^*$ and whose normal derivative vanishes on $\partial \Omega_k \cap (W_r^*-\overline{W}_{\rho_1}^*)$. Here we assume that $\Omega_1 \supset \partial W_{\rho_1}^*$. Then we have that $V_r=\lim_{k\to\infty} V_r^k$ on $W_r^*-\overline{W}_{\rho_1}^*$ [6]. Since V_r^k takes its minimum on $(\overline{W}_r^*-W_{\rho_1}^*)\cap \bar{\Omega}_k$ at a point of $\partial W_{\rho_1}^* \cup (\partial W_r^* \cap \Omega_k)$, V_r does not take its minimum on $\overline{W}_r^*-W_{\rho_1}^*$ at any inner point of $W_r^*-\overline{W}_{\rho_1}^*$. Hence we have that $\min_{\partial W_{\rho_1}^*} V_r<0$ and therefore we obtain an inequality

$$\min_{\partial W_{\rho_1}^*} \tilde{h}_r<\left(-\min_{\partial W_{\rho_2}^*} v_r\right)\frac{\rho_1-\rho_2}{r-\rho_2}.$$

Next let $\tilde{V}_r^k$ be the harmonic function in $(W_r^*-\overline{W}_{\rho_2}^*)\cap \Omega_k$ which has the boundary values V_r on $(\partial W_r^* \cap \Omega_k)\cup \partial W_{\rho_2}^*$ and whose normal derivative vanishes on $\partial \Omega_k \cap (W_r^*-\overline{W}_{\rho_2}^*)$. Here we assume that $\Omega_1 \supset \partial W_{\rho_2}^*$. Again, we have $V_r=\lim_{k\to\infty} \tilde{V}_r^k$ on $W_r^*-\overline{W}_{\rho_2}^*$. Since $\tilde{V}_r^k=0$ on $\partial W_r^* \cap \Omega_k$ and $\tilde{V}_r^k \geqq 0$ on $\partial W_{\rho_2}^*$ we obtain that $\tilde{V}_r^k>0$ on $(W_r^*-\overline{W}_{\rho_2}^*)\cap \Omega_k$ and that hence

$$\int_{\partial W_r^* \cap \Omega_k} \frac{\partial \tilde{V}_r^k}{\partial n} ds>0.$$

Therefore we obtain that

$$\int_{\partial W_{\rho_2}^*} \frac{\partial \tilde{V}_r^k}{\partial n} ds>0$$

for all k. Consequently we have that

$$\int_{\partial W_{\rho_2}^*} \frac{\partial V_r}{\partial n} ds \geqq 0$$

and so that

$$\int_{\partial W_{\rho_2}^*} \frac{\partial \tilde{h}_r}{\partial n} ds \geqq \left(-\min_{\partial W_{\rho_2}^*} v_r\right)\frac{1}{r-\rho_2}\int_{\partial W_{\rho_2}^*} \frac{\partial v_0}{\partial n} ds.$$

Since

$$\int_{\partial W_{\rho_2}^*}\frac{\partial\tilde{h}_r}{\partial n}ds=-\int_{\partial W_{\rho_2}^*}\frac{\partial v_0}{\partial n}ds=1,$$

by those two inequalities we obtain that

$$0<\min_{\partial W_{\rho_1}^*}\tilde{h}_r<\rho_2-\rho_1.$$

Hence by use of Harnack's principle we see that $\{\tilde{h}_r\}$ is normal and that hence there exists a sequence $\{r_j\}_{j=1}^{\infty}$ increasing to ∞ such that $\{\tilde{h}_{r_j}\}_{j=1}^{\infty}$ converges to a harmonic function h_0 uniformly on every compact subset of W.

