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# $\mathcal{E}$-REGENERATIVE PHENOMENA IN SOME STOCHASTIC PROCESSES

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**Summary.**

Feller's theory of recurrent events is a powerful tool for the study of regenerative stochastic processes in discrete time. The concept of "regeneration" in a stochastic process with continuous time parameter has been formulated in two ways. One is the theory of "regenerative stochastic processes" developed by Smith [10], and the other is the theory of "regenerative events" by Kingman [5], [6].

In this paper we consider the probabilities of an event in a stochastic process containing a regenerative event. Our event corresponds to the set $\mathcal{A}$ which is a subset of the state space in Smith's theory. However our difinition and discussions are based on Kingman's theory, and are applicable to the study of Markov processes with a continuous state space.

It is shown that the fact that a process contains a regenerative event imposes fairy strong conditions on the probabilities of an event.

Results obtained are compared with those in the papers [7], [9] by Kingman about the transition probabilities of Markov chains with a countable state space.

## § 1. Introduction.

Feller's theory of recurrent events is a powerful tool for the study of discrete time parameter Markov chains [3], [4]. A continuous time analogue of Feller's theory was given by Kingman in [5], [6]. Kingman's theory which is called the theory of regenerative events, together with its extentions by himself, provides a useful technique for continuous time parameter Markov processes with a countable state space [9].

Kingman defines a regenerative event $\mathcal{E}$ on a probability space $(\Omega, \mathcal{F}, P)$ to be a family $\{E(t), t>0\}$ of $\mathcal{F}$-measurable subsets of $\Omega$, such that, whenever

$$(1)\qquad 0<t_1<t_2<\cdots<t_k,$$

we have

$$(2)\qquad P\{E(t_1)E(t_2)\cdots E(t_k)\}=P\{E(t_1)\}P\{E(t_2-t_1)\cdots E(t_k-t_1)\}.$$

This difinition is appropriate to the study of the transition probabilities

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$$p_{ii}(t)=P\{X_t=i|X_0=i\},$$

in a Markov chain $\{X_t, t>0\}$ with stationary transition probabilities.

To deal with the off-diagonal elements $p_{ij}(t)\,(i\neq j)$ of the transition matrix, Kingman defines delayed regenerative events and linked systems of regenerative events as follows: A delayed regenerative event is a family $\{E(t), t>0\}$ of $\mathcal{F}$-measurable subsets of $\Omega$ for which there exist $p^0, p$ on $t>0$ such that, for all $t_1, \cdots, t_k$ satisfying (1), we have

$$P\{E(t_1)E(t_2)\cdots E(t_k)\}=p^0(t_1)p(t_2-t_1)\cdots p(t_k-t_{k-1}).$$

A linked system of $N$ regenerative events is a system

$$\varphi=(\Omega, \mathcal{F}, P_i, E_i(t);\ i=1,2,\cdots,N;\ t>0),$$

where $\Omega$ is any set, $\mathcal{F}$ is a $\sigma$-algebra of subsets of $\Omega$, $P_i$ is a (complete) probability measure on $\Omega$, and $E_i(t)$ is a member of $\mathcal{F}$, such that

$$E_i(t)\cap E_j(t)=\phi \qquad (i\neq j,\ t>0) \tag{3}$$

and such that, whenever

$$k\geqq 1,\quad i_0, i_1, \cdots, i_k\in\{1,2,\cdots N\},\quad 0<t_1<t_2\cdots<t_k,$$

we have

$$\begin{aligned} &P_{i_0}\{E_{i_1}(t_1)E_{i_2}(t_2)\cdots E_{i_k}(t_k)\}\\ &=P_{i_0}\{E_{i_1}(t_1)\}P_{i_1}\{E_{i_2}(t_2-t_1)\cdots E_{i_k}(t_k-t_1)\}. \end{aligned} \tag{4}$$

Applying Kingman's theories of delayed regenerative events and linked systems of regenerative events we can obtain many properties of $p_{ij}(t)\,(i\neq j)$ such as the continuity of $p_{ij}(t)$, the local bounded variation and almost everywhere differentiability of $p_{ij}(t)$, the existence of the limit of $p_{ij}(t)$ as $t\to\infty$, and so on [6], [7], [8].

Under the definitions, these results about $p_{ij}(t)\,(i\neq j)$ are consequences of the fact that the state $j$ is regenerative.

By the way, it will be also important to investigate what properties about $p_{ij}(t)\,(i\neq j)$ stem from only the fact that the state $i$ is regenerative.

The purpose of this paper is to give an answer to this question in a more general setting. The results obtained in this paper are applicable to the study of the transition probabilities of discontinuous Markov processes with a continuous state space.

Let us consider a discrete time parameter stochastic process which contains a recurrent event $\mathcal{E}$ in the sense of Feller [3], [4]. Let us denote by $E(n)$ the event that at time $n$ $\mathcal{E}$ occurs, and $A(n)$ the event that at time $n$ a certain phenomenon, which will be denoted by the symbol $\mathcal{A}$, occurs. Moreover we suppose $\mathcal{E}$ and $\mathcal{A}$ to be disjunct, that is,

$$E(n)\cap A(n)=\phi. \tag{5}$$

Let $\mathfrak{U}(n)$ be a field containing the $\omega$-sets $E(n)$ and $A(n)$. If the phenomenon

$\mathcal{A}$ is not pathological, from the definition of recurrent events, whenever

$$0<n_1<n_2<\cdots<n_i<\cdots<n_{i+j},$$
$$J(n_k)\in\mathfrak{U}(n_k), \qquad k=1,2,\cdots,i+j,$$

we shall have

$$\begin{aligned}(6)\qquad & P\{J(n_1)J(n_2)\cdots J(n_{i-1})E(n_i)J(n_{i+1})\cdots J(n_{i+j-1})E(n_{i+j})\}\\ &=P\{J(n_1)J(n_2)\cdots E(n_i)\}P\{J(n_{i+1}-n_i)\cdots E(n_{i+j}-n_i)\}.\end{aligned}$$

But as Doob pointed out, it would be more logical to omit the condition that the last term in the left side probability of (6) is $E(n_{i+j})$. (See [1].)

