KŌDAI MATH. SEM. REP.
19 (1967), 370—380

A THEORY OF RULED SURFACES IN E^4

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Introduction. In 4-dimensional Euclidean space E^4, a ruled surface is a surface generated by a moving straight line depending on one parameter. If we fix a point on such a straight line, we get a curve called the director curve. Using the expression of position vectors in E^4, we can write a ruled surface as $x=y(v)+u\xi(v)$, where $y(v)$ is a director curve and $\xi(v)$ is the unit tangent vector with the direction of generator through $y(v)$. On two adjacent generators corresponding to v and $v+\Delta v$, take P and Q, $\mathrm{P}=(u_1, v)$, $\mathrm{Q}=(u_2, v+\Delta v)$ such that PQ is common perpendicular for these generators, and let $\Delta\theta$ be the angle between $\xi(v)$ and $\xi(v+\Delta v)$. When Δv tends to zero, the limit point of P (if there exist) is called the center of the generator and its orbit the curve of striction of the ruled surface. If

$$\lim_{\Delta v\to 0}\frac{\mathrm{PQ}}{\Delta\theta}$$

exist, it is called the distribution parameter.

For a ruled surface in E^3, whose distribution parameter is not ∞, the ruled surface is, as is well known, completely determined by the Frenet-frame along its curve of striction, where there exist three functions characterize it, one of which is of course distribution parameter.

In § 1, we find the characteristic functions and the curve of striction of a ruled surface in E^4. In § 2, a few examples are shown by giving the special values to the characteristic functions. In § 3, we study relations between the characteristic functions and the invariants of a surface in E^4 for example, λ, μ, Gaussian curvature, torsion form, $\cdots$. In § 4, we study a condition that a surface in E^4 becomes a ruled surface.

§ 1. Let M^2 be a surface in E^4, and (p, e_1, e_2, e_3, e_4) be a Frenet-frame in the sense of Ōtsuki [1], then we have the following:

$$(1.1)\qquad \begin{cases} dp=\omega_1 e_1+\omega_2 e_2, \\ de_A=\sum\limits_B \omega_{AB}e_B, \qquad \omega_{AB}+\omega_{BA}=0, \\ d\omega_{AB}=\sum\limits_C \omega_{AC}\wedge\omega_{CB}, \qquad A, B, C=1, 2, 3, 4, \end{cases}$$

Received March 13, 1967.

$$(1.2)\qquad \begin{cases}\omega_{13}\wedge\omega_{24}+\omega_{14}\wedge\omega_{23}=0,\\ \omega_{13}\wedge\omega_{23}=\lambda\omega_1\wedge\omega_2, \qquad \omega_{14}\wedge\omega_{24}=\mu\omega_1\wedge\omega_2,\end{cases}$$

and

$$(1.3)\qquad \lambda+\mu=G, \qquad \lambda\geqq\mu,$$

where G is Gaussian curvature and ω_{34} is the torsion form of M^2.

Especially, if M^2 is a ruled surface, then we can take e_1 such that $e_1(p)$ has the direction of the generator through p. For the above defined e_1, $\omega_2=0$ implies $de_1=\sum \omega_{1i}e_i=0$, accordingly,

$$(1.4)\qquad \omega_{1i}=f_i\omega_2, \qquad i=2,3,4.$$

Making use of $d\omega_r=\omega_1\wedge\omega_{1r}+\omega_2\wedge\omega_{2r}=0$, $r=3,4$, we can put

$$(1.5)\qquad \omega_{23}=f_3\omega_1+h_3\omega_2,$$

$$(1.6)\qquad \omega_{24}=f_4\omega_1+h_4\omega_2,$$

and from (1.2), (1.4), (1.5), (1.6), we have

$$(1.7)\qquad f_3f_4=0.$$

Because $\lambda\geqq\mu$ and $\lambda=-f_3^2$, $\mu=-f_4^2$, (1.7) implies $f_3=0$. Then we get

$$(1.8)\qquad \lambda=0,$$

$$(1.9)\qquad \omega_{13}=0, \qquad \omega_{23}=h_3\omega_2.$$

On the other hand, $d\omega_1=\omega_{12}\wedge\omega_2=f_2\omega_2\wedge\omega_2=0$, hence we have locally

$$(1.10)\qquad \omega_1=du,$$

where u is a local function on M^2.

