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ON SOME ALMOST ANALYTIC TENSOR FIELDS IN ALMOST COMPLEX MANIFOLDS

BY YOSIO MUTŌ

§1. Introduction, contravariant almost analytic vector fields in the strict sense.

Almost analytic vectors and almost analytic tensors were studied by M. Ako, S. Kotō, I. Satô, S. Sawaki, S. Tachibana, K. Yano and others. In the present paper we define contravariant almost analytic vector fields *in the strict sense*. A contravariant vector field u^h is *almost analytic in the strict sense* when u^h is almost analytic in the ordinary sense and, moreover, u^h satisfies $N_{ji}{}^h u^j=0$ where $N_{ji}{}^h$ is the Nijenhuis tensor of the almost complex structure. An almost analytic tensor field in the strict sense is similarly defined. As a consequence of such definition we can treat these vector fields and tensor fields in a unified form. We also find that almost analytic tensors always appear in couples. In the last section some local property of an integral manifold of a distribution associated with contravariant almost analytic vector fields is studied.

Let M be a C^∞ manifold of dimension $2n$ admitting an almost complex structure J. The components of J are denoted by $F_i{}^h$ when we use natural frames associated with local coordinates ξ^h, where indices such as h, i, j, k run over the range $1, \cdots, 2n$. Let $N_{ji}{}^h$ be the Nijenhuis tensor of $F_i{}^h$, hence

$$N_{ji}{}^h=F_j{}^l(\partial_l F_i{}^h-\partial_i F_l{}^h)-F_i{}^l(\partial_l F_j{}^h-\partial_j F_l{}^h). \tag{1.1}$$

Then $N_{ji}{}^h$ satisfies [10]

$$N_{ji}{}^k F_k{}^h=-N_{ki}{}^h F_j{}^k=-N_{jk}{}^h F_i{}^k. \tag{1.2}$$

Let $\mathcal{L}_u$ be the symbol of Lie derivatives [8] with respect to a contravariant vector field u^h. Then contravariant almost analytic vector fields are defined as vector fields u^h satisfying

$$\mathcal{L}_u F_i{}^h\equiv u^k\partial_k F_i{}^h-F_i{}^k\partial_k u^h+F_k{}^h\partial_i u^k=0. \tag{1.3}$$

When u^h is a contravariant vector field, let us define $\tilde{u}^h$ by $\tilde{u}^h=F_i{}^h u^i$. Then we get

$$\mathcal{L}_{\tilde{u}} F_i{}^h=(\mathcal{L}_u F_i{}^l)F_l{}^h+N_{li}{}^h u^l, \tag{1.4}$$

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which is equivalent to

$$\mathcal{L}_{\tilde{u}} F_i^h - \frac{1}{2}\tilde{u}^l N_{li}^k F_k^h = \left(\mathcal{L}_u F_i^m - \frac{1}{2} u^l N_{li}^k F_k^m\right) F_m^h \tag{1. 5}$$

because of (1. 2).

(1. 4) shows that $\tilde{u}^h$ is in general not a contravariant almost analytic vector field when u^h is such a one.

A covariant almost analytic vector field [9] is defined to be a vector field w_i satisfying

$$(\partial_j F_i^k - \partial_i F_j^k) w_k - F_j^k \partial_k w_i + F_i^k \partial_j w_k = 0. \tag{1. 6}$$

In contrast with the case of contravariant vector fields, $\tilde{w}_i = F_i^k w_k$ is then also a covariant almost analytic vector field. Moreover w_i satisfies

$$N_{ji}^k w_k = 0. \tag{1. 7}$$

Under such circumstances we take an interest in almost analytic contravariant vector fields u^h satisfying

$$N_{ki}^h u^k = 0. \tag{1. 8}$$

DEFINITION. A contravariant vector field u^h satisfying (1. 3) and (1. 8) simultaneously is said to be a *contravariant almost analytic vector field in the strict sense.*

In the present paper we consider contravariant almost analytic vector fields *only in the strict sense.* Hence we drop the phrase "in the strict sense".

From this definition and (1. 4) contravariant almost analytic vector fields also appear in couples such as u^h and $\tilde{u}^h$.

As the main purpose of the present paper is to study local properties of vector and tensor fields, we consider such fields defined on some domain of M.

§ 2. Almost analytic pairs of scalars.

Let $\boldsymbol{u}, \tilde{\boldsymbol{u}}$ be a pair of contravariant almost analytic vector fields different from zero at every point of a domain $\mathcal{D}$. If a contravariant vector field $\boldsymbol{v}$ given by

$$v^h = l u^h - m \tilde{u}^h \tag{2. 1}$$

is also almost analytic, we have from $\mathcal{L}_v F_i^h = 0$

$$-(F_i^k \partial_k l - \partial_i m) u^h + (F_i^k \partial_k m + \partial_i l) \tilde{u}^h = 0.$$

As u^h and $\tilde{u}^h$ are linearly independent at each point of $\mathcal{D}$, we get

$$F_i^k \partial_k l - \partial_i m = 0 \qquad (F_i^k \partial_k m + \partial_i l = 0). \tag{2. 2}$$

This is also a sufficient condition for v^h to be almost analytic. Thus we obtain the following theorem.

THEOREM 2.1. *Let* $\boldsymbol{u}, \tilde{\boldsymbol{u}}$ *be a pair of contravariant almost analytic vector fields different from zero at every point of a domain* $\mathscr{D}$. *Then a necessary and sufficient condition that a vector field* $\boldsymbol{v}$ *given by* (2.1) *be a contravariant almost analytic vector field is that the functions* l, m *satisfy* (2.2).

DEFINITION. A pair of C^∞ scalar functions (l, m) satisfying (2.2) is called an almost analytic pair of scalars [6].

The following propositions are immediately proved.