Thus modifying the above, we make the following definition.

DEFINITION 1. *A discrete time parameter $\mathcal{E}$-regenerative phenomenon $\mathcal{A}$ is a family $\{A(n), n=1,2,\cdots\}$ of $\mathcal{F}$-measurable subsets of $\Omega$, such that, whenever*

$$(7)\qquad 0<n_1<n_2<\cdots<n_i<n_{i+1}<\cdots<n_{i+j},$$

*and*

$$(8)\qquad J(n_k)\in\mathfrak{U}(n_k), \qquad k=1,2,\cdots,i+j,$$

*where $\mathfrak{U}(n_k)$ is a field containing $E(n_k)$ and $A(n_k)$, we have*

$$\begin{aligned}(9)\qquad & P\{J(n_1)J(n_2)\cdots J(n_{i-1})E(n_i)J(n_{i+1})\cdots J(n_{i+j})\}\\ &=P\{J(n_1)J(n_2)\cdots E(n_i)\}P\{J(n_{i+1}-n_i)\cdots J(n_{i+j}-n_i)\}.\end{aligned}$$

It must be remarked that this definition implies that $\{E(n), n=1,2,\cdots\}$ is a recurrent event in Kingman's sense. Because if, for all $k$, $J(n_k)=E(n_k)$, (9) reduces to the equation

$$(10)\qquad P\{E(n_1)E(n_2)\cdots E(n_m)\}=P\{E(n_1)\}P\{E(n_2-n_1)\cdots E(n_m-n_1)\},$$

by which Kingman defines a recurrent event.

If $\mathcal{A}$ is an $\mathcal{E}$-regenerative phenomenon in Definition 1, define sequences $\{p_n\}$, $\{f_n\}$, $\{a_n\}$ and $\{w_n\}$ by

$$\begin{aligned}(11)\qquad & p_n=P\{E(n)\}, \qquad f_n=P\{\bar{E}(1)\bar{E}(2)\cdots\bar{E}(n-1)E(n)\},\\ & a_n=P\{A(n)\}, \qquad w_n=P\{\bar{E}(1)\bar{E}(2)\cdots\bar{E}(n-1)A(n)\},\end{aligned}$$

where $\bar{E}(k)$ denotes the complement of $E(k)$. Setting $p_0=1$, $a_0=0$, $f_0=0$, $w_0=0$, we have from (5) and (9)

$$(12)\qquad a_n=w_n+\sum_{r=0}^{n}f_r a_{n-r}=\sum_{r=0}^{n}p_r w_{n-r}. \qquad n=0,1,2,\cdots.$$

Equation (12) is equivalent to the power series relation

$$(13)\qquad \sum_{n=0}^{\infty}a_n z^n=\frac{\sum_{n=0}^{\infty}w_n z^n}{1-\sum_{n=0}^{\infty}f_n z^n}=\sum_{n=0}^{\infty}p_n z^n\cdot\sum_{n=0}^{\infty}w_n z^n \qquad (|z|<1).$$

The left hand equality of equation (12) is the so-called renewal equation. The sum $\sum_{n=0}^{\infty} w_n$ may diverge, but in the case of convergence we have the following theorem due to Feller [4].

THEOREM A. [Feller] *Suppose that* $\{f_n\}$ *is not periodic and that* $\sum w_n$ *is finite.*

(a) *If* $\sum f_n=1$, *then*

$$a_n \to \mu^{-1} \sum w_n \qquad where \qquad \mu = \sum n f_n. \tag{14}$$

*In particular,* $a_n \to 0$ *if* $\sum n f_n$ *diverges.*

(b) *If* $\sum f_n<1$, *then the series*

$$\sum a_n = \{1-\sum f_n\}^{-1} \sum w_n \tag{15}$$

*converges.*

## § 2. Continuous time parameter $\mathcal{E}$-regenerative phenomena.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. By analogy with Definition 1, we define continuous time parameter $\mathcal{E}$-regenerative phenomena. Let $E(t)$ and $A(t)$ be $\mathcal{F}$-measurable disjoint subsets of $\Omega$, and $\mathfrak{U}(t)$ be a field containing $E(t)$ and $A(t)$.

DEFINITION 2. *A continuous time parameter* $\mathcal{E}$*-regenerative phenomenon* $\mathcal{A}$ is a family $\{A(t),\ t>0\}$ of $\mathcal{F}$-measurable subsets of $\Omega$, such that, whenever

$$0<t_1<t_2<\cdots<t_i<\cdots<t_{i+j}, \tag{16}$$

and

$$J(t_k) \in \mathfrak{U}(t_k), \qquad k=1,2,\cdots,i+j, \tag{17}$$

we have,

$$\begin{aligned} &P\{J(t_1)J(t_2)\cdots J(t_{i-1})E(t_i)J(t_{i+1})\cdots J(t_{i+j})\} \\ &=P\{J(t_1)\cdots J(t_{i-1})E(t_i)\}P\{J(t_{i+1}-t_i)\cdots J(t_{i+j}-t_i)\}. \end{aligned} \tag{18}$$

As in discrete time parameter case, notice that the family $\mathcal{E}=\{E(t),\ t>0\}$ is a regenerative event, since $\mathcal{E}$ satisfies (2). Moreover it is clear that conditional on the occurrence of $\mathcal{E}$, the past and future are independent, i.e. $\mathcal{E}$ "regenerates" the process. (See Proposition 1.) Thus we call a phenomenon $\mathcal{A}$ satisfying the above conditions $\mathcal{E}$-regenerative.