In the following we assume that $\mu\neq0$, that is M^2 is not locally flat. By our assumption and (1.9), (1.4)

$$(1.11)\qquad d\omega_{13}=f_4\omega_2\wedge\omega_{43}=0,$$

it follows that

$$(1.12)\qquad \omega_{34}=\rho\omega_2.$$

By the structure equations, (1.4), (1.9) ane (1.12), it follows that

$$(1.13)\qquad \frac{\partial f_2}{\partial u}=-f_2^2+f_4^2,$$

$$\frac{\partial f_4}{\partial u}=-2f_2f_4, \tag{1.14}$$

$$\frac{\partial h_3}{\partial u}=-f_2h_3-\rho f_4, \tag{1.15}$$

$$\frac{\partial \rho}{\partial u}=-\rho f_2+h_3f_4. \tag{1.16}$$

(1.13), (1.14) and (1.15), (1.16) may be written as follows:

$$\frac{\partial(f_2+if_4)}{\partial u}=-(f_2+if_4)^2, \tag{1.17}$$

$$\frac{\partial(h_3+i\rho)}{\partial u}=-(f_2-if_4)(h_3+i\rho), \tag{1.18}$$

where $i^2=-1$. By integrating (1.17), we get

$$f_2+if_4=\frac{1}{u-c}, \tag{1.19}$$

where

$$c=p(v)+iq(v), \tag{1.20}$$

v will mean a parameter of some director curve of the ruled surface. By (1.18), (1.20) we get

$$h_3+i\rho=\frac{c_1}{u-\bar{c}}. \tag{1.21}$$

Putting $c_1=r(v)e^{i\theta(v)}$, we get the following:

$$f_2=\frac{u-p}{(u-p)^2+q^2}, \qquad f_4=\frac{q}{(u-p)^2+q^2}, \tag{1.22}$$

$$h_3=\frac{r[(u-p)\cos\theta+q\sin\theta]}{(u-p)^2+q^2}, \qquad \rho=\frac{r[(u-p)\sin\theta-q\cos\theta]}{(u-p)^2+q^2}. \tag{1.23}$$

Now the line element of M^2 is given by $ds^2=du^2+g_{22}dv^2$. We may consider that $\omega_2=\sqrt{g_{22}}\,dv$. By the structure equations and (1.22) we have

$$g_{22}=[(u-p)^2+q^2]l(v)^2, \qquad l(v)>0. \tag{1.24}$$

THEOREM 1. *For a ruled surface which is not locally flat, the curve given by $u=p(v)$ is its curve of striction and $|q|$ is the distribution parameter.*

Proof. Let v be the arc-length of the curve $u=p(v)$, then by (1.24)

$$ds^2=du^2+[(u-p)^2+q^2]l(v)^2dv^2. \tag{1.25}$$

By the hypothesis of v and $du=p'dv$, it follows that

$$\left(\frac{ds}{dv}\right)^2=p'^2+q^2l^2=1. \tag{1.26}$$

By using the expression of position vectors in E^4, we can put the curve

$$x=y(v). \tag{1.27}$$

By the definition of the curve of striction, it is sufficient to show that

$$\left\langle\frac{dy}{dv},\frac{de_1}{dv}\right\rangle=0$$

along $x=y(v)$, but along it we get by (1.22) and (1.25)

$$\frac{dy}{dv}=p'e_1+qle_2, \qquad \frac{de_1}{dv}=le_4, \tag{1.28}$$

which shows that $y(v)$ is the curve of striction. Putting

$$\tan\varphi(v)=\frac{ql}{p'}, \tag{1.29}$$

we get by (1.26) and (1.28),

$$\frac{de_1}{dv}=\frac{\sin\varphi(v)}{q}e_4, \tag{1.30}$$

which shows that $|q|$ is the distribution parameter of M^2. q.e.d.