PROPOSITION 2.2. *If* c, c_1, c_2 *are constants and* (l, m) *is an almost analytic pair of scalars, then the following pairs are almost analytic pairs of scalars,*

$$\text{(i)}\ (cl,\ cm), \qquad \text{(ii)}\ (l+c_1,\ m+c_2),$$

$$\text{(iii)}\ (m,\ -l), \qquad \text{(iv)}\ \left(\frac{l}{l^2+m^2},\ \frac{-m}{l^2+m^2}\right).$$

We do not consider (iv) at the points where $l=m=0$.

PROPOSITION 2.3. *If* (l, m) *and* (p, q) *are almost analytic pairs of scalars, then the following pairs are also almost analytic,*

$$\text{(i)}\ (l+p,\ m+q), \qquad \text{(ii)}\ (lp-mq,\ lq+mp).$$

These propositions show similarities of almost analytic pairs of scalars to complex numbers.

The next proposition shows a close resemblance between an almost analytic pair of scalars and a complex analytic function.

PROPOSITION 2.4. *Let* (l, m) *be a non constant almost analytic pair of scalars and let* $f(x, y)$, $g(x, y)$ *be* C^∞ *functions. A necessary and sufficient condition that a pair of functions* $(f(l, m), g(l, m))$ *be an almost analytic pair of scalars is that* f, g *satisfy the Cauchy-Riemann equations*

$$\frac{\partial f}{\partial l}=\frac{\partial g}{\partial m}, \qquad \frac{\partial f}{\partial m}=-\frac{\partial g}{\partial l}. \tag{2.3}$$

Proof. The system of equations

$$F_i{}^k\frac{\partial f}{\partial \xi^k}-\frac{\partial g}{\partial \xi^i}=0$$

is equivalent to the system

$$F_i{}^k\left(\frac{\partial f}{\partial l}\partial_k l+\frac{\partial f}{\partial m}\partial_k m\right)-\frac{\partial g}{\partial l}\partial_i l-\frac{\partial g}{\partial m}\partial_i m=0$$

and the latter is equivalent to

$$\left(\frac{\partial f}{\partial l}-\frac{\partial g}{\partial m}\right)\partial_i m-\left(\frac{\partial f}{\partial m}+\frac{\partial g}{\partial l}\right)\partial_i l=0$$

because of (2. 2). Since l, m are not constant, $\partial_i l$ and $\partial_i m$ are linearly independent. Hence we get (2. 3).

The following theorem and proposition are also immediately proved. Related theorems were obtained in [6].

THEOREM 2. 5. *Let $\boldsymbol{w}, \tilde{\boldsymbol{w}}$ be a pair of covariant almost analytic vector fields different from zero at every point of a domain $\mathcal{D}$. Then a necessary and sufficient condition that a vector field $lw_i-m\tilde{w}_i$ with C^∞ coefficients l, m be a covariant almost analytic vector field is that (l, m) be an almost analytic pair of scalars.*

PROPOSITION 2. 6. *Let u^h be a contravariant almost analytic vector field and w_i be a covariant almost analytic vector field, both defined on $\mathcal{D}$. Then the pair of scalars $(w_iu^i, \tilde{w}_iu^i)$ is an almost analytic pair on $\mathcal{D}$.*

Let (l, m) be an almost analytic pair of scalars. Differentiating (2. 2) and eliminating derivatives of m we get

$$(\partial_j F_i{}^k-\partial_i F_j{}^k)\partial_k l-F_j{}^k\partial_k\partial_i l+F_i{}^k\partial_j\partial_k l=0. \tag{2. 4}$$

This shows that $\partial_i l$ is a covariant almost analytic vector field. We immediately find that $\partial_i m$ is also a covariant almost analytic vector field and if we put $w_i=\partial_i l$, we have $\tilde{w}_i=\partial_i m$.

DEFINITION. Each of the scalars l, m in an almost analytic pair is called an almost harmonic scalar.

Hence the gradient vector of an almost harmonic scalar is a covariant almost analytic vector.

(2. 4) is equivalent to

$$\partial_j\partial_i l+F_j{}^l F_i{}^k\partial_l\partial_k l-F_i{}^l(\partial_j F_l{}^k-\partial_l F_j{}^k)\partial_k l=0 \tag{2. 5}$$

and also to

$$\partial_j\partial_i l+F_j{}^l F_i{}^k\partial_l\partial_k l+F_j{}^l(\partial_l F_i{}^k-\partial_i F_l{}^k)\partial_k l=0. \tag{2. 6}$$

$\partial_i l$ satisfies

$$N_{ji}{}^h\partial_h l=0. \tag{2. 7}$$

From (2. 5) we get

$$\sum_{i=1}^{2n}\partial_i\partial_i l+\sum_{i=1}^{2n}F_i{}^l F_i{}^k\partial_l\partial_k l-\sum_{i=1}^{2n}F_i{}^l(\partial_i F_l{}^k-\partial_l F_i{}^k)\partial_k l=0,$$

which can be written in the form

$$G^{kj}\partial_k\partial_j l+H^i\partial_i l=0$$

where

$$G^{kj}=\delta^{kj}+\sum_{i=1}^{2n}F_i{}^kF_i{}^j,$$

$$H^i=-\sum_{j=1}^{2n}F_j{}^k(\partial_j F_k{}^i-\partial_k F_j{}^i).$$

Since we have

$$G^{kj}X_kX_j=\sum_{i=1}^{2n}\{(X_i)^2+(F_i{}^kX_k)^2\},$$

the function l has no maximum in its domain according to Hopf's theorem [7]. This proves the following proposition.

PROPOSITION 2.7. *Any almost harmonic scalar has no maximum in its domain. If M is compact, every almost harmonic scalar which is defined over M globally is a constant* [6].

§ 3. Almost analytic tensors.

We have found that contravariant almost analytic vectors and covariant almost analytic vectors appear as pairs of such vectors, that is, that, if u^h and w_i are respectively contravariant and covariant almost analytic vectors, $\tilde{u}^h$ and $\tilde{w}_i$ are also almost analytic vectors.