There remains to be shown that h_0 so obtained meets the conditions of the lemma. Since

$$\int_c\frac{\partial\tilde{h}_r}{\partial n}ds=1$$

for sufficiently large r, we have that

$$\int_c\frac{\partial h_0}{\partial n}ds=1.$$

Since $v_0<\max_{\partial G_n^{(2)}}v_0$ on $W_2-\bar{G}_n^{(2)}$, the set $\{p\,|\,v_0(p)>r_j\}$ is contained in $G_n^{(2)}$ for sufficiently large j. By use of the maximum and minimum principle we have an inequality

$$\begin{aligned}|h_{n,k}-\tilde{h}_{r_j}|&\leqq\left|h_{n,k}-v_{n,k}^{(r_j)}+\min_{\partial W_{\rho_2}^*}v_{r_j}\right|+\left|v_{n,k}^{(r_j)}-\min_{\partial W_{\rho_2}^*}v_{r_j}-\tilde{h}_{r_j}\right|\\&\leqq\max\left\{\max_{\partial G_n^{(1)}}\left|h_0-v_{r_j}+\min_{\partial W_{\rho_2}^*}v_{r_j}\right|,\ \max_{\partial G_n^{(2)}}\left|h_0-v_{n,k}^{(r_j)}+\min_{\partial W_{\rho_2}^*}v_{r_j}\right|\right\}\\&\quad+\left|v_{n,k}^{(r_j)}-\min_{\partial W_{\rho_2}^*}v_{r_j}-\tilde{h}_{r_j}\right|\qquad\text{on}\quad(W-\bar{G}_n^{(1)}-\bar{G}_n^{(2)})\cap\Omega_k,\end{aligned}$$

where $v_{n,k}^{(r_j)}$ denotes the harmonic function in $(W_{r_j}^*-\bar{G}_n^{(1)})\cap\Omega_k$ which has the boundary values v_{r_j} on $(\partial W_{r_j}^*\cap\Omega_k)\cup\partial G_n^{(1)}$ and whose normal derivative vanishes on $\partial\Omega_k\cap(W_{r_j}^*-\bar{G}_n^{(1)})$.

Since $v_{r_j}=\lim_{k\to\infty}v_{n,k}^{(r_j)}$ on $W_{r_j}^*-\bar{G}_n^{(1)}$,

$$\left|\lim_{k\to\infty}h_{n,k}-\tilde{h}_{r_j}\right|\leqq\max\left\{\max_{\partial G_n^{(1)}}\left|h_0-v_{r_j}+\min_{\partial W_{\rho_2}^*}v_{r_j}\right|,\ \max_{\partial G_n^{(2)}}\left|h_0-v_{r_j}+\min_{\partial W_{\rho_2}^*}v_{r_j}\right|\right\}$$

on $W-\bar{G}_n^{(1)}-\bar{G}_n^{(2)}$. Further since $h_0=\lim_{j\to\infty}\tilde{h}_{r_j}$, we obtain that

$$h_0=\lim_{k\to\infty}h_{n,k}\qquad\text{on}\quad W-\bar{G}_n^{(1)}-\bar{G}_n^{(2)},$$

and so that

$$h_0=\lim_{n\to\infty}\lim_{k\to\infty}h_{n,k}\qquad\text{on}\quad W.$$

Now we can verify easily that $h_0-H_{h_0}^{W_1}$ and $-(h_0-H_{h_0}^{W_2})$ have the properties imposed in Lemma 2. Consequently, by Lemma 5 and Lemma 7 we have that

$$\lim_{p\to\beta_1} h_0(p) = -\lim_{p\to\beta_2} h_0(p) = +\infty.$$

Let h_0' be another harmonic function in W satisfying the conditions in the lemma. Then by Lemma 6 we have that

$$h_0 - H_{h_0}^{W_i} = h_0' - H_{h_0'}^{W_i} \qquad \text{on} \quad W_i \ (i=1,2).$$

This implies that $h_0 - h_0'$ is harmonic on W and takes its maximum on W at a point of γ. We conclude that $h_0 - h_0'$ is reduced to a constant by use of the maximum principle.