Now let w be the arc-length of the curve of the spherical image of e_1, then from (1.30) we get

$$dw=\frac{\sin\varphi(v)}{q}dv$$

which implies that

$$\frac{dy}{dw}=q(\cot\varphi\, e_1+e_2). \tag{1.31}$$

Now for the rest h_4, by using (1.22), (1.23), (1.24) and the structure equations we get the following:

$$\frac{\partial}{\partial u}\left(\sqrt{(u-p)^2+q^2}\cdot h_4\right)=\frac{1}{l}\frac{\partial f_4}{\partial v} \tag{1.32}$$

where v is the arc-length of the curve of striction. By (1. 22) and (1. 26),

$$h_4=\frac{-qq'(u-p)+p'(u-p)^2}{\sqrt{1-p'^2}\{(u-p)^2+q^2\}^{\frac{3}{2}}}+\frac{m(v)}{\sqrt{(u-p)^2+q^2}}. \tag{1. 33}$$

Thus $p(v)$, $q(v)$, $r(v)$, $\theta(v)$ and $m(v)$ are the characteristic functions of the ruled surface M^2 in E^4 which is not locally flat.

THEOREM 2. *For a ruled surface in E^4 which is not locally flat, the Frenet-frame along its curve of striction is given by*

$$\begin{cases} \dfrac{dy}{dv}=e_1p'+e_2lq, \\ \dfrac{de_1}{dv}= \qquad\qquad\qquad e_4l, \\ \dfrac{de_2}{dv}= \qquad\qquad\qquad e_3rl\sin\theta+e_4\left(\dfrac{p'}{q}+ml\right), \\ \dfrac{de_3}{dv}= \quad -e_2rl\sin\theta \qquad -e_4rl\cos\theta, \\ \dfrac{de_4}{dv}=-e_1l-e_2\left(\dfrac{p'}{q}+ml\right)+e_3rl\cos\theta, \end{cases} \tag{1. 34}$$

where $l=\sqrt{1-p'^2}/q$, conversely (1. 34) *determines a ruled surface for any given five characteristic functions p, q, r, θ, and m.*

For a ruled surface which is not locally flat, we can consider two asymptotic lines with respect to $\Phi_4=\sum A_{4ij}\omega_i\omega_j$, as $\sum A_{4ij}\omega_i\omega_j=0$. Since $\Phi_3=h_3\omega_2\omega_2$ and $A_{411}=0$, the second fundamental form Φ with respect to any unit normal vector $e=e_3\cos\phi+e_4\sin\phi$ is given by $\Phi=2f_4\sin\phi\omega_1\omega_2+(h_3\cos\phi+h_4\sin\phi)\omega_2\omega_2$, which shows that a generator is an asymptotic line with respect to the second fundamental form Φ defined by any unit normal vector e. Let us call the asymptotic line with respect to Φ_4 which is not generator, the half-asymptotic line. It is defined by $2f_4\omega_1+h_4\omega_2=0$, which is written as

$$\frac{du}{dv}=\frac{-qq'(u-p)+p'(u-p)^2}{2q\sqrt{1-p'^2}}+\frac{m}{2q}\{(u-p)^2+q^2\}$$

by (1. 10), (1. 22), (1. 24) and (1. 33). Since the above differential equation is a Riccati equation, it is clear that the following theorem is true:

THEOREM 3. *The compound ratio of four points at which four half-asymptotic lines intersect a generator, is constant.*

§2. We give a few examples of ruled surfaces. In this section v is always taken the arc-length of the curve of striction $u=p(v)$.

EXAMPLE 1. We consider *the case of locally flat*, that is $\mu=0$. Because (1. 13) holds under $\mu=0$ and $\mu=-f_4^2=0$, we get $f_2=0$ or $f_2=1/(u-p)$.