DEFINITION. A pair of contravariant vectors $(u^h, \tilde{u}^h)$ is said to be an *almost analytic pair* of contravariant vectors or simply an almost analytic pair when u^h or $\tilde{u}^h$ is a contravariant almost analytic vector. Similarly, a pair of covariant vectors $(w_i, \tilde{w}_i)$ is said to be an almost analytic pair of covariant vectors or simply an almost analytic pair when w_i or $\tilde{w}_i$ is a covariant almost analytic vector.

Remark that we consider contravariant almost analytic vectors only in the strict sense.

We have defined almost analytic pairs for scalars, contravariant vectors and covariant vectors. If (l, m), $(u^h, \tilde{u}^h)$, $(w_i, \tilde{w}_i)$ are almost analytic pairs, $(-m, l)$, $(lu^h-m\tilde{u}^h,\ l\tilde{u}^h+mu^h)$, $(lw_i-m\tilde{w}_i,\ l\tilde{w}_i+mw_i)$ are also almost analytic pairs. Moreover we find immediately that $(u^hw_h,\ \tilde{u}^hw_h)$ or equivalently $(-\tilde{w}_i\tilde{u}^i,\ \tilde{w}_iu^i)$ and hence

$$(u^hw_h-\tilde{u}^h\tilde{w}_h,\ \tilde{u}^hw_h+u^h\tilde{w}_h)$$

are almost analytic pairs of scalars. These results suggest a rule for multiplication of pairs.

We can generalize this rule and apply it to construct tensors from vectors. For example we can construct from almost analytic pairs $(u^h, \tilde{u}^h)$, $(w_i, \tilde{w}_i)$ a pair

$$(u^h w_i - \tilde{u}^h \tilde{w}_i,\ u^h \tilde{w}_i + \tilde{u}^h w_i)$$

of pure (1, 1)-tensors. If

$$(f, g),\ (\underset{1}{u}^h, \underset{1}{\tilde{u}}^h), \cdots, (\underset{p}{u}^h, \underset{p}{\tilde{u}}^h),\ (\underset{1}{w}_i, \underset{1}{\tilde{w}}_i), \cdots, (\underset{q}{w}_i, \underset{q}{\tilde{w}}_i)$$

are almost analytic pairs, we can construct a pair of pure (p, q)-tensors by making a formal product

$$(f+\sqrt{-1}\, g)(\underset{1}{u}^{h_1}+\sqrt{-1}\,\underset{1}{\tilde{u}}^{h_1})\cdots(\underset{p}{u}^{h_p}+\sqrt{-1}\,\underset{p}{\tilde{u}}^{h_p})(\underset{1}{w}_{i_1}+\sqrt{-1}\,\underset{1}{\tilde{w}}_{i_1})\cdots(\underset{q}{w}_{i_q}+\sqrt{-1}\,\underset{q}{\tilde{w}}_{i_q})$$

and arranging its real part and imaginary part.

Now, if $p+q\geqq 2$, a pure (p, q)-tensor is a tensor of the form

$$T_{(i)}{}^{(h)} = T_{i_q\cdots i_1}{}^{h_p\cdots h_1}$$

such that

$$F_{i_r}{}^k T_{i_q\cdots k\cdots i_1}{}^{(h)} = F_k{}^{h_s} T_{(i)}{}^{h_p\cdots k\cdots h_1}.$$

We define $\tilde{T}$ by

$$\begin{aligned}\tilde{T}_{(i)}{}^{(h)} &= F_{i_r}{}^k T_{i_q\cdots k\cdots i_1}{}^{(h)}\\ &= F_{i_q}{}^k T_{k i_{q-1}\cdots i_1}{}^{(h)}.\end{aligned}$$

A pure (p, q)-tensor $T_{(i)}{}^{(h)}$ satisfying

$$(3.1)_{p,q} \qquad \begin{aligned}&F_j{}^k \partial_k T_{(i)}{}^{(h)} - \partial_j \tilde{T}_{(i)}{}^{(h)}\\ &+\sum_{r=1}^{q}(\partial_{i_r}F_j{}^k)T_{i_q\cdots k\cdots i_1}{}^{(h)}\\ &+\sum_{s=1}^{p}(\partial_j F_k{}^{h_s}-\partial_k F_j{}^{h_s})T_{(i)}{}^{h_p\cdots k\cdots h_1}=0\end{aligned}$$

is called a Φ-tensor with respect to $F_i{}^h$ [5].

DEFINITION. A pure $(0, q)$-tensor $T_{i_q\cdots i_1}$ $(q\geqq 2)$ satisfying $(3.1)_{0,q}$ is called an almost analytic $(0, q)$-tensor. A pure (p, q)-tensor $T_{i_q\cdots i_1}{}^{h_p\cdots h_1}$ $(p\geqq 1,\ q+p\geqq 2)$ satisfying $(3.1)_{p,q}$ and

$$(3.2)_{p,q} \qquad N_{jk}{}^{h_s} T_{(i)}{}^{h_p\cdots k\cdots h_1}=0 \qquad (s=1, \cdots, p)$$

is called an almost analytic (p, q)-tensor in the strict sense.

Since we consider almost analytic (p, q)-tensors only in the strict sense, we drop the phrase "in the strict sense" in the sequel.

From (3.1) and (3.2) we get

THEOREM 3.1. *An almost analytic tensor* $T_{(i)}{}^{(h)}$ $(p\geqq 0,\ p+q\geqq 2,\ q\geqq 1)$ *satisfies*

$$N_{ji_r}{}^{k}T_{i_q\cdots k\cdots i_1}{}^{(h)}=0. \tag{3.3}$$

If $T_{(i)}{}^{(h)}$ *is an almost analytic tensor, then so is* $\tilde{T}_{(i)}{}^{(h)}$.