Our main interest lies in the study of continuous time parameter $\mathcal{E}$-regenerative phenomena. Of prime importance is the functions

$$a(t)=P\{A(t)\}, \tag{19}$$

$$p(t)=P\{E(t)\}. \tag{20}$$

DEFINITION 3. These functions $a(t)$ and $p(t)$ will be called the *a-function* and the *p-function* of the continuous time parameter $\mathcal{E}$-regenerative phenomenon $\mathcal{A}$.

For brebity, in the following, we shall use the term "$\mathcal{E}$-regenerative phenomenon" in stead of "continuous time parameter $\mathcal{E}$-regenerative event".

An $\mathcal{E}$-regenerative phenomenon can easily be shown to have following three elementary properties. Proposition 1 states that if $\mathcal{E}$ occurs, then the process starts anew.

PROPOSITION 1. *Let B and D be any ω-sets belonging to the smallest σ-fields generated by the sets* $\{A(t), E(t), t<T\}$, *and generated by the sets* $\{A(t), E(t), t>T\}$, *respectively. Then we have for each* $T>0$,

$$P\{B\cdot E(T)\cdot D\}P\{E(T)\}=P\{B\cdot E(T)\}P\{E(T)\cdot D\}. \tag{21}$$

*Proof.* We can prove this proposition by the argument analogous to Kingman [6]. So we shall omit the details.

PROPOSITION 2. *Let A be an* $\mathcal{E}$*-regenerative phenomenon. Then for any* $h>0$,

$$\{A(nh),\ n=1, 2, \cdots\}$$

*is a discrete time parameter* $\mathcal{E}$*-regenerative phenomenon.*

Now Definition 2 imposes some regularity conditions on $a$-functions. For example, we have

PROPOSITION 3. *For any* $s, t>0$,

$$\max[p(s)a(t),\ p(t)a(s)]\leqq a(t+s)\leqq 1-\max[p(s)\{1-a(t)\},\ p(t)\{1-a(s)\}]. \tag{22}$$

*Proof.* From (18), we have, for $t_1<t_2$,

$$P\{\bar{E}(t_1)A(t_2)\}=P\{A(t_2)\}-P\{E(t_1)A(t_2)\}=a(t_1)-p(t_1)a(t_2-t_1),$$

and

$$P\{\bar{E}(t_1)\bar{A}(t_2)\}=1-p(t_1)-a(t_2)+p(t_1)a(t_2-t_1).$$

These lead to (22).

Let $\{X_t, t\geqq 0\}$ be a Markov process on a continuous state space $S$. We assume that $X_0=\xi$ a.e., and $A\subset S$, and that the transition probabilities

$$p(t;\xi, A)=P\{X_{t+s}\in A|X_s=\xi\}$$

are independent of $s$. Consider a phenomenon $\mathcal{A}$ defined by

$$A(t)=\{\omega;\ X_t\in A\} \qquad \text{and} \qquad E(t)=\{\omega;\ X_t=\xi\}.$$

Then $\mathcal{A}$ is an $\mathcal{E}$-regenerative phenomenon, since Definition 2 is satisfied. Hence we can deal with Markov processes on a continuous state space.

## § 3. Standard phenomena.

In Kingman's theory a regenerative event is called standard if

$$p(t)=P\{E(t)\}\to 1 \qquad (t\to 0) \tag{23}$$

and the class of all the $p$-functions of standard events is denoted by $\mathcal{P}$. We also assume (23) for our phenomena.

DEFINITION 4. *A standard phenomenon* $\mathcal{A}$ is an $\mathcal{E}$-regenerative phenomenon $\mathcal{A}$

satisfying (23). Let $\mathcal{H}$ be the class of all the standard $\mathcal{E}$-regenerative phenomena.

For later uses, we quote here some results from the theory of regenerative events [6], [7].

THEOREM B. [Kingman] *If $p(t)\in\mathcal{P}$, then* (i) *$p(t)$ is strictly positive and uniformly continuous, in $0<t<\infty$. Moreover* (ii) *$p(t)$ is of bounded variation in every finite interval, and is thus differentiable almost everywhere.* (iii) *The limit*

$$q=\lim_{t\to 0}\frac{1-p(t)}{t} \tag{24}$$

*exists (it is possibly infinite), and if $q<\infty$ then*

$$p(t)\geqq e^{-qt} \tag{25}$$

*for all $t>0$.* (iv) *The limit*

$$\tilde{\omega}=\lim_{t\to\infty}p(t) \tag{26}$$

*exists.*

We begin with showing that $a$-functions of phenomena in $\mathcal{H}$ have similar analytic properties as $p$-functions in $\mathcal{P}$.

THEOREM 1. *Let $\mathcal{A}\in\mathcal{H}$. Then*

(i) *$a(t)$ of the phenomenon $\mathcal{A}$ is uniformly continuous in $0<t<\infty$.*

(ii) *If for some $t_0$, $a(t_0)$ is positive, then $a(t)$ is strictly positive in $t_0\leqq t<\infty$. Moreover if the limit* (24) *of the $p$-function of $\mathcal{A}$ is finite, then*

$$a(t)\geqq a(t_0)e^{-q(t-t_0)}\qquad (t\geqq t_0). \tag{27}$$

(iii) *If $q<\infty$, then $a(t)$ satisfies the Lipschitz condition*

$$|a(t_1)-a(t_2)|\leqq 1-\exp(-q|t_1-t_2|)\leqq q|t_1-t_2|. \tag{28}$$

*Proof.* (i) From (22) we have for any $h, t>0$,

$$-a(t)\{1-p(h)\}\leqq a(t+h)-a(t)\leqq(1-p(h))(1-a(t)).$$

Therefore

$$|a(t+h)-a(t)|\leqq 1-p(h). \tag{29}$$

From (23) it follows that $a(t)$ is uniformly continuous in $0<t<\infty$.

