KŌDAI MATH. SEM. REP.
19 (1967), 289—298

ON $|C, 1|_k$ SUMMABILITY FACTORS OF FOURIER SERIES

BY NIRANJAN SINGH

1. 1. Let $\sum a_n$ be a given infinite series with its n-th partial sum s_n, and let $t_n=t_n^0=na_n$. By $\{\sigma_n^\alpha\}$ and $\{t_n^\alpha\}$ we denote the n-th Cesàro means of order α ($\alpha>-1$) of the sequences $\{s_n\}$ and $\{t_n\}$ respectively. The series $\sum a_n$ is said to be absolutely summable (C, α) with index k, or simply summable $|C, \alpha|_k$ ($k\geqq 1$), if

$$\sum n^{k-1}|\sigma_n^\alpha-\sigma_{n-1}^\alpha|^k<\infty. \tag{1. 1. 1}$$

Summability $|C, \alpha|_1$ is the same as summability $|C, \alpha|$. Since

$$t_n^\alpha=n(\sigma_n^\alpha-\sigma_{n-1}^\alpha),$$

condition (1. 1. 1) can also be written as

$$\sum \frac{|t_n^\alpha|^k}{n}<\infty. \tag{1. 1. 2}$$

A sequence $\{\lambda_n\}$ is said to be convex if $\Delta^2\lambda_n\geqq 0$, $n=1, 2, \cdots$, where $\Delta\lambda_n=\lambda_n-\lambda_{n+1}$ and $\Delta^2\lambda_n=\Delta(\Delta\lambda_n)$.

1. 2. Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let the fourier series of $f(t)$ be given by

$$f(t)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)=\sum_{n=0}^{\infty}A_n(t),$$

where we can assume, without loss of generality, that $a_0=0$.

We shall use throughout this paper the following notations and identities:

$$\varphi(t)=\frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\},$$

$$D_n(t)=\frac{1}{2}+\cos t+\cos 2t+\cdots+\cos nt=\frac{\sin(n+1/2)t}{2\sin(t/2)},$$

$$s_n(x)=\sum_{\nu=0}^{n}A_\nu(x)=\frac{1}{\pi}\int_0^\pi\{f(x+t)+f(x-t)\}D_n(t)dt,$$

$$s_n(x)-f(x)=\frac{1}{\pi}\int_0^\pi\{f(x+t)+f(x-t)-2f(x)\}D_n(t)dt=\frac{2}{\pi}\int_0^\pi\varphi(t)D_n(t)dt,$$

and

Received January 12, 1967.

$$K_n(t)=\frac{1}{n}\sum_{\nu=1}^{n}\nu\lambda_\nu\cos\nu t.$$

1. 3. Pati [3] has recently proved the following theorem.

THEOREM. *If $\{\lambda_n\}$ be a convex sequence such that $\sum\lambda_n/n<\infty$, then a necessary and sufficient condition that $\sum\lambda_n A_n(x)$ be summable $|C,1|$, when*

$$\int_0^t|\varphi(u)|du=o(t), \tag{1. 3. 1}$$

is that

$$\sum\frac{\lambda_n}{n}|s_n(x)-f(x)|<\infty. \tag{1. 3. 2}$$

The object of this paper is to find a necessary and sufficient condition in order that the series $\sum\lambda_n A_n(t)$ be summable $|C,1|_k$, $k\geqq 1$, at the point $t=x$, under a suitable condition.

1. 4. In what follows, we shall prove the following theorem.

THEOREM. *If $\{\lambda_n\}$ be a convex sequence such that $\sum\lambda_n/n<\infty$, then a necessary and sufficient condition for $\sum\lambda_n A_n(x)$ to be summable $|C,1|_k$, $k\geqq 1$, when*

$$\int_0^t|\varphi(u)|^k du=o(t), \qquad \text{as} \quad t\to 0, \tag{1. 4. 1}$$

is that

$$\sum\frac{\lambda_n^k}{n}|s_n(x)-f(x)|^k<\infty.$$

For $k=1$, it may be observed that the theorem of Pati, mentioned above, is a particular case of our theorem.

1. 5. For the proof of our theorem we require the following Lemmas.