If (l, m) is an almost analytic pair of scalars, we call l an almost analytic $(0,0)$-tensor and write $m=\tilde{l}$. If $(u^h, \tilde{u}^h)$ is an almost analytic pair of contravariant vectors, we call u^h an almost analytic $(1,0)$-tensor. If $(w_i, \tilde{w}_i)$ is an almost analytic pair of covariant vectors, we call w_i an almost analytic $(0,1)$-tensor. We regard them as pure tensors. We find immediately that they satisfy (3.1) and (3.2).

The following theorem is also immediately obtained.

THEOREM 3.2. *If T and U are almost analytic tensors, $TU-\tilde{T}\tilde{U}$ and $T\tilde{U}+\tilde{T}U$ are almost analytic tensors.*

Thus we find that almost analytic tensors also appear as pairs. Moreover Theorem 3.2 gives a method to construct an almost analytic pair of tensors from two almost analytic pairs of tensors.

If $T_{(i)}{}^{(h)}=T_{i_q\cdots i_1}{}^{h_p\cdots h_1}$ is a pure tensor and $p\geqq 1$, $q\geqq 1$, then

$$V_{(i)}{}^{(h)}=T_{ki_{q-1}\cdots i_1}{}^{kh_{p-1}\cdots h_1}$$

is also a pure tensor. As we get

$$(\partial_j F_l{}^k)T_{ki_{q-1}\cdots i_1}{}^{lh_{p-1}\cdots h_1}=0$$

from

$$(\partial_j F_s{}^t)(A_t^i A_h^s-F_t{}^i F_h{}^s)=0,$$

we find after straightforward calculation that, if $T_{(i)}{}^{(h)}$ satisfies $(3.1)_{p,q}$, $V_{(i)}{}^{(h)}$ satisfies $(3.1)_{p-1,q-1}$. Moreover we find that, if $p\geqq 2$ and $T_{(i)}{}^{(h)}$ satisfies $(3.2)_{p,q}$, then $V_{(i)}{}^{(h)}$ satisfies $(3.2)_{p-1,q-1}$.

Thus we obtain the following

THEOREM 3.3. *Let* $T_{(i)}{}^{(h)}$ *be an almost analytic* (p, q)*-tensor where* $p\geqq 1$, $q\geqq 1$. *Then a* $(p-1,\ q-1)$*-tensor* $V_{(i)}{}^{(h)}$ *obtained from* $T_{(i)}{}^{(h)}$ *by contracting with respect to one contravariant index and one covariant index is also an almost analytic tensor. Let T be a pure (p, q)-tensor where $p\geqq 1$, $q\geqq 1$ and let $(T, \tilde{T})$ be an almost analytic pair. Then a pair of $(p-1,\ q-1)$-tensors $(V, \tilde{V})$ where V is obtained by contracting T with respect to one covariant index and one contravariant index is an almost analytic pair.*

§ 4. Lie derivatives of almost analytic tensors.

Let $T_{(i)}{}^{(h)}$ be an almost analytic (p, q)-tensor and u^h be an almost analytic contravariant vector. Hence $T_{(i)}{}^{(h)}$ satisfies $(3.1)_{p,q}$, $(3.2)_{p,q}$ and u^h satisfies

$$\mathcal{L}_u F_i{}^h = 0, \tag{4.1}$$

$$N_{ji}{}^h u^i = 0. \tag{4.2}$$

Then we can prove that $\mathcal{L}_u T_{(i)}{}^{(h)}$ is also an almost analytic (p, q)-tensor.

As we have

$$\mathcal{L}_u T_{(i)}{}^{(h)} = u^k \partial_k T_{(i)}{}^{(h)} - \sum_{s=1}^{p} (\partial_k u^{h_s}) T_{(i)}{}^{h_p \cdots k \cdots h_1} + \sum_{r=1}^{q} (\partial_{i_r} u^k) T_{i_q \cdots k \cdots i_1}{}^{(h)}$$

and $T_{(i)}{}^{(h)}$ is pure, we get

$$F_{i_q}{}^m F_l{}^{h_p} \mathcal{L}_u T_{m\cdots}{}^{l\cdots} = F_{i_q}{}^m F_l{}^{h_p} u^k \partial_k T_{m\cdots}{}^{l\cdots} + \sum_{s=1}^{p-1} (\partial_k u^{h_s}) T_{(i)}{}^{\cdots k \cdots}$$

$$- F_{i_q}{}^m F_l{}^{h_p} (\partial_k u^l) T_{m\cdots}{}^{k\cdots} - \sum_{r=1}^{q-1} (\partial_{i_r} u^k) T_{\cdots k \cdots}{}^{(h)} + F_{i_q}{}^m F_l{}^{h_p} (\partial_m u^k) T_{k\cdots}{}^{l\cdots}$$

where it is understood that $T_{m\cdots}{}^{l\cdots}$ means $T_{m i_{q-1} \cdots i_1}{}^{l h_{p-1} \cdots h_1}$ and so on. As u^h satisfies (4.1), the second member becomes

$$= F_{i_q}{}^m F_l{}^{h_p} u^k \partial_k T_{m\cdots}{}^{l\cdots} + \sum_{s=1}^{p-1} (\partial_k u^{h_s}) T_{(i)}{}^{\cdots k \cdots}$$

$$- F_{i_q}{}^m (\partial_l u^{h_p}) F_k{}^l T_{m\cdots}{}^{k\cdots} + F_{i_q}{}^m (u^l \partial_l F_k{}^{h_p}) T_{m\cdots}{}^{k\cdots}$$

$$- \sum_{r=1}^{q-1} (\partial_{i_r} u^k) T_{\cdots k \cdots}{}^{(h)} + F_l{}^{h_p} (\partial_{i_q} u^m) F_m{}^k T_{k\cdots}{}^{l\cdots} + F_l{}^{h_p} (u^m \partial_m F_{i_q}{}^k) T_{k\cdots}{}^{l\cdots}$$