(ii) This follows from (i), (iii) of Theorem B and (22).

(iii) If $q<\infty$,

$$1-p(h)\leqq 1-e^{-qh}. \tag{30}$$

From (29) and (30), we obtain (28). Q.E.D.

Further analytical properties of $a$-functions, such as bounded variation, can be proved, but we shall carry it out in Section 5 after deriving an important integral formula in the next section.

## § 4. The integral formula.

In this section we derive an integral formula which is the continuous time analogue of the right hand equality of the equation (12). We proceed in several steps.

Let $\mathcal{A}$ be a standard $\mathcal{E}$-regenerative phenomenon. Then from Proposition (13) we can write, for any $h>0$, and $|z|<1$, setting $p(0)=1$ and $a(0)=0$,

$$\sum_{n=0}^{\infty} a(nh)z^n = \sum_{n=0}^{\infty} p(nh)z^n \cdot \sum_{n=0}^{\infty} w_n(h)z^n$$

where

$$w_n(h)=P\{\bar{E}(h)\bar{E}(2h)\cdots\bar{E}((n-1)h)A(nh)\}.$$

Let $\theta$ be any complex number with strictly positive real part, and put

$$H_h(\theta)=\sum_{n=0}^{\infty} w_n(h)e^{-\theta nh}.$$

Then we have

$$h\sum_{n=0}^{\infty} a(nh)e^{-\theta nh}=h\sum_{n=0}^{\infty} p(nh)e^{-\theta nh}\cdot H_h(\theta).$$

The left hand converges, as $h\to 0$, to

$$s(\theta)=\int_0^\infty a(t)e^{-\theta t}dt.$$

Hence it follows

$$H_h(\theta)\to\frac{s(\theta)}{r(\theta)} \qquad (h\to 0), \tag{31}$$

where

$$r(\theta)=\int_0^\infty p(t)e^{-\theta t}dt.$$

Now we introduce a measure $\tau_h$ on $[0,\infty)$ with mass $w_n(h)$ at $x=nh$ for each $n=1,2,\cdots$. Then

$$H_h(\theta)=\int_{[0,\infty)} e^{-\theta x}\tau_h(dx).$$

Let $[x]$ denote the largest integer which does not exceed $x$. If there exists a function $M(x)$ such that

$$\sum_{n=0}^{[x/h]} w_n(h)\leqq M(x)<\infty, \tag{32}$$

it follows that there exist a sequence $\{h_j\}$ $(j=1,2,\cdots)$, tending to zero, and a measure $\tau$ on $[0,\infty)$ such that

(33) $$\int e^{-\theta x}\tau_{h_j}(dx)\to\int e^{-\theta x}\tau(dx) \qquad (j\to\infty).$$

Then from (31) and (33), we have

(34) $$\frac{s(\theta)}{r(\theta)}=\int_{[0,\infty)} e^{-\theta x}\tau(dx).$$

It is clear that $\tau$ has no atom at $x=0$, since $\theta r(\theta)\to 1$, $\theta s(\theta)\to 0$, as $\theta\to\infty$, by monotone convergence principle we have

$$0=\lim_{\theta\to\infty}\frac{s(\theta)}{r(\theta)}=\lim_{\theta\to\infty}\left(\tau\{0\}+\int_0^\infty e^{-\theta x}\tau(dx)\right)=\tau\{0\}$$

where $\tau\{0\}$ is the mass of the measure $\tau$ at $x=0$. If measures $\tau_1$, $\tau_2$ satisfy (33), then

$$\int_0^\infty e^{-\theta x}\tau_1(dx)=\int_0^\infty e^{-\theta x}\tau_2(dx).$$

Therefore $\tau_1=\tau_2$, that is, the measure $\tau$ is unique. We now define for each $x\geqq 0$,

$$K(x)=\tau[0, x].$$

Summing up these, we have the following

LEMMA 1. *If a finite-valued function* $M(x)$ *satisfies* (32), *then there exists a unique* (*finite or infinite*) *measure* $\tau$ *with,*

(35) $$s(\theta)=r(\theta)\int_0^\infty e^{-\theta x}\tau(dx),$$

*for all* $\theta$ *in* $\operatorname{Re}\theta>0$. *Or,*

(36) $$a(t)=\int_0^t p(t-s)dK(s).$$

Notice that $K(t)$ may diverge as $t\to\infty$, and that in the deduction of (36) we used the continuity of $a(t)$ and $p(t)$.

In the next step, we shall prove that for any $\mathcal{E}$-regenerative phenomenon in $\mathcal{H}$, there exists a function $M(x)$ satisfying (32). Following a theorem by Kingman [6], we shall prove it.

