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ON A FINITE MODIFICATION OF AN ULTRA-HYPERELLIPTIC SURFACE

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1. Let R and S be two ultrahyperelliptic surfaces defined by two equations $y^2=g(z)$ and $y^2=G(z)$, where $g(z)$ and $G(z)$ are two entire functions having no zero other than an infinite number of simple zeros respectively. If $g(z)$ and $G(z)$ have the same zeros for $|z|\geqq R_0$ for a suitable R_0, then we call S as a finite modification of R.

In the present paper we shall prove the following

Theorem 1. *If there is a non-trivial analytic mapping from R into S, which is a finite modification of R, then it reduces to a conformal mapping from R onto S whose projection has the form $az+b$.*

By this theorem we have the following non-existence criterion of non-trivial analytic mappings from R into its finite modification S: If the group of conformal automorphisms $A(R)$ of R is not isomorphic to that $A(S)$ of S, then there is no non-trivial analytic mapping from R into S.

In the case of $R=S$, Hiromi and Mutō [3] proved the following interesting result: Every non-trivial analytic mapping from R into itself is an automorphism whose projection has the form $e^{2\pi ip/q}z+b$ with a suitable rational number p/q.

We shall extend Hiromi-Mutō's theorem to a more general case.

Theorem 2. *Let S be a finite modification of R. If G and g have the same number of zeros in $|z|<R_0$, then any non-trivial analytic mapping from R into S reduces to a conformal automorphism whose projection has the form $e^{2\pi ip/q}z+b$ with a suitable rational number p/q.*

2. *Proof of theorem* 1. Assume there is a non-trivial analytic mapping φ from R into S. By our previous result [4] we have the existence of two entire functions $h(z)$ and $f(z)$ satisfying

$$G\circ h(z)=f(z)^2g(z). \tag{1}$$

Here $g(z)$ has the following form

$$G(z)\prod_{j=1}^{q}(z-a_j)\prod_{j=1}^{p}\frac{1}{z-b_j}.$$

For simplicity's sake we shall put this $G(z)F(z)$. Hence (1) reduces to

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$$G\circ h(z)=f(z)^2F(z)G(z). \tag{2}$$

Making the n-th iteration of the above equation, we have

$$\begin{aligned} G\circ h_n(z)&=(f\circ h_{n-1}(z))^2F\circ h_{n-1}(z)\,G\circ h_{n-1}(z)\\ &=\prod_{\nu=0}^{n-1}(f\circ h_\nu(z))^2\prod_{\nu=0}^{n-1}F\circ h_\nu(z)G(z), \end{aligned} \tag{3}$$

where $h_\nu(z)$ is the ν-th iteration of $h(z)$, that is, $h_\nu(z)=h\circ h_{\nu-1}(z)$, $h_0(z)=z$.

We discuss the problem along the same line in [3]. First of all we shall prove that $h(z)$ is a polynomial. Assume that $h(z)$ is a transcendental entire function. Fatou [1] proved that $h(z)=z$ or $h\circ h(z)=z$ has an infinite number of roots. According to the cases, we can consider the iteration h_n or h_{2n}. Hence we may asume that the equation $h(z)=z$ has an infinite number of roots. Let z_0 be an arbitrary non-zero root of $h(z)=z$. Now we select twelve complex numbers $w_1^*,\cdots,w_{12}^*$ from the set of zeros of $G(z)$ in such a manner that $w_j^*\neq z_0$, $|w_j^*|>R_0$.

If $h(z)=z_0+c(z-z_0)^k+\cdots$, $c\neq 0$, then

$$h_n(z)=z_0+c^{\Sigma_0^{n-1}k^\nu}(z-z_0)^{k^n}+\cdots.$$

Since $h(z)$ is transcendental, for any positive number K

$$T(r,h)>K\log r$$

for $r>r_1$. By Pólya's result [5]

$$T(r,h_n)\geqq\frac{1}{3}T(r^{6k},h_{n-1})$$

for $r\geqq r_2$ and $k>0$, where r_2 depends on k and h but does not depend on n. Hence for all n and for $r\geqq r_3=\max(2,r_1,r_2)$

$$T(r,h_n)>K(2k)^{n-1}\log r.$$

Let $H_n(z)=h_n(z+z_0)$. Then if $r\geqq 2(r_3+|z_0|)$

$$\begin{aligned} T(r,H_n(z))&\geqq\frac{1}{3}\overset{+}{\log} M\left(\frac{r}{2},H_n(z)\right)\geqq\frac{1}{3}\overset{+}{\log} M\left(\frac{r}{2}-|z_0|,h_n(z)\right)\\ &\geqq\frac{1}{3}T\left(\frac{r}{2}-|z_0|,h_n(z)\right). \end{aligned}$$