LEMMA 1 [3]. *If $\sum_{n,t}\equiv\sum_{\nu=1}^{n}\nu\cos\nu t$, we have the following order estimates of $\sum_{n,t}$:*

$$O(n^2), \tag{1. 5. 1}$$

$$O(1)+O(nt^{-1}), \tag{1. 5. 2}$$

$$O(1)+O(t^{-2})+nD_n(t). \tag{1. 5. 3}$$

LEMMA 2 [3]. *If $K_n(t)$ is defined as before, then we have the following estimates for it.*

$$O\left(n^{-1}\sum_{\nu=1}^{n-1}\nu^2\Delta\lambda_\nu\right)+O(n\lambda_n), \tag{1. 5. 4}$$

$$O(n^{-1})+O(n^{-1}\lambda_n)+O(n^{-1}\lambda_n t^{-2})+\left(t^{-1}n^{-1}\sum_{\nu=1}^{n-1}\nu\Delta\lambda_\nu\right)+\lambda_n D_n(t). \tag{1. 5. 5}$$

LEMMA 3 [2]. *If $\{\lambda_n\}$ is a convex sequence such that $\sum\lambda_n/n<\infty$, then*

(1. 5. 6) $$\sum_{n=1}^{m} \log (n+1)\Delta\lambda_n = O(1), \qquad m\to\infty$$

and

(1. 5. 7) $$\lambda_m \log m = o(1), \qquad m\to\infty.$$

LEMMA 4. *If* (1. 4. 1) *holds, then*

(1. 5. 8) $$\left\{\int_0^{1/n} |\varphi(t)|\, dt\right\}^k = O(n^{-k}),$$

(1. 5. 9) $$\left\{\int_{1/n}^{\pi} |\varphi(t)|\, dt\right\}^k = O(1),$$

(1. 5. 10) $$\left\{\int_{1/n}^{\pi} \frac{|\varphi(t)|}{t}\, dt\right\}^k = O\{(\log n)^k\},$$

(1. 5. 11) $$\left\{\int_{1/n}^{\pi} \frac{|\varphi(t)|}{t^2}\, dt\right\}^k = O(n^k).$$

Proof. We have

$$\left(\int_0^{1/n} |\varphi(t)|\, dt\right)^k \leqq \int_0^{1/n} |\varphi(t)|^k dt \left(\int_0^{1/n} dt\right)^{k-1}$$
$$= O\left(\frac{1}{n}\right)\cdot n^{1-k} = O(n^{-k}).$$

Thus (1. 5. 8) holds. (1. 5. 9) is evident. Applying Hölder's inequality we get

$$\left(\int_{1/n}^{\pi} \frac{|\varphi(t)|}{t} dt\right)^k \leqq \left(\int_{1/n}^{\pi} \frac{|\varphi(t)|^k}{t} dt\right)\left(\int_{1/n}^{\pi} \frac{dt}{t}\right)^{k/k'}$$
$$= O\left[(\log n)^{k-1}\left\{\left[\frac{\Phi(t)}{t}\right]_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi(t)}{t^2} dt\right\}\right]$$
$$= O\left[(\log n)^{k-1}\left\{O(1) + O\left(\int_{1/n}^{\pi} \frac{dt}{t}\right)\right\}\right]$$
$$= O[(\log n)^{k-1}\cdot \log n] = O[(\log n)^k].$$

Thus (1. 5. 10) holds. Similarly (1. 5. 11) can be proved.

1. 6. *Proof of the theorem.* Let T_n denote the n-th Cesàro mean of order 1 of the sequence $\{n\lambda_n A_n(x)\}$. Then summability $|C, 1|_k$ of $\sum \lambda_n A_n(x)$ by (1. 1. 2) is the same as the convergence of $\sum |T_n(x)|^k/n$.