THEOREM C. [Kingman] *Let* $p(t)\in\mathcal{P}$. *Then there exists a unique measure* $\mu$ *on* $(0,\infty]$ *with*

(37) $$\int_{(0,\infty]}(1-e^{-x})\mu(dx)<\infty$$

*such that the Laplace transform* $r(\theta)$ *of* $p(t)$ *satisfies*

(38) $$\frac{1}{r(\theta)}=\theta+\int_{(0,\infty]}(1-e^{-\theta x})\mu(dx)$$

*for all* $\theta$ *in* $\operatorname{Re}\theta>0$. *And* $p(t)$ *satisfies the Volterra equation*

(39) $$1-p(t)=\int_0^t p(t-s)\mu(s,\infty]ds.$$

Now we define a new $\mathcal{E}$-regenerative phenomenon $\mathcal{A}^*$ by

(40) $$E^*(t)=E(t), \qquad A^*(t)=\bar{E}(t) \qquad (t>0),$$

and $w_n^*(h)$ by

$$w_n^*(h)=P\{\bar{E}^*(h)\bar{E}^*(2h)\cdots\bar{E}^*((n-1)h)A^*(nh)\}.$$

Then from (11)

(41) $$w_n(h)\leqq 1-\sum_{i=1}^{n} f_i(h)=w_n^*(h).$$

(12) leads to

(42) $$\begin{aligned}1-p\left(\left[\frac{x}{h}\right]h\right)&=\sum_{i=0}^{[x/h]}p\left(\left[\frac{x}{h}\right]h-ih\right)w_i^*(h)\\&=\int_{[0,[x/h]h]}p\left(\left[\frac{x}{h}\right]h-s\right)dV_h(s)\end{aligned}$$

where $V_h(t)$ is defined as follows:

$$V_h(t)=\sum_{i=1}^{[x/h]}w_i^*(h).$$

On the other hand, from (39)

(43) $$1-p\left(\left[\frac{x}{h}\right]h\right)=\int_0^{[x/h]h}p\left(\left[\frac{x}{h}\right]h-s\right)\mu(s,\infty]ds.$$

Since $p(t)$ is strictly positive and continuous in $0\leqq t<\infty$, it follows from (42) and (43)

(44) $$V_h\left(\left[\frac{x}{h}\right]h\right)\leqq\frac{1}{\min\limits_{0\leqq s\leqq x}p(s)}\int_0^x\mu(s,\infty]ds.$$

By the way, (37) of Theorem C implies

(45) $$\int_0^x\mu(s,\infty]ds<\infty.$$

This completes the proof of

LEMMA 2. *For any $h>0$, it satisfies*

(46) $$\sum_{n=0}^{[x/h]}w_n(h)\leqq\frac{1}{\min\limits_{0\leqq s\leqq x}p(s)}\int_0^x\mu(s,\infty)ds.$$

Consider the phenomenon $\mathcal{A}^*$. From (36) and (39) it is easily shown that for almost all $t$, $K^*(t)$ is differentiable and $dK^*/dt=\mu(t,\infty]$ where $K^*(t)$ denotes the $K(t)$ of $\mathcal{A}^*$ defined above, LEMMA 3. *For any $x_1$, $x_2$ with*

$$0 \leqq x_1 \leqq x_2 < \infty,$$

$$K(x_2) - K(x_1) \leqq \int_{x_1}^{x_2} \mu(s, \infty] ds \tag{47}$$

*holds.*

*Proof.* The continuity of $a(t)$ and (36) imply the continuity of $K(x)$, hence for any $x_1$, $x_2$ with $0 \leqq x_1 \leqq x_2 < \infty$, we have

$$K(x_2) - K(x_1) = \lim_{h \to 0} \sum_{n=[x_1/h]+1}^{[x_2/h]} w_n(h).$$

From (41),

$$\lim_{h \to 0} \sum_{n=[x_1/h]+1}^{[x_2/h]} w_n(h) \leqq \lim_{h \to 0} \sum_{n=[x_1/h]+1}^{[x_2/h]} w_n^*(h). \tag{48}$$

Since $K^*(t)$ is a function of bounded variation and $dK^*/dt = \mu(t, \infty]$ a.e., we have

$$K^*(x_2) - K^*(x_1) = \int_{x_1}^{x_2} \mu(s, \infty] ds. \tag{49}$$

Thus from (48) and (49), (47) is concluded.

We are now in the position to prove that $K(t)$ is absolutely continuous.

LEMMA 4. *$K(t)$ is absolutely continuous and there is a measurable function $u(t)$ such that for all $t > 0$*

$$0 \leqq u(t) \leqq \mu(t, \infty], \tag{50}$$

*and*

$$K(t) = \int_0^t u(s) ds. \tag{51}$$

*Proof.* (45) implies that for any $\varepsilon > 0$, there is a positive number $\delta$ with

$$\int_0^\delta \mu(s, \infty] ds < \varepsilon. \tag{52}$$

Let $I_1 = [a_1, b_1)$, $I_2 = [a_2, b_2)$, $\cdots$ $I_n = [a_n, b_n)$ be disjoint intervals in $[0, \infty)$ with

$$\sum_{k=1}^n (b_k - a_k) < \delta.$$

Because of the monotonicity of $\mu(s, \infty]$ with respect to $s$, and Lemma 3, we have

$$\sum_{k=1}^n |K(b_k) - K(a_k)| \leqq \int_0^\delta \mu(s, \infty] ds.$$

This and (52) mean that $K(t)$ is absolutely continuous in $t \geqq 0$. For almost all $t > 0$ (50) holds, since for almost all $t > 0$

$$0 \leqq \frac{dK(t)}{dt} = \lim_{h \downarrow 0} \frac{K(t+h)-K(t)}{h} \leqq \lim_{h \downarrow 0} \int_t^{t+h} \frac{\mu(s, \infty]}{h} ds = \mu(t, \infty].$$

Therefore we can take $u(t)$ such that (50) holds for all $t>0$. Hence the proof is complete.