Hence

$$T(r,H_n(z))>\frac{1}{3}K(2k)^{n-1}\log\left(\frac{r}{2}-|z_0|\right)>(2k)^{n-1}K$$

for $r\geqq 2(r_3+|z_0|)$.

Again by Pólya's result

$$T(r,H_n)\geqq\frac{1}{3}T(r^{6k},H_{n-1})\geqq KT(r,H_{n-1})\geqq K_1N(r;a_j,H_{n-1})$$

for any given K and $K_1=K-\varepsilon$ and for all $r\geqq r_2$ and for any j. Similarly

$$T(r, H_{n-1})\geqq K^{\nu}T(r, H_{n-1-\nu})\geqq K_1^{\nu}N(r; a_j, H_{n-1-\nu})$$
$$\geqq 2^{\nu}qN(r; a_j, H_{n-1-\nu})$$

for any $K_1=K-\varepsilon>2q^{1/\nu}$ and for all $r\geqq r_2$ and for any j. Hence

$$\sum_{j=1}^{q}\sum_{\nu=0}^{n-1}N(r; a_j, H_\nu)\leqq 2T(r, H_{n-1})<T(r, H_n)$$

for any $r\geqq r_2$ and for any n.

By applying the second fundamental theorem for H_n

$$\begin{aligned}11T(r, H_n)\leqq&\sum_{\nu=1}^{12}N(r; w_\nu^*, H_n)-N(r; 0, H_n')\\ &+\overset{+}{\log}\left(\frac{1}{|c|}\right)^{1+k+\cdots+k^{n-1}}+K_1\log T(r, H_n)+K_2\log r+K_3\end{aligned}\tag{4}$$

outside a set E_n of r of linear measure at most 2, where K_1, K_2, K_3 are constants which depend on z_0, w_ν^* but do not depend on n. On the other hand by the equation (3) we have

$$G\circ H_n(z-z_0)=\prod_{\nu=0}^{n-1}(f\circ H_\nu(z-z_0))^2\prod_{\nu=0}^{n-1}F\circ H_\nu(z-z_0)G(z+z_0).$$

Hence

$$\begin{aligned}\sum_{\nu=1}^{12}N(r; w_\nu^*, H_n)-N(r; 0, H_n')&\leqq\sum_{\nu=1}^{12}\bar{N}(r; w_\nu^*, H_n)\\ &\leqq\sum_{\nu=1}^{12}\bar{N}_1(r; w_\nu^*, H_n)+\sum_{\nu=1}^{12}N_2(r; w_\nu^*, H_n)\\ &\leqq\frac{1}{2}\sum_{\nu=1}^{12}N(r; w_\nu^*, H_n)+N(r; 0, G(z+z_0))+\sum_{\nu=0}^{n-1}N_2(r; 0, F\circ H_\nu(z-z_0))\\ &\leqq 6T(r, H_n)+N(r; 0, G(z+z_0))+\sum_{\nu=0}^{n-1}\sum_{j=1}^{q}N(r; a_j, H_\nu(z-z_0)),\end{aligned}$$

where $N_2(r; A, T)$ indicates the N-function of simple A-points of T. Thus (4) reduces to

$$\begin{aligned}5T(r, H_n)\leqq& N(r; 0, G(z+z_0))+\overset{+}{\log}\left(\frac{1}{|c|}\right)^{1+k+\cdots+k^{n-1}}+K_1\log T(r, H_n)\\ &+K_2\log r+K_3+\sum_{\nu=0}^{n-1}\sum_{j=1}^{q}N(r; a_j, H_\nu(z-z_0)).\end{aligned}\tag{5}$$