LEMMA 9. *If φ is a member of $A(\beta_1, \beta_2)$, then either $h_0 \circ \varphi = h_0 + \lambda_\varphi$ or else $h_0 \circ \varphi = -h_0 + \lambda_\varphi$, where λ_φ is a constant. Further*

(1) *there exists an integer n such that φ_n is the identity mapping if and only if either $h_0 \circ \varphi = h_0$ or $h_0 \circ \varphi = -h_0 + \lambda_\varphi$,*

(2) *φ_n is not the identity mapping for any integer $n (\neq 0)$ if and only if $h_0 \circ \varphi = h_0 + \lambda_\varphi$, $\lambda_\varphi \neq 0$.*

Proof. The former part of the lemma is an obvious consequence of Lemma 8.

Suppose that $h_0 \circ \varphi = h_0$. Assume that φ_m is not the identity mapping for any integer $m (\neq 0)$. Let γ be a Jordan closed curve separating β_1 from β_2 and let $\{\tilde{\Omega}_n\}_{n=1}^{\infty}$ be an exhaustion of W such that $G_n^{(i)}$ $(i=1,2)$ is a component of $W - \bar{\tilde{\Omega}}_n$. Applying Lemma 1 there exists an integer m such that $\varphi_m(\gamma) \cap \tilde{\Omega}_n = \phi$. Since $\varphi_m(\gamma)$ separates β_1 from β_2, $\varphi_m(\gamma)$ must be contained in $G_n^{(1)} \cup G_n^{(2)}$. Thus we can select a sequence $\{\varphi_{m_k}(p)\}_{k=1}^{\infty}$ tending to β_1 or β_2 for a point $p \in \gamma$. For such a sequence

$$\lim_{k\to\infty} h_0 \circ \varphi_{m_k}(p) = +\infty \qquad \text{or} \qquad \lim_{k\to\infty} h_0 \circ \varphi_{m_k}(p) = -\infty.$$

On the other hand $h_0 \circ \varphi_m(p) = h_0(p)$ for all integers m. This is a contradiction.

Suppose that $h_0 \circ \varphi = -h_0 + \lambda_\varphi$. Since $h_0 \circ \varphi_2 = h_0$, we conclude that there exists an integer n such that φ_n is the identity mapping, as before.

Suppose that $h_0 \circ \varphi = h_0 + \lambda_\varphi$, $\lambda_\varphi \neq 0$. Evidently φ_n is not the identity mapping for any integer $n (\neq 0)$ because of the equation

$$h_0 \circ \varphi_n = h_0 + n\lambda_\varphi.$$

Finally we prove a lemma which completes the proof of Theorem 4. We denote by $A^1(\beta_1, \beta_2)$ the class of members φ of $A(\beta_1, \beta_2)$ such that $h_0 \circ \varphi = h_0$, by $A^2(\beta_1, \beta_2)$ the class of members φ of $A(\beta_1, \beta_2)$ such that $h_0 \circ \varphi = h_0 + \lambda_\varphi$, $\lambda_\varphi \neq 0$, and by $A^3(\beta_1, \beta_2)$ the class of members φ of $A(\beta_1, \beta_2)$ such that $h_0 \circ \varphi = -h_0 + \lambda_\varphi$. For these classes we conclude

LEMMA 10. (1) *$A^1(\beta_1, \beta_2)$ is a finite group.*

(2) *There exists a member $\tilde{\varphi}$ of $A^2(\beta_1, \beta_2)$ such that each member of $A^2(\beta_1, \beta_2)$ is expressed as the composition of a member of $A^1(\beta_1, \beta_2)$ and an iterate of $\tilde{\varphi}$. $\tilde{\varphi}$ is determined uniquely up to a member of $A^1(\beta_1, \beta_2)$.*