Combining Lemma 1, 2 and 4, we have the following main

THEOREM 2. *Let* $\mathcal{A} \in \mathcal{H}$. *Then there is a measurable function* $u(t)$ *which is unique except for the values of Lebesgue measure zero of* $t$ *such that*

$$0 \leqq u(t) \leqq \mu(t, \infty] \tag{53}$$

*and*

$$a(t) = \int_0^t p(t-s)u(s)ds \tag{54}$$

*for all* $t \geqq 0$. *Thus the Laplace transform* $s(\theta)$ *of* $a(t)$ *is written as follows*:

$$s(\theta) = r(\theta)u^*(\theta) \tag{55}$$

*where* $\operatorname{Re} \theta > 0$ *and*

$$u^*(\theta) = \int_0^\infty e^{-\theta x} u(x) dx.$$

## § 5. Analytic properties of $a$-functions.

Kingman obtained the following result in [8].

THEOREM D. [Kingman] *Let* $p(t)$ *in* $\mathcal{P}$ *have*

$$\tilde{\omega} = \lim_{t \to \infty} p(t) > 0. \tag{56}$$

*Then* $p(t)$ *has totally bounded variation on* $(0, \infty]$.

THEOREM 3. *Let* $\mathcal{A} \in \mathcal{H}$. *Then the* $a$*-function of* $\mathcal{A}$ *is of bounded variation in every finite interval, and is thus differentiable almost everywhere in* $t>0$. *Moreover if the* $p$*-function of* $\mathcal{A}$ *satisfies* (56), *then* $a(t)$ *has totally bounded variation on* $(0, \infty)$.

*Proof.* Let $I=(0, t]$, $I-s=(-s, t-s]$ $(s<t)$ and $\operatorname{Var}(f, I)$ be the variation of the function $f$ in $I$. Then from (54), it follows

$$\operatorname{Var}(a, I) \leqq \int_0^t \operatorname{Var}(p, I-s)u(s)ds. \tag{57}$$

By Theorem B (ii), $\operatorname{Var}(p, I)$ is finite. Hence the right hand of (57) is finite. Moreover if the limit $\tilde{\omega}$ is positive, then clearly $\lim_{x\to\infty} K(x) = K(\infty)$ is finite. Now (57) shows for any interval $I=(0, t]$

$$\operatorname{Var}(a, I) \leq \operatorname{Var}(p, I) K(\infty).$$

Applying theorem D, we have the assertions to be proved.

We can easily show that the differentiabilities of $a(t)$ and $K(t)$ at $t=0$ are equivalent. If these are differentiable, then $a'(0)=K'(0)\leqq q=\lim_{h\to 0}h^{-1}(1-p(h))$, since $p(t)\to 1$ as $t\to 0$ and (36) holds. If so, then we shall have to ask under what conditions $a'(0)$ exists. It suffices to consider the case in which $a(t)$ is strictly positive in $t>0$. Otherwise the derivative $a'(0)$ exists and is equal to zero because of Theorem 1 (ii).

THEOREM 4. *Let $\mathcal{A}\in\mathcal{H}$. Assume that the a-function of A is strictly positive in* $0<t<\infty$. *If*

$$P\{A(t+\delta)|A(t)\}\to 1 \qquad (\delta\to 0), \tag{58}$$

*holds uniformly in t near the origin, and the limit $q=\lim_{h\to 0}h^{-1}(1-p(h))$ is finite, then $a'(0)$ exists and finite.*

*Proof.* $a(nh)=P\{A(nh)\}\geqq \sum_{\nu=1}^{n-1} P\{E(h)E(2h)\cdots E(\nu h)A(\nu h+h)A(nh)\}$

$$=P\{A(h)A(nh)\}+\sum_{\nu=0}^{n-1} P\{E(h)\cdots E(\nu h)\}P\{A(h)A(nh-\nu h)\}$$

$$=\sum_{\nu=0}^{n-1}\{p(h)\}^{\nu}a(h)P\{A(nh-\nu h)|A(h)\}.$$

From (58), for any $\varepsilon>0$, there exists a $t_0(\varepsilon)$ such that, whenever $nh\leqq t_0$, then we have

$$a(nh)\geqq(1-\varepsilon)\frac{1-\{p(h)\}^n}{1-p(h)}a(h).$$

Meanwhile every standard regenerative event can be regarded as separable. Then it follows

$$a(t)\geqq(1-\varepsilon)\frac{1-e^{-qt}}{q}\varlimsup_{h\to 0}\frac{a(h)}{h}. \tag{59}$$

Consequently $\varlimsup_{h\to 0}(a(h)/h)$ is finite. Dividing both side of (59) by $t$, and letting $t\to 0$, we have

$$\varliminf_{\to 0}\frac{a(t)}{t}\geqq\varlimsup_{h\to 0}\frac{a(h)}{h}.$$

## § 6. Limit probability.

We shall now examine the limiting behaviour as $t\to\infty$ of $a$-function of $\mathcal{E}$-regenerative phenomena in $\mathcal{H}$. Theorem B (iv) asserts that any $p$-function in $\mathcal{P}$ has the limit $\tilde{\omega}=\lim_{t\to\infty}p(t)$. But Theorem A for the discrete time parameter case suggests that $a$-function has not always the limit $\tilde{\alpha}=\lim_{t\to\infty}a(t)$. A sufficient condition for the limit to exist is a condition corresponding to the finiteness of the sum $\sum_{n=0}^{\infty}w_n$.

THEOREM 5. *Let* $\mathcal{A}\in\mathcal{H}$.

(i) *If* $\tau$ *is a finite measure, or* $K(\infty)=\lim_{x\to\infty}K(x)$ *is finite, then*

$$a(t)\to\tilde{\omega}\cdot K(\infty)=\tilde{\omega}\int_0^\infty u(s)ds \tag{60}$$

*where* $\tilde{\omega}=\lim_{t\to\infty}p(t)$.