Sufficiency. In this part of the proof we assume that

$$\sum_n \frac{\lambda_n^k}{n} |s_n(x) - f(x)|^k < \infty,$$

and proceed to show that $\sum |T_n(x)|^k/n < \infty$. Now

$$\sum_{2}^{\infty}\frac{|T_n|^k}{n}=\sum_{n=2}^{\infty}\frac{1}{n}\left|\frac{1}{(n+1)}\sum_{\nu=1}^{n}\nu\lambda_\nu A_\nu(x)\right|^k$$
$$=\sum_{n=2}^{\infty}\frac{1}{n}\left|\frac{1}{(n+1)}\sum_{\nu=1}^{n}\nu\lambda_\nu\frac{2}{\pi}\int_0^\pi\varphi(t)\cos\nu t\,dt\right|^k$$
$$=A\sum_{n=2}^{\infty}\frac{1}{n}\left|\int_0^\pi\varphi(t)\,\frac{1}{(n+1)}\sum_{\nu=1}^{n}\nu\lambda_\nu\cos\nu t\,dt\right|^k$$
$$\leqq A\sum_{n=2}^{\infty}\frac{1}{n}\left|\int_0^\pi\varphi(t)K_n(t)dt\right|^k$$
$$=A\sum_{n=2}^{\infty}\frac{1}{n}\left|\int_0^{1/n}\varphi(t)K_n(t)dt+\int_{1/n}^{\pi}\varphi(t)K_n(t)dt\right|^k$$
$$\leqq A\sum_{n=2}^{\infty}\frac{1}{n}\left\{\int_0^{1/n}|\varphi(t)|\,|K_n(t)|dt+\left|\int_{1/n}^{\pi}\varphi(t)K_n(t)dt\right|\right\}^k$$
$$\leqq A\sum_{n=1}^{\infty}\frac{1}{n}\left\{\int_0^{1/n}|\varphi(t)||K_n(t)|dt\right\}^k+A\sum_{n=2}^{\infty}\frac{1}{n}\left\{\left|\int_{1/n}^{\pi}\varphi(t)K_n(t)dt\right|\right\}^k$$
$=\Sigma_1+\Sigma_2$, say.[1)]

Now

$$\Sigma_1\leqq A\sum_{n=2}^{\infty}\frac{1}{n}\left[\int_0^{1/n}|\varphi(t)|\left\{\frac{1}{n}\sum_{\nu=1}^{n-1}\nu^2\Delta\lambda_\nu+n\lambda_n\right\}dt\right]^k$$
$$=A\sum_{n=2}^{\infty}\frac{1}{n}\left\{\int_0^{1/n}|\varphi(t)|\frac{1}{n}\sum_{\nu=1}^{n-1}\nu^2\Delta\lambda_\nu\,dt+\int_0^{1/n}|\varphi(t)|n\lambda_n dt\right\}^k$$
$$\leqq A\sum_{n=2}^{\infty}\frac{1}{n}\left\{\int_0^{1/n}|\varphi(t)|\frac{1}{n}\sum_{\nu=1}^{n-1}\nu^2\Delta\lambda_\nu dt\right\}^k+A\sum_{n=2}^{\infty}\frac{1}{n}\left\{\int_0^{1/n}|\varphi(t)|n\lambda_n dt\right\}^k$$
$=I_1+I_2$, say.

By virtue of (1. 5. 8) we have

$$I_2\leqq A\sum_{n=2}^{\infty}n^{k-1}\cdot\lambda_n^k\left\{\int_0^{1/n}|\varphi(t)|\,dt\right\}^k$$
$$\leqq A\sum_{n=2}^{\infty}n^{k-1}\cdot\lambda_n^k\cdot n^{-k}$$
$$\leqq A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}<A.$$

Again by virtue of (1. 5. 8) we get

$$I_1=A\sum_{n=2}^{\infty}\frac{1}{n}\left\{\frac{1}{n}\sum_{\nu=1}^{n-1}\nu^2\Delta\lambda_\nu\cdot\int_0^{1/n}|\varphi(t)|\,dt\right\}^k$$
$$=A\sum_{n=2}^{\infty}\frac{1}{n^{k+1}}\left(\sum_{\nu=1}^{n-1}\nu^2\Delta\lambda_\nu\right)^k\left(\int_0^{1/n}|\varphi(t)|\,dt\right)^k$$

1) A is a positive finite constant but is not necessarily the same at each occurrence.