$$= F_{i_q}{}^m F_l{}^{h_p} u^k \partial_k T_{m\cdots}{}^{l\cdots} + F_{i_q}{}^m (u^k \partial_k F_l{}^{h_p}) T_{m\cdots}{}^{l\cdots} + (u^k \partial_k F_{i_q}{}^m) F_l{}^{h_p} T_{m\cdots}{}^{l\cdots}$$

$$+ \sum_{s=1}^{p} (\partial_k u^{h_s}) T_{(i)}{}^{\cdots k \cdots} - \sum_{r=1}^{q} (\partial_{i_r} u^k) T_{\cdots k \cdots}{}^{(h)}$$

$$= - u^k \partial_k T_{(i)}{}^{(h)} + \sum_{s=1}^{p} (\partial_k u^{h_s}) T_{(i)}{}^{\cdots k \cdots} - \sum_{r=1}^{q} (\partial_{i_r} u^k) T_{\cdots k \cdots}{}^{(h)}$$

$$= - \mathcal{L}_u T_{(i)}{}^{(h)},$$

which proves that $\mathcal{L}_u T_{(i)}{}^{(h)}$ is a pure tensor.

We get

$$\mathcal{L}_u N_{ji}{}^h = 0$$

from $\mathcal{L}_u F_i{}^h = 0$. On the other hand we get $\mathcal{L}_u(N_{jk}{}^{hs} T_{(i)}{}^{\cdots k \cdots}) = 0$ from $(3.2)_{p,q}$. Hence we have

$$N_{jk}{}^{hs} \mathcal{L}_u T_{(i)}{}^{h_p \cdots k \cdots h_1} = 0. \tag{4.3}$$

In order to calculate the Lie derivative of $(3.1)_{p,q}$ we write $(3.1)_{p,q}$ in the form

$$\begin{aligned} & F_j{}^k \nabla_k T_{(i)}{}^{(h)} - \nabla_j (F_{i_q}{}^k T_{k i_{q-1} \cdots i_1}{}^{(h)}) \\ & + \sum_{r=1}^{q} (\nabla_{i_r} F_j{}^k) T_{i_q \cdots k \cdots i_1}{}^{(h)} \\ & + \sum_{s=1}^{p} (\nabla_j F_k{}^{h_s} - \nabla_k F_j{}^{h_s}) T_{(i)}{}^{h_p \cdots k \cdots h_1} = 0, \end{aligned} \tag{4.4}$$

where ∇ denotes covariant differentiation with respect to an arbitrary symmetric connection $\Gamma^h_{ji} = \Gamma^h_{ij}$. Then the Lie derivative of the left hand side of (4.4) is equal to

$$\begin{aligned} & F_j{}^k \mathcal{L}_u \nabla_k T_{(i)}{}^{(h)} - (\mathcal{L}_u \nabla_j F_{i_q}{}^k) T_{k \cdots}{}^{(h)} \\ & - (\nabla_j F_{i_q}{}^k) \mathcal{L}_u T_{k \cdots}{}^{(h)} - F_{i_q}{}^k \mathcal{L}_u \nabla_j T_{k \cdots}{}^{(h)} \\ & + \sum_{r=1}^{q} (\mathcal{L}_u \nabla_{i_r} F_j{}^k) T_{\cdots k \cdots}{}^{(h)} + \sum_{r=1}^{q} (\nabla_{i_r} F_j{}^k) \mathcal{L}_u T_{\cdots k \cdots}{}^{(h)} \\ & + \sum_{s=1}^{p} (\mathcal{L}_u \nabla_j F_k{}^{h_s} - \mathcal{L}_u \nabla_k F_j{}^{h_s}) T_{(i)}{}^{\cdots k \cdots} + \sum_{s=1}^{p} (\nabla_j F_k{}^{h_s} - \nabla_k F_j{}^{h_s}) \mathcal{L}_u T_{(i)}{}^{\cdots k \cdots} \\ = {} & F_j{}^k \nabla_k \mathcal{L}_u T_{(i)}{}^{(h)} - \nabla_j (F_{i_q}{}^k \mathcal{L}_u T_{k \cdots}{}^{(h)}) + \sum_{r=1}^{q} (\nabla_{i_r} F_j{}^k) \mathcal{L}_u T_{\cdots k \cdots}{}^{(h)} \\ & + \sum_{s=1}^{p} (\nabla_j F_k{}^{h_s} - \nabla_k F_j{}^{h_s}) \mathcal{L}_u T_{(i)}{}^{\cdots k \cdots} + F_j{}^k (\mathcal{L}_u \nabla_k - \nabla_k \mathcal{L}_u) T_{(i)}{}^{(h)} \\ & - (\mathcal{L}_u \nabla_j F_{i_q}{}^k) T_{k \cdots}{}^{(h)} - F_{i_q}{}^k (\mathcal{L}_u \nabla_j - \nabla_j \mathcal{L}_u) T_{k \cdots}{}^{(h)} \\ & + \sum_{r=1}^{q} (\mathcal{L}_u \nabla_{i_r} F_j{}^k) T_{\cdots k \cdots}{}^{(h)} + \sum_{s=1}^{p} (\mathcal{L}_u \nabla_j F_k{}^{h_s} - \mathcal{L}_u \nabla_k F_j{}^{h_s}) T_{(i)}{}^{\cdots k \cdots}. \end{aligned} \tag{4.5}$$

But we have

$$\mathcal{L}_u \nabla_j F_i^h = (\mathcal{L}_u \nabla_j - \nabla_j \mathcal{L}_u) F_i^h$$
$$= (\mathcal{L}_u \Gamma_{jl}^h) F_i^l - (\mathcal{L}_u \Gamma_{ji}^l) F_l^h$$

and

$$(\mathcal{L}_u \nabla_k - \nabla_k \mathcal{L}_u) T_{(i)}{}^{(h)}$$
$$= \sum_{s=1}^{p} (\mathcal{L}_u \Gamma_{kl}^{h_s}) T_{(i)}{}^{\cdots l \cdots} - \sum_{r=1}^{q} (\mathcal{L}_u \Gamma_{ki_r}^{l}) T_{\cdots l \cdots}{}^{(h)}.$$