(ii) *And if*

$$\int_0^\infty p(t)dt<\infty,$$

*then*

$$\int_0^\infty a(t)dt=K(\infty)\cdot\int_0^\infty p(t)dt<\infty. \tag{61}$$

*Proof*. Applying the dominated convergence principle to (36) and (54), we obtain (i). Taking $\theta\to 0$ in (55) leads to (ii).

Let $h>0$ be fixed, then we have from (22)

$$\int_{nh}^{(n+1)h}a(t)dt=\int_0^h a(nh+u)du\geqq a(nh)\int_0^h p(u)du\equiv a(nh)B, \qquad \text{say,}$$

$$\int_{(n-1)h}^{nh}a(t)dt=\int_0^h a(nh-u)du\leqq a(nh)\int_0^h \frac{du}{p(u)}\equiv a(nh)C, \qquad \text{say,}$$

where from Theorem B (i), $0<B$, $C<\infty$. And so

$$B\sum_{n=0}^\infty a(nh)\leqq\int_0^\infty a(t)dt\leqq C\sum_{n=0}^\infty a(nh).$$

This yields the first part of the following theorem.

THEOREM 6. *Let* $\mathcal{A}\in\mathcal{H}$. *Then*

(i) $\int_0^\infty a(t)dt$ *converges if and only if, for all* $h>0$, $\sum_{n=0}^\infty a(nh)$ *converges.*

(ii) *If* $\int_0^\infty p(t)dt$ *diverges, then either* $a(t)\equiv 0$ *in* $0<t<\infty$ *or* $\int_0^\infty a(t)dt$ *diverges.*

(iii) *In the last case, with probability one, the phenomenon* $A$ *occurs after an arbitrary time.*

*Proof*. (ii) can easily be derived from monotone convergence principle.

(iii) In view of Proposition 2 and (i) of this theorem, it suffices to consider the second assertion of (ii) in the discrete case. If $\int_0^\infty p(t)dt=\infty$, then for a fixed positive $h$ $\sum_{n=0}^\infty p(nh)=\infty$. This means that with probability one $\mathcal{E}$ occurs at $nh$ for infinitely many $n$. Recalling that if $\mathcal{E}$ occurs, then with a positive probability, $\mathcal{A}$ occurs, we obtain the required result.

Some results concerning the rate at which the $p$-function converges as $t\to\infty$ have been obtained by Kingman [2].

THEOREM E. [Kingman] *Let* $p(t)\in\mathcal{P}$.

(i) *The limit*

(62) $$\beta=-\lim_{t\to\infty} t^{-1}\log p(t)$$

*exists and is finite and non-negative. For all* $t\geqq 0$,

(63) $$p(t)\leqq e^{-\beta t}, \qquad q\geqq\beta.$$

(ii) *If* $\int_0^\infty p(t)dt=\infty$, $\lim_{t\to\infty}p(t)=0$ *and for all* $a>0$,

(64) $$\overline{\lim_{t\to\infty}}\frac{p(t+a)}{p(t)}\leqq 1,$$

*then, for all* $a>0$,

(65) $$\lim_{t\to\infty}\frac{p(t+a)}{p(t)}=1.$$

(iii) *Suppose that the limit* $\tilde{\omega}=\lim_{t\to\infty}p(t)$ *is positive. Then*

(66) $$p(t)=\tilde{\omega}+O(e^{-\beta t})$$

*holds for some* $\beta>0$ *if and only if there exists a* $\lambda>0$ *such that*

$$\int_{(1,\infty)} e^{\lambda x}\mu(dx)>\infty$$

*where* $\mu$ *is the measure of Theorem* C.

Easy calculations prove the analogous theorem about $a(t)$.

THEOREM 7. *Let* $\mathscr{A}\in\mathscr{H}$.

(i) *If the limit* $\beta$ *of* (62) *is positive and for some* $\nu>0$

$$\int_0^\infty e^{\nu x}dK(x)<\infty,$$

*then* $a(t)$ *satisfies for some* $\gamma>0$

(67) $$a(t)=O(e^{-\gamma t}), \qquad (t\to\infty).$$

(ii) *If*

$$\int_0^\infty p(t)dt=\infty, \qquad \lim_{t\to\infty}p(t)=0, \qquad K(\infty)<\infty,$$

*and for all* $a>0$

$$\overline{\lim_{t\to\infty}}\frac{p(t+a)}{p(t)}\leqq 1,$$

*then we have*

(68) $$\lim_{t\to\infty}\frac{a(t)}{p(t)}=K(\infty),$$

*and for all* $m>0$

(69) $$\lim_{t\to\infty}\frac{a(t+m)}{a(t)}=1.$$

(iii) *If*

(70) $$\tilde{\omega}=\lim_{t\to\infty} p(t)>0,$$

*for some* $\rho>0$

(71) $$p(t)=\tilde{\omega}+O(e^{-\rho t})$$

*and for some* $\delta>0$

(72) $$\int_0^\infty e^{\delta x}dK(x)<\infty,$$

*then there exists a* $\gamma>0$ *such that*

(73) $$a(t)=\tilde{\alpha}+O(e^{-\gamma t})$$

*where* $\tilde{\alpha}=\lim_{t\to\infty}a(t)$.

*Proof.* (i) From (36), (63) and the assumption, we have (67) with $\gamma=\min(\beta, \nu)$.

(ii) Fatou's lemma and (65) lead to (68). (69) is derived from (68) and (65).

(iii) When $\tilde{\omega}>0$, $K(\infty)$ is finite. Hence the limit $\tilde{\alpha}=\lim_{t\to\infty}a(t)$ exists (Theorem 5 (i)).