$$\leqq A \sum_{n=2}^{\infty} \frac{1}{n^{2k+1}} \left(\sum_{\nu=1}^{n-1} \nu^2 \Delta\lambda_\nu \right)^k$$

$$= A \sum_{n=2}^{\infty} \frac{1}{n^{2k+1}} \left(\sum_{\nu=1}^{n-1} \nu^2 (\Delta\lambda_\nu)^{1/k} (\Delta\lambda_\nu)^{1/k'} \right)^k$$

$$\leqq A \sum_{n=2}^{\infty} \frac{1}{n^{2k+1}} \left(\sum_{\nu=1}^{n-1} \nu^{2k} \Delta\lambda_\nu \right) \left(\sum_{\nu=1}^{n-1} \Delta\lambda_\nu \right)^{k/k'}$$

$$\leqq A \sum_{n=2}^{\infty} \frac{1}{n^{2k+1}} \sum_{\nu=1}^{n-1} \nu^{2k} \Delta\lambda_\nu$$

$$= A \sum_{\nu=1}^{\infty} \nu^{2k} \Delta\lambda_\nu \sum_{n=\nu+1}^{\infty} \frac{1}{n^{2k+1}}$$

$$\leqq A \sum_{\nu=1}^{\infty} \Delta\lambda_\nu \cdot \nu^{2k} \cdot \frac{1}{(\nu+1)^{2k}}$$

$$\leqq A \sum_{\nu=1}^{\infty} \Delta\lambda_\nu < A.$$

Therefore $\Sigma_1 = O(1)$.

Now we have to show that $\Sigma_2 = O(1)$. Making use of (1. 5. 5) we have

$$\Sigma_2 = A \sum_{n=2}^{\infty} \frac{1}{n} \left\{ \left| \int_{1/n}^{\pi} \varphi(t) K_n(t) dt \right| \right\}^k$$

$$\leqq A \sum_{n=2}^{\infty} \frac{1}{n} \Bigg[\lambda_n \left| \int_{1/n}^{\pi} \varphi(t) D_n(t) dt \right| + \int_{1/n}^{\pi} |\varphi(t)| \left\{ O(n^{-1}) + O\left(\frac{\lambda_n}{n}\right) + \left(\frac{\lambda_n}{n} t^{-2}\right) + O\left(t^{-1} n^{-1} \sum_{\nu=1}^{n-1} \nu \Delta\lambda_\nu \right) \right\} \Bigg]^k$$

$$\leqq A \sum_{n=2}^{\infty} \frac{1}{n} \left\{ \lambda_n \left| \int_{1/n}^{\pi} \varphi(t) D_n(t) dt \right| \right\}^k + O\left[\sum_{n=2}^{\infty} \frac{1}{n} \left\{ \int_{1/n}^{\pi} |\varphi(t)| \cdot \frac{1}{n} \right\}^k \right]$$

$$+ O\left[\sum_{n=2}^{\infty} \frac{1}{n} \left\{ \int_{1/n}^{\pi} |\varphi(t)| \, dt \cdot \frac{\lambda_n}{n} \right\}^k \right] + O\left[\sum_{n=2}^{\infty} \frac{1}{n} \left\{ \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t^2} \cdot \frac{\lambda_n}{n} \right\}^k \right]$$

$$+ O\left[\sum_{n=2}^{\infty} \frac{1}{n} \left\{ \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t} dt \cdot \frac{1}{n} \sum_{\nu=1}^{n-1} \nu \Delta\lambda_\nu \right\}^k \right]$$

$$= M_1 + M_2 + M_3 + M_4 + M_5, \text{ say.}$$

Hence we have to show that $M_r = O(1)$, $r = 1, 2, 3, 4, 5$.

$$M_1=A\sum_{n=2}^{\infty}\frac{1}{n}\left\{\lambda_n\left|\int_{1/n}^{\pi}\varphi(t)D_n(t)dt\right|\right\}^k$$
$$=A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}\left\{\left|\int_{1/n}^{\pi}\varphi(t)D_n(t)dt\right|\right\}^k$$
$$\leqq A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}\left\{\left|\int_0^{\pi}\varphi(t)D_n(t)dt\right|+\int_0^{1/n}|\varphi(t)|\,|D_n(t)|\,dt\right\}^k$$
$$\leqq A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}\left\{|s_n(x)-f(x)|+\int_0^{1/n}|\varphi(t)|\,|D_n(t)|\,dt\right\}^k$$
$$=A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}|s_n(x)-f(x)|^k+A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}\left\{\int_0^{1/n}|\varphi(t)|\,|D_n(t)|\,dt\right\}^k$$
$$=O(1)+A\sum_{n=2}^{\infty}\frac{\lambda_n^k\cdot n^k}{n}\left\{\int_0^{1/n}|\varphi(t)|dt\right\}^k$$
$$=O(1)+A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}\cdot n^k\cdot n^{-k}$$
$$=O(1)+A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}=O(1),$$

by virtue of the hypothesis, (1. 5. 8) and the fact that $D_n(t)=O(n)$. Again

$$M_2=O\left[\sum_{n=2}^{\infty}\frac{1}{n}\left\{\int_{1/n}^{\pi}|\varphi(t)|\,dt\cdot\frac{1}{n}\right\}^k\right]$$
$$=O\left[\sum_{n=2}^{\infty}\frac{1}{n^{k+1}}\left\{\int_{1/n}^{\pi}|\varphi(t)|\,dt\right\}^k\right]$$
$$=O\left[\sum_{n=2}^{\infty}\frac{1}{n^{k+1}}\right]=O(1).$$