Besides, $T_{(i)}{}^{(h)}$ is pure and Γ_{ji}^h is symmetric. Hence in the second member of (4.5) the sum of the last five terms vanishes and we get

$$F_j{}^k \nabla_k \mathcal{L}_u T_{(i)}{}^{(h)} - \nabla_j (F_{i_q}{}^k \mathcal{L}_u T_{k\cdots}{}^{(h)})$$
$$+ \sum_{r=1}^{q} (\nabla_{i_r} F_j{}^k) \mathcal{L}_u T_{\cdots k \cdots}{}^{(h)} + \sum_{s=1}^{p} (\nabla_j F_k{}^{h_s} - \nabla_k F_j{}^{h_s}) \mathcal{L}_u T_{(i)}{}^{\cdots k \cdots} = 0$$

from (4.4). This proves that $\mathcal{L}_u T_{(i)}{}^{(h)}$ is also an almost analytic tensor.

Thus we get

THEOREM 4.1. *If $T_{(i)}{}^{(h)}$ and u^h are almost analytic, $\mathcal{L}_u T_{(i)}{}^{(h)}$ is also almost analytic.*

It must be remarked that our almost analytic tensors (vectors) are almost analytic tensors (vectors) in the strict sense. From Theorem 4.1 we see easily that the set of all almost analytic contravariant vectors is a Lie algebra. But we are considering local properties and our fields are local. Hence we must state this result in the following form.

THEOREM 4.2. *Let P be a point of M and U be a neighbourhood of P. The set of all almost analytic contravariant vector fields over U is denoted by $\mathfrak{g}_U$. Then $\mathfrak{g}_U$ is a Lie algebra. If $V \supset U$, $\mathfrak{g}_V \subset \mathfrak{g}_U$.*

As we have $\mathcal{L}_{\tilde{u}} u^h = -\mathcal{L}_u \tilde{u}^h = -\mathcal{L}_u (u^i F_i^h) = 0$, we get $[\boldsymbol{u}, \tilde{\boldsymbol{u}}] = 0$. Hence an almost analytic pair of contravariant vector fields spans a distribution which is involutive in the domain where the pair does not vanish [6].

§ 5. Almost analytic distributions.

Let U be an open set in M and

$$\underset{1}{u^h},\ \underset{2}{u^h} = \underset{1}{\tilde{u}^h},\ \underset{3}{u^h}, \cdots, \underset{2m-1}{u^h},\ \underset{2m}{u^h} = \underset{2m-1}{\tilde{u}^h}$$

be $2m$ almost analytic contravariant vector fields on U such that the $2m$ vectors $\underset{1}{u^h}, \cdots, \underset{2m}{u^h}$ are linearly independent at every point of U. For any given open set U, let $m(U)$ be the number such that, if $m \leqq m(U)$, such a set of $2m$ vector fields exists and, if $m > m(U)$, there exists no such set of $2m$ vector fields. It will be easily seen that $m(U) \leqq n$ and that, if $V \subset U$, $m(V) \geqq m(U)$. Let P be a point of M. Then there exists a neighbourhood U_P of P such that $m(U_P) \geqq m(U)$ for every neighbourhood U of P. We define $m(P)$ by $m(P) = m(U_P)$.

If $Q \in U_P$ we see easily that $m(Q) \geqq m(P)$. Let us put

$$m_0 = \max\{m(P),\ P \in M\},$$

$$M_0 = \{P \in M,\ m(P) = m_0\}.$$

Then M_0 is an open set of M.

We get

PROPOSITION 5.1. *Let P be a point of M_0. Then for a suitable neighbourhood U of P we can take $2m_0$ contravariant almost analytic vector fields*

$$(5.1)\qquad \begin{aligned} &\underset{\alpha}{u}^h && (\alpha = 1, 2, \cdots, 2m_0), \\ &\underset{2p}{u}^h = \underset{2p-1}{\tilde{u}}^h && (p = 1, 2, \cdots, m_0), \end{aligned}$$

such that (5.1) *are linearly independent at every point of U and such that, if u^h is any contravariant almost analytic vector field on some $V \subset U$, the $2m_0+2$ vector fields $\underset{1}{u}^h, \cdots, \underset{2m_0}{u}^h, u^h, \tilde{u}^h$ are linearly independent at no point of V.*

Hence we have

$$(5.2)\qquad au^h - b\tilde{u}^h + \sum_\alpha a_\alpha (-1)^{\alpha+1} \underset{\alpha}{u}^h = 0$$

where a, b, a_α are functions and $a^2+b^2>0$. We get from (5.2)

$$bu^h + a\tilde{u}^h + \sum_\alpha a_\alpha (-1)^{\alpha+1} \underset{\alpha}{\tilde{u}}^h = 0,$$

and, eliminating $\tilde{u}^h$, we get an expression of the form

$$(5.3)\qquad u^h = \sum_\alpha f_\alpha (-1)^{1+\alpha} \underset{\alpha}{u}^h.$$

As the vector u^h is almost analytic, we have $\mathcal{L}_u F_i{}^h = 0$ from which we get

$$\sum_\alpha (\partial_i f_\alpha)(-1)^{1+\alpha} \underset{\alpha}{\tilde{u}}^h + \sum_\alpha (F_i{}^l \partial_l f_\alpha)(-1)^\alpha \underset{\alpha}{u}^h = 0.$$

This proves that

$$(f_{2p-1}, f_{2p}), \qquad p = 1, \cdots, m_0,$$

are almost analytic pairs of scalars. Thus we have

PROPOSITION 5.2. *Let $\underset{\alpha}{u}^h$ $(\alpha = 1, \cdots, 2m_0)$ be the contravariant almost analytic* *vector fields in Proposition* 5.1, *and let* u^h *be an arbitrary contravariant almost analytic vector field on* $V\subset U$. *Then* u^h *is a linear combination of* $\overset{\alpha}{u}{}^h$ *of the form* (5.3) *where* (f_{2p-1}, f_{2p}) $(p=1, \cdots, m_0)$ *are almost analytic pairs of scalars.*

Let U be as in Proposition 5.1 and let $\boldsymbol{u}, \boldsymbol{v}$ be arbitrary contravariant almost analytic vector fields on U. Then by Theorem 4.2 $[\boldsymbol{u}, \boldsymbol{v}]$ is also a contravariant almost analytic vector field on U. $\boldsymbol{u}, \boldsymbol{v}, [\boldsymbol{u}, \boldsymbol{v}]$ are linear combinations of vector fields (5.1). Hence (5.1) spans an involutive distribution $\mathcal{D}_U$ on U [1].