Therefore we can write

$$|a(t)-\tilde{\alpha}|\leqq\int_0^t |p(t-s)-\tilde{\omega}|dK(s)+\tilde{\omega}\int_t^\infty dK(s).$$

From (71), it follows for some $M>0$

$$|a(t)-\tilde{\alpha}|\leqq\int_0^t Me^{-\rho(t-s)}dK(s)+\tilde{\omega}e^{-\delta t}\int_t^\infty e^{\delta s}dK(s).$$

Thus setting $\min(\rho, \delta)=\gamma$, from (72) we have (73).

## § 7. Further remarks.

REMARK 1. Let $\mathcal{A}$ be a standard $\mathcal{E}$-regenerative phenomenon, and let $\{Z_t(\omega), t>0\}$ be the stochastic process defined by

$$\begin{aligned} Z_t(\omega)&=0 \quad \text{if} \quad \omega\in E(t),\\ &=1 \quad \text{if} \quad \omega\in A(t),\\ &=2 \quad \text{if} \quad \omega\in\Omega-\{E(t)\cup A(t)\}\equiv C(t). \end{aligned}$$

For any $t>0$, $h>0$, we have, from (18),

$$\begin{aligned} P\{Z_t(\omega)\neq Z_{t+h}(\omega)\}&=P\{E(t)\bar{E}(t+h)\}+P\{\bar{E}(t)E(t+h)\}\\ &\quad+P\{A(t)C(t+h)\}+P\{C(t)A(t+h)\}\\ &=p(t)+p(t+h)-2p(t)p(h)+a(t)+a(t+h)\\ &\quad-2P\{A(t)A(t+h)\}-P\{A(t)E(t+h)\}-p(t)a(h). \end{aligned}$$

If for each $t>0$ such that $P\{A(t)\}>0$,

$$P\{A(t+\delta)|A(t)\}\to 1 \qquad (\delta\to 0), \tag{74}$$

then the above probability tends to zero as $h\to 0$. Therefore the process $Z_t$ is continuous in probability. It follows from a well-known theorem by Doob [2] that there exists a measurable, separable process $\{Z_t^*(\omega), t>0\}$ defined on the probability space $\Omega$, such that

$$P\{Z_t^*(\omega)=Z_t(\omega)\}=1 \qquad (t>0).$$

Clearly, the families $\{E^*(t), t>0\}$ and $\{A^*(t), t>0\}$, defined by

$$E^*(t)=\{\omega;\ Z_t^*(\omega)=0\}, \qquad A^*(t)=\{\omega;\ Z_t^*(\omega)=1\},$$

satisfy Definition 2, and

$$p(t)=p^*(t)=P\{E^*(t)\},$$

$$a(t)=a^*(t)=P\{A^*(t)\}.$$

Let us say that two $\mathcal{E}$-regenerative phenomena $\mathcal{A}_1$ and $\mathcal{A}_2$ are $p$-equivalent if, for all $t>0$,

$$p_1(t)=p_2(t)$$

and

$$a_1(t)=a_2(t).$$

Moreover, let us say that two $\mathcal{E}$-regenerative phenomena $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent if, for all $t>0$,

$$P\{E_1(t)\varDelta E_2(t)\}=0,$$

$$P\{A_1(t)\varDelta A_2(t_2)\}=0,$$

where $\varDelta$ denotes the symmetric difference of sets. Two equivalent $\mathcal{E}$-regenerative phenomena are necessarily $p$-equivalent, but the converse is false. Then we have the following

PROPOSITION 4. *Let $\mathcal{A}\in\mathcal{H}$. Suppose that $A$ satisfies* (74) *in each $t>0$ such that $a(t)>0$. Then there exists a measurable and separable standard $\mathcal{E}$-regenerative phenomenon $\mathcal{A}^*$ which is equivalent to $\mathcal{A}$, hence is also $p$-equivalent to $\mathcal{A}$.*

REMARK 2. In his papers [7], [9], Kingman classified the theorems about Markov chains with transition probabilities $p_{ij}(t)$ into three types. We can roughly say that his classification is as follows:

Type 1. Those results which can be deduced from the fact that the state $j$ is regenerative. For example, we can list up the following analytical properties of $p_{ij}(t)$.

1) The continuity of $p_{ii}(t)$, and the existence of $p'_{ii}(0)$.

2) The local bounded variation and almost everywhere differentiability of $p_{ij}(t)$.

3) The existence of the limit of $p_{ij}(t)$ as $t\to\infty$.

Type 2. Those results which are not of Type 1 but which follow from the fact that the state $i$ and the state $j$ are both regenerative.

4) The continuity of $p_{ij}(t)$.

5) The existence and finiteness of $p'_{ij}(0)$ for $i\neq j$.

6) The last exit decomposition

$$p_{ij}(t)=\int_0^t p_{ii}(t-s)g_{ij}(s)ds. \tag{75}$$

Type 3. Those results which cannot be deduced from either of the above facts.

7) The continuous differentiability of $p_{ij}(t)$.

8) The Austin-Ornstein theorem that $p_{ij}(t)$ is either strictly positive or identically zero.

Applying our theorems to Markov chains, we can conclude that the following results about $p_{ij}(t)$ can be deduced from the fact that the state $i$ is regenerative.

4) The continuity of $p_{ij}(t)$ for $i\neq j$.

2) The local bounded variation and almost everywhere differentiability of $p_{ij}(t)$ for $i\neq j$.

6) The last exit decomposition (75).

These show that in Markov chains the fact that the state $i$ is regenerative imposes fairly strong regularity conditions on $p_{ij}(t)$. However, it seems that 3) and 5) require the state $j$ to be regenerative.

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