Let us firstly assume that $f_2=0$. Then we have $de_1=0$, which shows that the ruled surface is a cylinder. In general, a complete surface in E^4 which has the curvatures $\lambda=\mu=0$ is a cylinder [3].

Let us secondary assume that $f_2=1/(u-p)$. Since $\lambda=\mu=0$, we can take $\omega_{34}=0$. And similarly we get by the structure equations as following:

$$h_3=\frac{c(v)}{u-p}, \qquad h_4=\frac{m(v)}{u-p}, \tag{2. 1}$$

$$\omega_2=(u-p)l(v)dv. \tag{2. 2}$$

Therefore we get the following:

$$\begin{cases} dp=e_1du+e_2(u-p)dv, \\ de_1= \quad e_2l(v)dv, \\ de_2=-e_1l(v)dv \quad +e_3l(v)c(v)dv+e_4l(v)m(v)dv, \\ de_3= \quad -e_2l(v)c(v)dv, \\ de_4= \quad -e_2l(v)m(v)dv. \end{cases} \tag{2. 3}$$

If $p(v)$ is constant, then the curve defined by $u=p(v)$ is a constant curve. Consequently this ruled surface is a cone in E^4. Now suppose that $p(v)$ is not constant, i.e., $p'(v)\neq 0$, then it is clear that this ruled surface is a torse whose edge of regression is defined by $u=p(v)$.

EXAMPLE 2. Let us consider the *case of not locally flat and $p=0$, $q=const.$* $\neq 0$, $m=0$. We may consider that $q=1$ by a suitable similar transformation. Then we have

$$\begin{cases} dp= \quad e_1du \quad +e_2\sqrt{u_2+1}\,dv, \\ de_1= \quad e_2\dfrac{u}{\sqrt{u_2+1}}dv \quad +e_4\dfrac{1}{\sqrt{u^2+1}}dv, \\ de_2=-e_1\dfrac{u}{\sqrt{u^2+1}}dv \quad +e_3\dfrac{r(u\cos\theta+\sin\theta)}{\sqrt{u^2+1}}dv+e_4\dfrac{1}{u^2+1}du, \\ de_3= \quad -e_2\dfrac{r(u\cos\theta+\sin\theta)}{\sqrt{u^2+1}}dv \quad +e_4\dfrac{r(u\sin\theta-\cos\theta)}{\sqrt{u^2+1}}dv, \\ de_4=-e_1\dfrac{1}{\sqrt{u^2+1}}dv-e_2\dfrac{1}{u^2+1}du-e_3\dfrac{r(u\sin\theta-\cos\theta)}{\sqrt{u^2+1}}dv. \end{cases} \tag{2. 4}$$

Therefore we get along the curve of striction $x=y(v)$ defined by $u=p(v)$,

$$(2.5)\qquad \begin{cases} dy=e_2dv, \\ de_2= \quad e_3r\sin\theta dv, \\ de_3=-e_2r\sin\theta dv \quad -e_4r\cos\theta dv, \\ de_4= \quad e_3r\cos\theta dv \quad -e_1dv, \\ de_1= \quad e_4dv. \end{cases}$$

Morever let us assume that $p=m=0$, $q=1$ and $r=0$. By virtue of (2.4) it is clear that this ruled surface is a helicoid in a hyperplane E^3 perpendicular to a fixed unit vector e_3, which is written as follows:

$$(2.6)\qquad x(u, v)=vY+u(X\cos v+Z\sin v)$$

where X, Y, Z is an orthomormal base of E^3. And if $p=m=\theta=0$, $q=1$, then it is a helicoid in E^4 in the sense that it is generated by a moving straight line perpendicular to a fixed straight line that the ratio of the velocity of the moving point of intersection and the angular velocity of its direction is constant. Moreover if $p=m=0$, $q=1$ and $\theta=\pi/2$, then we get a sort of helicoid in E^4 which is defined as follows:

$$(2.7)\qquad X(u, v)=y(v)+u(X\cos v+Y\sin v),$$

where $y(v)$ is a plane curve and X, Y are orthogonal unit vectors each of which is perpendicular to the plane containing the curve $x=y(v)$.