Let P_1 and P_2 be points of M_0 and let U_1 and U_2 be suitable neighbourhoods of P_1 and P_2 respectively such that there exist involutive distributions $\mathcal{D}_{U_1}$ and $\mathcal{D}_{U_2}$ as stated above. The two distributions coincide in $U_1\cap U_2$. Hence we have

THEOREM 5.3. *The* $2m_0$*-dimensional distribution on* M_0 *spanned by contravariant almost analytic vector fields is involutive.*

§6. Integral manifolds of contravariant almost analytic vector fields.

In this paragraph we study some local properties of integral manifolds X^{2m_0} of the distribution in Theorem 5.3.

THEOREM 6.1. *The number* m_0 *satisfies* $m_0\neq n-1$, *that is,* $m_0<n-1$ *except the case of* $N_{ji}{}^h=0$.

Proof. Suppose $m_0=n-1$. Then there exist $2n-2$ linearly independent vectors u^h satisfying $u^jN_{ji}{}^h=0$. Hence we can write

$$N_{ji}{}^h=A_jP_i{}^h+B_jQ_i{}^h.$$

Substituting this into $N_{ji}{}^h+N_{ij}{}^h=0$ we get

$$A_jP_i{}^h+B_jQ_i{}^h+A_iP_j{}^h+B_iQ_j{}^h=0. \tag{6.1}$$

If A_i and B_i are not linearly independent, we can put $N_{ji}{}^h=A_jP_i{}^h$ and get $A_jP_i{}^h+A_iP_j{}^h=0$ from which we can conclude that $N_{ji}{}^h=0$. Hence we assume that A_i and B_i are linearly independent. Transvecting C^j such that $A_iC^i=1$, $B_iC^i=0$ we get

$$P_i{}^h=A_iS^h+B_iT^h$$

and similarly

$$Q_i{}^h=A_iU^h+B_iV^h.$$

Substituting these into (6.1) we get

$$2A_jA_iS^h+2B_jB_iV^h$$
$$+(A_jB_i+B_jA_i)(T^h+U^h)=0.$$

Transvecting $C^j C^i$ we get $S^h=0$ and similarly $V^h=0$. Hence we get

$$T^h+U^h=0, \qquad P_i{}^h=B_i T^h, \qquad Q_i{}^h=-A_i T^h$$

and $N_{ji}{}^h$ takes the form

$$N_{ji}{}^h=(A_j B_i - A_i B_j)T^h.$$

As $N_{ji}{}^h$ must satisfy $F_j{}^k N_{ki}{}^h=-N_{ji}{}^k F_k{}^h$, we get $T^h=0$, $N_{ji}{}^h=0$. This contradicts $m_0=n-1$.

Take a point P of M_0 and a neighbourhood U of P such that U admits a set of $2m_0$ contravariant almost analytic vector fields

(6.2) $$\underset{\alpha}{u}{}^h, \qquad \alpha=1, \cdots, 2m_0,$$

linearly independent at each point of U. In general such a set of vectors (6.2) satisfies

(6.3) $$\underset{\beta}{\tilde{u}}{}^h = \underset{\beta}{\overset{\alpha}{f}} \underset{\alpha}{u}{}^h$$

in stead of $\underset{2p-1}{\tilde{u}}{}^h = \underset{2p}{u}{}^h$ $(p=1, \cdots, m_0)$. The vectors (6.2) also satisfy

(6.4) $$\underset{\gamma}{u}{}^k \partial_k \underset{\beta}{u}{}^h - \underset{\beta}{u}{}^k \partial_k \underset{\gamma}{u}{}^h = \varphi_{\gamma\beta}{}^\alpha \underset{\alpha}{u}{}^h$$

by virtue of Theorem 4.2.

Hence the system of partial differential equations

$$\underset{\beta}{u}{}^k \partial_k f=0$$

in an unknown function f admits $2n-2m_0$ independent solutions f^x which we write

$$f^x=\eta^x(\xi^1, \cdots, \xi^{2n}) \qquad x=2m_0+1, \cdots, 2n.$$

We can take $2m_0$ functions

$$\eta^1(\xi^1, \cdot\cdot, \xi^{2n}), \cdots, \eta^{2m_0}(\xi^1, \cdots, \xi^{2n})$$

such that the system of $2n$ equations

$$\eta^h=\eta^h(\xi^1, \cdots, \xi^{2n})$$

can be solved in the form

$$\xi^h=\xi^h(\eta^1, \cdots, \eta^{2n})$$

in a suitable neighbourhood V of P, $V\subset U$.