Next

$$M_3=O\left[\sum_{n=2}^{\infty}\frac{1}{n}\left\{\int_{1/n}^{\pi}|\varphi(t)|\,dt\cdot\frac{\lambda_n}{n}\right\}^k\right]$$
$$=O\left[\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n^{k+1}}\left\{\int_{1/n}^{\pi}|\varphi(t)|dt\right\}^k\right]$$
$$=O\left[\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n^{k+1}}\right]=O(1).$$

Also by (1. 5. 11)

$$M_4=O\left[\sum_{n=2}^{\infty}\frac{1}{n}\left\{\int_{1/n}^{\pi}\frac{|\varphi(t)|}{t^2}dt\cdot\frac{\lambda_n}{n}\right\}^k\right]$$
$$=O\left[\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n^{k+1}}\left\{\int_{1/n}^{\pi}\frac{|\varphi(t)|}{t^2}\,dt\right\}^k\right]$$
$$=O\left[\sum_{n=2}^{\infty}\frac{\lambda_n^k\cdot n^k}{n^{k+1}}\right]=O\left[\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}\right]=O(1).$$

Lastly, we have by virtue of (1. 5. 10) and (1. 5. 6)

$$M_5=O\left[\sum_{n=2}^{\infty}\frac{1}{n}\left\{\int_{1/n}^{\pi}\frac{|\varphi(t)|}{t}dt\cdot\frac{1}{n}\sum_{\nu=1}^{n-1}\nu\varDelta\lambda_\nu\right\}^k\right]$$

$$=O\left[\sum_{n=2}^{\infty}\frac{1}{n^{k+1}}\left(\sum_{\nu=1}^{n-1}\nu\varDelta\lambda_\nu\right)^k\left\{\int_{1/n}^{\pi}\frac{|\varphi(t)}{t}dt\right\}^k\right]$$

$$=O\left[\sum_{n=2}^{\infty}\frac{1}{n^{k+1}}\left(\sum_{\nu=1}^{n-1}\nu\varDelta\lambda_\nu\right)^k(\log n)^k\right]$$

$$=O\left[\sum_{n=2}^{\infty}\frac{(\log n)^k}{n^{k+1}}\left(\sum_{\nu=1}^{n-1}\nu\varDelta\lambda_\nu\right)^k\right]$$

$$=O\left[\sum_{n=2}^{\infty}\frac{(\log n)^k}{n^{k+1}}\left(\sum_{\nu=1}^{n-1}\nu\varDelta\lambda_\nu\right)\left(\sum_{\nu=1}^{n-1}\nu\varDelta\lambda_\nu\right)^{k-1}\right]$$

$$=O\left[\sum_{n=2}^{\infty}\frac{(\log n)^k}{n^{k+1}}\left(\sum_{\nu=1}^{n-1}\nu\varDelta\lambda_\nu\right)\left(\sum_{\nu=1}^{n-1}\frac{\log(\nu+1)\cdot\nu\varDelta\lambda_\nu}{\log(\nu+1)}\right)^{k-1}\right]$$

$$=O\left[\sum_{n=2}^{\infty}\frac{(\log n)^k}{n^{k+1}}\cdot\frac{n^{k-1}}{(\log n)^{k-1}}\cdot\sum_{\nu=1}^{n-1}\nu\varDelta\lambda_\nu\left(\sum_{\nu=1}^{n-1}\log(\nu+1)\varDelta\lambda_\nu\right)^{k-1}\right]$$

$$=O\left[\sum_{n=2}^{\infty}\frac{\log n}{n^2}\sum_{\nu=1}^{n-1}\nu\varDelta\lambda_\nu\right]$$

$$=O\left[\sum_{\nu=1}^{\infty}\nu\varDelta\lambda_\nu\sum_{n=\nu+1}^{\infty}\frac{\log n}{n^2}\right]$$

$$=O\left[\sum_{\nu=1}^{\infty}\nu\varDelta\lambda_\nu\frac{\log(\nu+1)}{(\nu+1)}\right]=O\left[\sum_{\nu=1}^{\infty}\log(\nu+1)\varDelta\lambda_\nu\right]=O(1).$$

Hence $\Sigma_2=O(1)$.