§3. We study some relations between characteristic functions and the invariants of M^2 in E^4. By (1.8) and (1.22), we get at once

THEOREM 4. *For a ruled surface in E^4, it follows that*

$$(3.1)\qquad \lambda=0,$$

$$(3.2)\qquad \mu=G=\frac{-q^2}{(u-p)^2+q^2}\leqq 0.$$

Hence there does not exist a ruled surface in E^4 with constant negative curvature.

The torsion form ω_{34} defines a covariant vector field $Z=(Z_1, Z_2)$, and by (1.12) it follows that

$$(3.3)\qquad Z_1=0, \qquad Z_2=\rho.$$

Therefore we get

(3.4) $$\|Z\|=|\rho|.$$

THEOREM 5. *The divergence and the rotation of torsion vector Z are given as follows*:

(3.5) $$\operatorname{div} Z=\frac{1}{\sqrt{g_{22}}}\frac{\partial\rho}{\partial v},$$

(3.6) $$-\frac{1}{G}(\operatorname{rot} Z)^2+\|Z\|^2=\frac{\sqrt{-G}}{|q|}r^2.$$

Proof. For a vector field $Z=(Z_1, Z_2)$, we have the following:

(3.7) $$\operatorname{div} Z=Z_{1,1}+Z_{2,2},$$

(3.8) $$\operatorname{rot} Z=Z_{1,2}-Z_{2,1},$$

where $DZ_i=Z_{i,1}\omega_1+Z_{i,2}\omega_2$ $(i=1,2)$ and $DZ_i=dZ_i+\omega_{ji}Z_j$. By (3.3) we have

(3.9) $$Z_{1,1}=0, \qquad Z_{1,2}=-\rho f_2, \qquad Z_{2,1}=\frac{\partial\rho}{\partial u}, \qquad Z_{2,2}=\frac{1}{\sqrt{g_{22}}}\frac{\partial\rho}{\partial v},$$

which imply that

$$\operatorname{div} Z=\frac{1}{\sqrt{g_{22}}}\frac{\partial\rho}{\partial v}, \qquad \text{and} \qquad \operatorname{rot} Z=-\rho f_2-\frac{\partial\rho}{\partial u}.$$

From (1.22) and (1.23) we have the following:

(3.10) $$-\operatorname{rot} Z=\frac{rq[(u-p)\cos\theta+q\sin\theta]}{[(u-p)^2+q^2]^2}=\frac{q}{(u-p)^2+q^2}h_3.$$

But (1.23) shows that

(3.11) $$h_3^2+\rho^2=\frac{r^2}{(u-p)^2+q^2},$$

from which we get (3.6).

§4. In this section, we study a necessary and sufficient condition that a surface in E^4 becomes a ruled surface. Let (p, e_1, e_2, e_3, e_4) be a Frenet-frame in the sense of Ōtsuki for a surface in E^4. Put $\omega_{34}=Z_1\omega_1+Z_2\omega_2$. We shall introduce two vector fields P and Q by using torsion form ω_{34} and the second fundamental forms $\Phi_3=\sum A_{3ij}\omega_i\omega_j$, $\Phi_4=\sum A_{4ij}\omega_i\omega_j$, where $\omega_{ir}=\sum A_{rij}\omega_j$, $(r=3,4,\ i,j=1,2)$. For the torsion vector $Z=Z_1e_1+Z_2e_2$, let $\bar{Z}=\bar{Z}_1e_1+\bar{Z}_2e_2$ be as follows:

(4.1) $$\bar{Z}_1=-Z_2, \qquad \bar{Z}_2=Z_1.$$

We can write $\bar{Z}=iZ$, where $i^2=-1$. Putting $P_h=\sum A_{3hk}\bar{Z}_k$ and $Q_h=\sum A_{4hk}\bar{Z}_k$ where $h, k=1, 2$, we obtain two vector fields P and Q by contracting $\Phi_3, \bar{Z}$ and $\Phi_4, \bar{Z}$ respectively, i.e., we have the following:

$$P=P_1e_1+P_2e_2=(\Phi_3, \bar{Z})=(\Phi_3, iZ), \tag{4. 2}$$

$$Q=Q_1e_1+Q_2e_2=(\Phi_4, \bar{Z})=(\Phi_4, iZ). \tag{4. 3}$$

Now suppose that M^2 is a ruled surface, then (1. 8) and (1. 9) hold. Let us define two sets:

$$M_0=\{p\in M^2: \mu(p)=0\}, \tag{4. 4}$$

$$M_1=\{p\in M^2: \mu(p)\neq 0\}. \tag{4. 5}$$

For any point of M_1 we have (1. 12), accordingly $Z_1=0$, $Z_2=\rho$ and $A_{311}=A_{312}=0$, $A_{322}=h_3$. Therefore it follows that

$$P=(\Phi_3, iZ)=0. \tag{4. 6}$$

For any point of the interior $\mathring{M}_0$ of M_0, we have $\lambda=\mu=0$ by the definition of M_0 and THEOREM 4. Then we can chose a torsionless Frenet-frame. Hence we get the following:

$$P=(\Phi_3, iZ)=0, \qquad Q=(\Phi_4, iZ)=0. \tag{4. 7}$$

In the following we consider a surface ih E^4 with the properties:

$$\lambda=0, \qquad P=(\Phi_3, iZ)=0. \tag{4. 8}$$

Let p be a fixed point in M_1, and e_1 be the asymptotic direction with respect to Φ_3. Then we have by the definition of e_1, $A_{311}=0$. Since $\lambda=A_{311}A_{322}-A_{312}A_{321}=0$, it follows that $A_{321}=0$, from which we have $\omega_{13}=0$, $\omega_{23}=h_3\omega_2$. Because $P=0$, it follows that $P_1=A_{312}Z_1-A_{311}Z_2=0$, $P_2=A_{322}Z_1-A_{321}Z_2=0$ from which we have

$$h_3Z_1=0. \tag{4. 9}$$

Suppose that $h_3(p)\neq 0$ for $p\in M_1$. Then by (4. 9) we have $Z_1=0$, i.e., $\omega_{34}=\rho\omega_2$. From (1. 2) we have $\omega_{14}\wedge h_3\omega_2=0$, i.e., $\omega_{14}=f_4\omega_2$. On the other hand, $d\omega_{13}=\omega_{12}\wedge\omega_{23}+\omega_{14}\wedge\omega_{43}=0$, from which we get $\omega_{12}=f_2\omega_2$. The above fact shows that the asymptotic line with respect to Φ_3 is a straight line segment in M_1.

Suppose that there exists an open set U of M_1 in which $h_3=0$. Then it follows that $\Phi_3\equiv 0$ in U, consequently the hypothesis $(\Phi_3, iZ)=0$ is trivial in U. Because $\mu\neq 0$, there are two asmptotic directions with respect to Φ_4. Let e_1 be one of these asymptotic direction, it follows that

$$\omega_{14}=f_4\omega_2, \qquad \omega_{24}=f_4\omega_1+h_4\omega_2, \qquad f_4\neq 0. \tag{4. 10}$$

Since $d\omega_{13}=\omega_{14}\wedge\omega_{43}=f_4\omega_2\wedge\omega_{43}=0$, it follows that $\omega_{34}=\rho\omega_2$. But $d\omega_{23}=\omega_{24}\wedge\omega_{43}=-\rho f_4\omega_1\wedge\omega_2=0$. Consequently $\rho=0$ or $\omega_{34}=0$, from which we have $de_3=\sum\omega_{3i}e_i=0$. Therefore U is contained in a hyperplane E^3 of E^4 which is perpendicular to a constant unit vector e_3. Since $\omega_{14}=f_4\omega_2$, the condition that the asymptotic lines become straight lines or straight line segments, is equivalent to $\omega_{12}=f_2\omega_2$.