Hence $(\eta^1, \cdots, \eta^{2n})$ is a local coordinate system in V. The components of the vectors $\underset{\alpha}{u}$ with respect to this system will be denoted by $\underset{\alpha}{h}^1, \cdots, \underset{\alpha}{h}^{2n}$. As we have

$$\underset{\beta}{u}^k \partial_k \eta^x = 0, \tag{6.5}$$

we obtain

$$\underset{\beta}{h}^x = 0, \qquad \underset{\beta}{h}^\alpha = \underset{\beta}{u}^k \partial_k \eta^\alpha, \tag{6.6}$$
$$\underset{\beta}{u}^h = \underset{\beta}{h}^\alpha \frac{\partial \xi^h}{\partial \eta^\alpha}.$$

The components of the vectors $\underset{\beta}{\tilde{u}}$ with respect to the coordinate system (η) will be denoted by $\underset{\beta}{\tilde{h}}^h$ where $\underset{\beta}{\tilde{h}}^x = 0$. Then we have

$$\underset{\beta}{\tilde{h}}^\alpha = \overset{\gamma}{\underset{\beta}{f}} \underset{\gamma}{h}^\alpha$$

from (6.3). On the other hand, since we have $\det(\underset{\beta}{h}^\alpha) \neq 0$, there must be a relation of the form

$$\underset{\beta}{\tilde{h}}^\alpha = 'F_\gamma{}^\alpha \underset{\beta}{h}^\gamma$$

between $\underset{\beta}{\tilde{h}}^\alpha$ and $\underset{\beta}{h}^\alpha$. From these two relations we get

$$\underset{\beta}{h}^\gamma \, 'F_\gamma{}^\alpha = \overset{\gamma}{\underset{\beta}{f}} \underset{\gamma}{h}^\alpha. \tag{6.7}$$

It will be easily seen that $\overset{\alpha}{\underset{\beta}{f}}$ are the components of an almost complex structure $'J$ of X^{2m_0} with respect to the frame $\underset{\alpha}{u}$ and that $'F_\beta{}^\alpha$ are the components of the same structure $'J$ with respect to the local coordinate system (η). We say that $'J$ is induced from J.

If we use the local coordinate system (η) in M the structure J has the components

$$G_i{}^h = F_l{}^k \frac{\partial \xi^l}{\partial \eta^i} \frac{\partial \eta^h}{\partial \xi^k}.$$

We get

$$G_\beta{}^\alpha = 'F_\beta{}^\alpha, \qquad G_\beta{}^x = 0, \tag{6.8}$$

for we have

$$\underset{\beta}{u}^i F_i{}^h = \overset{\gamma}{\underset{\beta}{f}} \underset{\gamma}{u}^h$$

which is equivalent to

$$\underset{\beta}{h}^{\gamma} G_{\gamma}{}^{\alpha} = \underset{\beta}{\overset{\gamma}{f}} \underset{\gamma}{h}^{\alpha},$$

$$\underset{\beta}{h}^{\gamma} G_{\gamma}{}^{x} = \underset{\beta}{\overset{\gamma}{f}} \underset{\gamma}{h}^{x} = 0.$$

Let us consider the components of the Nijenhuis tensor of J with respect to (η). As we have $G_{\beta}{}^{x}=0$, we get

$$N_{\gamma\beta}{}^{\alpha} = G_{\gamma}{}^{\varepsilon}(\partial_{\varepsilon} G_{\beta}{}^{\alpha} - \partial_{\beta} G_{\varepsilon}{}^{\alpha}) - G_{\beta}{}^{\varepsilon}(\partial_{\varepsilon} G_{\gamma}{}^{\alpha} - \partial_{\gamma} G_{\varepsilon}{}^{\alpha}).$$

The components $'N_{\gamma\beta}{}^{\alpha}$ of the Nijenhuis tensor of $'J$ with respect to (η) are given by

$$'N_{\gamma\beta}{}^{\alpha} = 'F_{\gamma}{}^{\varepsilon}(\partial_{\varepsilon} 'F_{\beta}{}^{\alpha} - \partial_{\beta} 'F_{\varepsilon}{}^{\alpha}) - 'F_{\beta}{}^{\varepsilon}(\partial_{\varepsilon} 'F_{\gamma}{}^{\alpha} - \partial_{\gamma} 'F_{\varepsilon}{}^{\alpha}).$$

Hence we get

$$(6.9) \qquad 'N_{\gamma\beta}{}^{\alpha} = N_{\gamma\beta}{}^{\alpha}.$$

Let us write (1.3) and (1.8) with respect to the local coordinates (η). Since the components of the almost analytic vector are h^{α} and h^{x} where $h^{x}=0$ and $G_{\beta}{}^{\alpha} = 'F_{\beta}{}^{\alpha}$, $G_{\beta}{}^{x}=0$, we get from (1.3)

$$h^{\gamma}\partial_{\gamma} 'F_{\beta}{}^{\alpha} - 'F_{\beta}{}^{\gamma}\partial_{\gamma} h^{\alpha} + 'F_{\gamma}{}^{\alpha}\partial_{\beta} h^{\gamma} = 0.$$

We get from (1.8) and (6.9)

$$h^{\gamma} {'N_{\gamma\beta}}{}^{\alpha} = 0.$$

This proves that a contravariant vector field of an integral manifold X^{2m_0} obtained by restricting a contravariant almost analytic vector field of M to X^{2m_0} is an almost analytic vector field of X^{2m_0} with respect to the induced structure $'J$.

But we have

$$\underset{\gamma}{h}^{\varepsilon} N_{\varepsilon\beta}{}^{\alpha} = 0, \qquad \det \underset{\beta}{h}^{\alpha} \neq 0.$$

Hence we get $N_{\gamma\beta}{}^{\alpha}=0$ and the integral manifold X^{2m_0} is a complex space by the theorem of Newlander and Nirenberg.

Thus we have proved

THEOREM 6.2. *The distribution on M_0 spanned by the contravariant almost analytic vector fields is involutive. Let $'M$ be any integral manifold of this distribution,* dim $'M=2m_0$. *Then $'M$ is a complex manifold by virtue of the almost complex structure $'J$ induced from the almost complex structure J of M. Any contravariant vector field obtained by restricting to $'M$ a contravariant almost analytic vector field of M is an analytic vector field of $'M$.*

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DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF ENGINEERING,
YOKOHAMA NATIONAL UNIVERSITY.