Thus the sufficiency of the theorem is proved.

Necessity. To prove the necessity of the theorem we assume that $\sum\lambda_nA_n(x)$ is summable $|C,1|_k$, and we have to show that

$$\sum\frac{\lambda_n^k}{n}|s_n(x)-f(x)|<\infty.$$

Now we observed that summability $|C,1|_k$ of the above series is the same as the convergence of the series $\sum|T_n|^k/n$, that is

$$\sum_n\frac{1}{n}\left|\int_0^{\pi}\varphi(t)K_n(t)dt\right|^k<\infty.$$

We first show that

$$\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}\left\{\left|\int_{1/n}^{\pi}\varphi(t)D_n(t)dt\right|\right\}^k<\infty.$$

Now by Lemma 2

$$\sum_{n=2}^{\infty}\frac{1}{n}\left\{\left|\int_{1/n}^{\pi}\varphi(t)\lambda_n D_n(t)dt\right|\right\}^k$$

$$\leqq A\sum_{n=2}^{\infty}\frac{1}{n}\left\{\left|\int_{1/n}^{\pi}\varphi(t)K_n(t)dt\right|+\frac{1}{n}\int_{1/n}^{\pi}|\varphi(t)|dt\right.$$

$$\left.+\frac{\lambda_n}{n}\int_{1/n}^{\pi}|\varphi(t)|dt+\frac{\lambda_n}{n}\int_{1/n}^{\pi}\frac{|\varphi(t)|}{t^2}dt+\frac{1}{n}\sum_{\nu=1}^{n-1}\nu\Delta\lambda_\nu\int_{1/n}^{\pi}\frac{|\varphi(t)}{t}dt\right\}^k$$

$$\leqq\sum_{n=2}^{\infty}\frac{1}{n}\left[\left|\int_{1/n}^{\pi}\varphi(t)K_n(t)dt\right|\right]^k$$

$$+\sum_{n=2}^{\infty}\frac{1}{n^{k+1}}\left\{\int_{1/n}^{\pi}|\varphi(t)|dt\right\}^k$$

$$+\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n^{k+1}}\left\{\int_{1/n}^{\pi}|\varphi(t)|dt\right\}^k+\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n^{k+1}}\left\{\int_{1/n}^{\pi}\frac{|\varphi(t)|}{t^2}dt\right\}^k$$

$$+\sum_{n=2}^{\infty}\frac{1}{n^{k+1}}\left\{\sum_{\nu=1}^{n-1}\nu\Delta\lambda_\nu\int_{1/n}^{\pi}\frac{|\varphi(t)|}{t}dt\right\}^k$$

$$=L_1+L_2+L_3+L_4+L_5, \text{ say.}$$

Clearly $L_2=O(1)$ and $L_3=O(1)$. Also by (1. 5. 11)

$$L_4=O\left[\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n^{k+1}}\cdot n^k\right]$$

$$=O\left[\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}\right]=O(1).$$

Since L_5 is the same as M_5 of the sufficiency part, we have $L_5=O(1)$. We have now only to show that $L_1=O(1)$. Now by Lemma 2 and (1. 5. 8),

$$L_1=\sum_{n=2}^{\infty}\frac{1}{n}\left[\left|\int_{1/n}^{\pi}\varphi(t)K_n(t)dt\right|\right]^k$$

$$\leqq A\sum_{n=2}^{\infty}\frac{1}{n}\left[\left|\int_0^{\pi}\varphi(t)K_n(t)dt\right|+\int_0^{1/n}|\varphi(t)||K_n(t)|dt\right]^k$$

$$\leqq A\sum_{n=2}^{\infty}\frac{1}{n}|T_n|^k+A\sum_{n=2}^{\infty}\frac{1}{n}\left\{\int_0^{1/n}|\varphi(t)||K_n(t)|dt\right\}^k$$