In the following we study the condition $\omega_{12}=f_2\omega_2$ in $U\subset M_1$. Let e_1, e_2 be the principal directions of the second fundamental form Φ_4. We have

$$\Phi_4=A_{411}\omega_1\omega_1+A_{422}\omega_2\omega_2. \tag{4.11}$$

Putting

$$A_{411}=\varepsilon B_1^2, \qquad A_{422}=-\varepsilon B_2^2, \tag{4.12}$$

where $\varepsilon=\pm1$. We have the following:

$$\Phi_4=\varepsilon(B_1^2\omega_1\omega_1-B_2^2\omega_2\omega_2). \tag{4.13}$$

The asymptotic directions $\bar{e}_1$ and $\bar{e}_2$ with respect to Φ_4 is written as

$$\begin{aligned}\bar{e}_1&=e_1\cos\theta+e_2\sin\theta,\\ \bar{e}_2&=-e_1\sin\theta+e_2\cos\theta,\end{aligned} \tag{4.14}$$

where

$$\cos\theta=\frac{B_2}{\sqrt{B_1^2+B_2^2}}, \qquad \sin\theta=\frac{\pm B_1}{\sqrt{B_1^2+B_2^2}}. \tag{4.15}$$

It follows that

$$\begin{aligned}\bar{\omega}_{12}&=\langle d\bar{e}_1, \bar{e}_2\rangle=d\theta+\omega_{12},\\ \bar{\omega}_2&=-\omega_1\sin\theta+\omega_2\cos\theta.\end{aligned} \tag{4.16}$$

Then $\bar{\omega}_{12}\wedge\bar{\omega}_2=0$ is equivalent to

$$[(B_2dB_1-B_1dB_2)\pm(B_1^2+B_2^2)\omega_{12}]\wedge[B_1\omega_1\mp B_2\omega_2]=0. \tag{4.17}$$

But we have

$$\begin{aligned}&[(B_2dB_1-B_1dB_2)+(B_1^2+B_2^2)\omega_{12}]\wedge[B_1\omega_1-B_2\omega_2]\\ &=[B_2DB_1-B_1DB_2]\wedge[B_1\omega_1-B_2\omega_2]\\ &=[B_1(-B_2B_{1,2}+B_1B_{2,2})-B_2(B_2B_{1,1}-B_1B_{2,1})]\omega_1\wedge\omega_2\\ &=[\varepsilon A_{411}B_{2,2}+\varepsilon A_{422}B_{1,1}-B_1B_2\operatorname{rot}B]\omega_1\wedge\omega_2,\end{aligned}$$

where $B=B_1e_1+B_2e_2$. Similarly we get

$$[(B_2dB_1-B_1dB_2)-(B_1^2+B_2^2)\omega_{12}]\wedge[B_1\omega_1+B_2\omega_2]$$
$$=[\varepsilon A_{411}B^*_{2,2}+\varepsilon A_{422}B^*_{1,1}-B^*_1B^*_2 \operatorname{rot} B^*]\omega_1\wedge\omega_2,$$

where $B^*_1=B_1$, $B^*_2=-B_2$ and $B^*=B^*_1e_1+B^*_2e_2$.

Consequently U is a piece of ruled surface if the following holds:

$$[\varepsilon A_{411}B_{2,2}+\varepsilon A_{422}B_{1,1}-B_1B_2 \operatorname{rot} B]\cdot[\varepsilon A_{411}B^*_{2,2}+\varepsilon A_{422}B^*_{1,1}-B^*_1B^*_2 \operatorname{rot} B^*]=0. \tag{4.18}$$

On the other hand, it is clear that the interior of M_0 is a piece of a cylinder or a torse by Example 1 in § 2.

THEOREM 6. *If a surface in E^4 satisfies $\lambda=0$ and $(\Phi_3, iZ)=0$, then it is locally a ruled surface except U and if* (4. 18) *holds in addition to the above conditions in U, then U becomes locally a ruled surface where U is the interior point of* $\{p\colon \mu(p)<0,\ \Phi_3\equiv 0\}$.

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