(3) *Let $\tilde{\Phi}$ be an arbitrary member of $A^3(\beta_1, \beta_2)$. Then each member of $A^3(\beta_1, \beta_2)$ is expressed as the proper composition of a member of $A^1(\beta_1, \beta_2)$, $\tilde{\Phi}$ and an iterate of $\tilde{\varphi}$.*

Proof. (1) Assume that $A^1(\beta_1, \beta_2)$ contains infinitely many members. Then applying the same method as in the proof of Lemma 3, there exists a member φ of $A^1(\beta_1, \beta_2)$ such that $\{\varphi_n(p)\}_{n=1}^{\infty}$ tends to β_1 or β_2. On the other hand $h_0 \circ \varphi_n(p) = h_0(p)$. It is a contradiction.

(2) First we show that $\lambda_0 = \inf_{A^2(\beta_1, \beta_2) \ni \varphi} \lambda_\varphi$ is positive and there exists a member $\tilde{\varphi}$ of $A^2(\beta_1, \beta_2)$ such that $\lambda_{\tilde{\varphi}} = \lambda_0$.

If λ_0 were zero, then there exists a sequence $\{\varphi^{(\nu)}\}_{\nu=1}^{\infty}$ of distinct members of $A^2(\beta_1, \beta_2)$ satisfying the condition that the sequence $\{\lambda_{\varphi^{(\nu)}}\}_{\nu=1}^{\infty}$ converges to zero. Hence we see that

$$\lim_{\nu \to \infty} h_0 \circ \varphi^{(\nu)}(p) = \lim_{\nu \to \infty} \{h_0(p) + \lambda_{\varphi^{(\nu)}}\} = h_0(p).$$

On the other hand we may assume that $\{\varphi^{(\nu)}(p)\}_{\nu=1}^{\infty}$ tends to β_1 or β_2 for $\nu \to \infty$ and so that $\lim_{\nu \to \infty} h_0 \circ \varphi^{(\nu)}(p) = +\infty$ or $\lim_{\nu \to \infty} h_0 \circ \varphi^{(\nu)}(p) = -\infty$. It is a contradiction. Suppose that $\lambda_\varphi \neq \lambda_0$ for all members φ of $A^2(\beta_1, \beta_2)$, then we are led to a contradiction by the same reasoning as before. Thus we obtain that there exists a member $\tilde{\varphi}$ of $A^2(\beta_1, \beta_2)$ such that $\lambda_{\tilde{\varphi}} = \lambda_0$.

Let φ be an arbitrary member of $A^2(\beta_1, \beta_2)$ and suppose that $\lambda_\varphi \neq n\lambda_0$ for any integer n. There exists an integer m such that $0 < \lambda_\varphi - m\lambda_0 < \lambda_0$. This implies that $\varphi \circ \tilde{\varphi}_{-m} \in A^2(\beta_1, \beta_2)$ and that $\lambda_{\varphi \circ \tilde{\varphi}_{-m}} = \lambda_\varphi - m\lambda_0 < \lambda_0$. It is a contradiction. Hence $\lambda_\varphi = n\lambda_0$ for an integer n. Thus we conclude that $\varphi \circ \tilde{\varphi}_{-n}$ is a member of $A^1(\beta_1, \beta_2)$.

(3) Let $\tilde{\varphi}$ be an arbitrary member of $A^3(\beta_1, \beta_2)$ and let φ be another arbitrary member of $A^3(\beta_1, \beta_2)$. Then we see that $\varphi \circ \tilde{\varphi}_{-1}$ is a member of $A^1(\beta_1, \beta_2)$ or $A^2(\beta_1, \beta_2)$ because of the equation

$$h_0 \circ \varphi \circ \tilde{\varphi}_{-1} = h_0 + \lambda_\varphi + \lambda_{\tilde{\varphi}}.$$

This lemma infers that $A(\beta_1, \beta_2)$ is finitely generated and we have proved Theorem 4.

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