$$\leqq A+A\sum_{n=2}^{\infty}\frac{1}{n}\left\{\int_0^{1/n}|\varphi(t)|\,dt\cdot\frac{1}{n}\sum_{\nu=1}^{n-1}\nu^2\Delta\lambda_\nu+\int_0^{1/n}|\varphi(t)|\,dt\cdot n\lambda_n\right\}^k$$

$$\leqq A+A\sum_{n=2}^{\infty}\frac{1}{n}\left\{\int_0^{1/n}|\varphi(t)|\,dt\right\}^k\left(\frac{1}{n}\sum_{\nu=1}^{n-1}\nu^2\Delta\lambda_\nu\right)^k$$

$$+A\sum_{n=2}^{\infty}\frac{1}{n}\left\{\int_0^{1/n}|\varphi(t)|\,dt\right\}^k\cdot n^k\lambda_n^k$$

$$\leqq A+A\sum_{n=2}^{\infty}\frac{1}{n^{2k+1}}\left(\sum_{\nu=1}^{n-1}\nu^2\Delta\lambda_\nu\right)^k+A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}$$

$$\leqq A+A\sum_{n=2}^{\infty}\frac{1}{n^{2k+1}}\left(\sum_{\nu=1}^{n-1}\nu^{2k}\Delta\lambda_\nu\right)\left(\sum_{\nu=1}^{n-1}\Delta\lambda_\nu\right)^{k/k'}$$

$$\leqq A+A\sum_{n=2}^{\infty}\frac{1}{n^{2k+1}}\sum_{\nu=1}^{n-1}\nu^{2k}\Delta\lambda_\nu$$

$$\leqq A+A\sum_{\nu=1}^{\infty}\nu^{2k}\Delta\lambda_\nu\sum_{n=\nu+1}^{\infty}\frac{1}{n^{2k+1}}$$

$$\leqq A+A\sum_{\nu=1}^{\infty}\nu^{2k}\Delta\lambda_\nu\cdot\frac{1}{(\nu+1)^{2k}}$$

$$\leqq A+A\sum_{\nu=1}^{\infty}\Delta\lambda_\nu\leqq A.$$

Therefore we have

$$\sum_n\frac{\lambda_n^k}{n}\left\{\left|\int_{1/n}^{\pi}\varphi(t)D_n(t)dt\right|\right\}^k<\infty. \tag{1.6.1}$$

In order to complete the proof of the necessity part of the theorem we have to show that

$$\sum\frac{\lambda_n^k}{n}|s_n(x)-f(x)|<\infty.$$

Now by (1.6.1)

$$\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}|s_n(x)-f(x)|^k$$

$$=A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}\left|\int_0^{\pi}\varphi(t)D_n(t)dt\right|^k$$

$$\leqq A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}\left|\int_{1/n}^{\pi}\varphi(t)D_n(t)dt\right|^k+A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}\left\{\int_0^{1/n}|\varphi(t)||D_n(t)|dt\right\}^k$$

$$\leqq A+A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}\left\{\int_0^{1/n}|\varphi(t)||D_n(t)|dt\right\}^k$$

$$\leqq A+A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}\left\{\int_0^{1/n}|\varphi(t)|n\,dt\right\}^k$$

$$\leqq A+A\sum_{n=2}^{\infty}\lambda_n^k\cdot\frac{n^k}{n}\left\{\int_0^{1/n}|\varphi(t)|\,dt\right\}^k$$

$$\leqq A+A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}\cdot n^k\cdot n^{-k}\leqq A+A\sum_{n=2}^{\infty}\frac{\lambda_n^k}{n}$$

$$\leqq A.$$

This completes the proof of the theorem.

The author would like to express his warmest thanks to Dr. S. M. Mazhar for his kind help and valuable suggestions during the preparation of this paper.

REFERENCES

[1] FLETT, T. M., On an extension of absolute summability and some theorems of Littlewood and Paley. Proc. London Math. Soc. **7** (1957), 113–141.

[2] PATI, T., Absolute Cesàro summability factors of infinite series. Math. Z. **78** (1962), 293–297.

[3] PATI, T., On an unsolved problem in the theory of absolute summability factors of Fourier series. Math. Z. **82** (1963), 106–114.

DEPARTMENT OF MATHEMATICS,
ALIGARH MUSLIM UNIVERSITY, ALIGARH, INDIA.