The exceptional set E_n^* has linear measure at most $2+r_3$. Now firstly we select an r^* such that

$$T(r, H_n)>2\sum_{\nu=0}^{n-1}\sum_{j=1}^{q}N(r; a_j, H_\nu(z-z_0))$$

for any positive integer n and for $r \geqq r^*$, $r \notin E_n^*$. Next fix r. Then take n sufficiently large in such a manner that

$$N(r; 0, G(z+z_0)) - T(r, H_n) < 0, \qquad \overset{+}{\log} \left(\frac{1}{|c|} \right)^{1+k+\cdots+k^{n-1}} - T(r, H_n) < 0,$$

$$K_1 \log T(r, H_n) - T(r, H_n) < 0, \qquad K_2 \log r + K_3 - T(r, H_n) < 0.$$

Then we have a contradiction by (5). Thus $h(z)$ must be a polynomial.

Next assume $h(z)$ is a polynomial of degree at least two. If w satisfies $|w| > K_0$ for sufficiently large K_0, then $h(z)$ has d simple w-points in $(|w|/|a_0|)^{1/d}(1-\varepsilon) < |z| < (|w|/|a_0|)^{1/d}(1+\varepsilon)$, where $h(z) = a_0 z^d + \cdots + a_d$. However

$$N_2(r; 0, G \circ h) = N(r; 0, G) + O(\log r)$$

and

$$N_2(r; 0, G \circ h) \geqq N_2(r^d(1-\varepsilon)|a_0|, 0, G) - O(\log r),$$

which lead to a contradiction. Hence $h(z)$ must be a linear function $az+b$.

Therefore we have the desired result.

3. Our theorem 1 is best possible. Let R be an ultrahyperelliptic surface defined by $y^2 = g(z)$,

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{b \dfrac{a^n - 1}{a - 1}} \right), \qquad |a| > 1, \quad b \neq 0.$$

Let S be the surface defined by $y^2 = G(z)$,

$$G(z) = \prod_{n=2}^{\infty} \left(1 - \frac{z}{b \dfrac{a^n - 1}{a - 1}} \right) = \frac{g(z)}{1 - \dfrac{z}{b}}.$$

Then

$$G(az+b) = \frac{a}{a-1} g(z).$$

Hence there is an analytic mapping from R into S whose projection is $az+b$, which implies the conformal equivalence of R and S.

4. *Proof of theorem* 2. Assume that the projection has the form $az+b$, $|a| \neq 1$. We may assume that $|a| > 1$. Start from a branch point w_1 of S and consider its counter-image z_1 in R. By the modification z_1 corresponds to a branch point w_2 of S. Consider its counter-image z_2 in R. Continue this process. Then we have two sequences $\{w_j\}$ and $\{z_j\}$. If in the sequence $\{w_j\}$ there are two indices j and k such that $w_j = w_k$, then $z_{j-1} = z_{k-1}$, $w_{j-1} = w_{k-1}, \cdots$ and finally $w_1 = w_{k-j+1}$ and $z_1 = z_{k-j+1}$. We shall call this sequence as a cycle. Make all cycles starting from every branch point in $|z| < R_0$. Evidently the number of these cycles is finite. Now we shall start from a branch point w_1^* of S, which does not belong to the set of above cycles. Make a similar sequence $w_1^*, w_2^*, \cdots$. If w_1^* is sufficiently large, then the set of antecedents of w_1^* makes an infinite sequence. Hence $\{w_j^*\}$

does not make a cycle, that is, $\{w_j^*\}$ is an infinite sequence such that $w_j^* \neq w_k^*$ for $j \neq k$. By a simple calculation we have that w_n^* tends to $-b/(a-1)$, which is a contradiction.

Assume that $a=\exp(2\pi i\theta)$ with an irrational number θ. Similarly we can find an infinite sequence such that $w_j^* \neq w_k^*$ for $j \neq k$ and w_n^* clusters on the circle

$$\frac{b}{1-e^{2\pi i\theta}}+e^{i\alpha}\left(w_1^*-\frac{b}{1-e^{2\pi i\theta}}\right).$$

This is also a contradiction.

References

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