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CONVERGENCE OF NORMAL OPERATORS

BY BURTON RODIN AND LEO SARIO

The first purpose of the present paper is to show that elementary linear operator theory can be used to give an elegant proof of the fundamental existence theorem of principal functions corresponding to given normal operators L [17]. When L is defined by a limiting process a harmonic function may also be obtained by applying the main theorem to each approximating operator and forming a limit of resulting functions. It is important to know when these processes commute. We shall give a general criterion to this effect and show that it applies to operators L_0 and L_1. Earlier literature on normal operators and their applications is compiled in the Bibliography.

1. The q-lemma. We start with a slight sharpening of the q-lemma [17]:

LEMMA 1. *Let K be a compact subset of a Riemann surface W. There exists a positive constant $q<1$ such that all harmonic functions u on W satisfy the inequality*

$$q \inf_W u+(1-q) \max_K u \leqq u|K \leqq (1-q) \min_K u+q \sup_W u. \tag{1}$$

Proof. Harnack's inequality for positive harmonic functions v in the unit disk reads

$$\frac{1-|z|}{1+|z|} v(0) \leqq v(z) \leqq \frac{1+|z|}{1-|z|} v(0).$$

An easy consequence is that to any compact set K in a Riemann surface W there corresponds a constant $c>0$ such that

$$c^{-1} \leqq \frac{v(\mathrm{P})}{v(\mathrm{Q})} \leqq c \tag{2}$$

for all points P and Q in K and all positive harmonic functions v. To see this note first that K may be assumed to be connected, thanks to the existence of an exhaustion for W. K can be covered by a finite number n of parametric disks $\mathcal{V}_i$ with centers V_i such that the subdisks $\mathcal{V}_i'$ corresponding to $\{z\colon |z|<1/2\}$ also form an open cover of K. By Harnack's inequality $1/3<v(\mathrm{P}_i)/v(\mathrm{Q}_i)<3$ for $\mathrm{P}_i \in \mathcal{V}_i'$, and conse-

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quently $1/9<v(P_i)/v(Q_i)<9$ if P_i and Q_i are in $\mathcal{V}_i'$. To any pair of points P and Q in K there is a sequence $P=R_1, R_2, \cdots, R_n=Q$ from K such that R_j and R_{j+1} belong to some $\mathcal{V}_i'$ $(j=1, \cdots, n-1)$; in fact for fixed P the set of such Q's is seen to be open and closed in K. Thus 9^{n-1} can be a value for c in (2).

An application of (2) to a function $1-u$ gives $u(P)\leqq c^{-1}u(Q)+(1-c^{-1})$ if $u\leqq 1$ on W. Equivalently, there is a $q<1$ such that $u(P)\leqq(1-q)u(Q)+q\sup_W u$ for any harmonic function u on W. This last formula may be applied to $-u$, P, and some $Q'\in K$ to yield

$$q\inf_W u+(1-q)u(Q')\leqq u(P)\leqq(1-q)u(Q)+q\sup_W u,$$

from which (1) follows immediately.

2. The main existence theorem. Let W' be a union of disjoint, bordered Riemann surfaces. Let α denote the border of $W', C(\alpha)$ the space of continuous real-valued functions on α, and $H(W')$ the space of real-valued functions which are continuous on W' and harmonic in its interior. In a natural way we may consider $C(\alpha)$ and $H(W')$ as real vector spaces.

By definition [17], a normal operator L for W' is a linear transformation of $C(\alpha)$ into $H(W')$ such that for all $f\in C(\alpha)$

$$(Lf)|\alpha=f, \tag{3}$$

$$\min f\leqq Lf\leqq\max f, \tag{4}$$

$$\int_\beta dLf^*=0. \tag{5}$$

In condition (5) β is any cycle on W' homologous to α, and

$$dLf^*=\frac{\partial}{\partial n}(Lf)ds$$

where n is the outer normal to β. The integral is then well-defined and is called the flux of Lf.

We call a subset W' of a Riemann surface W a regular boundary neighborhood if $W-W'$ is a regular subregion of W. Let $H(W)$ be the vector space of harmonic functions on W.

We recall that the cokernel of a transformation L is the quotient space of the range of L by the image of L; in symbols $\operatorname{coker} L=\operatorname{rg} L/\operatorname{im} L=H(W')/\operatorname{im} L$.

The main existence theorem of principal functions [17] can be given the following algebraic formulation:

THEOREM 1. *The linear transformation of $H(W)$ into* coker L *by $p\to p|W'+\operatorname{im} L$ has as its kernel the space of real numbers, and as its image the space of cosets $s+\operatorname{im} L$ with $\int_\beta ds^*=0$.*

We shall give a proof of this important theorem by making use of linear operators. Suppose p is harmonic on W and is the kernel of the mapping $p \to p|W' + \operatorname{im} L$. Then $p|W' \in \operatorname{im} L$ and so achieves its maximum on $\alpha = \partial W'$ according to (4). This maximum also dominates $p|W$. Since p has a maximum at an interior point of W it is constant. By (3) every constant is in $\operatorname{im} L$, so the kernel of the mapping in question is precisely the space of real numbers.

The flux $\int_\beta ds^*$ of a coset $s + \operatorname{im} L$ is independent of the representative s according to (5). By Green's formula $\int_\beta dp^* = 0$ for any $p \in H(W)$. Hence the image of the mapping $p \to p|W' + \operatorname{im} L$ is contained in the subspace of $\operatorname{coker} L$ consisting of cosets with vanishing flux. To complete the proof we must show that every coset with vanishing flux has a global representative.

3. Reduction of the problem. Given a coset $s + \operatorname{im} L$ in $\operatorname{coker} L$ with $\int_\beta ds^* = 0$ we must find a p harmonic on W such that $p - s$ is in the image of L. Without loss of generality we may assume s vanishes on α since $s - Ls$ does and it is a representative of the same coset. Let Ω be a regular subregion of W which contains $\overline{W - W'}$. We shall now show that our problem can be reduced to that of finding a function $\tilde{p}$ on $\partial\Omega$ with the property

$$\tilde{p} = LD\tilde{p} + s, \tag{6}$$

where D is the "Dirichlet operator" which associates to each continuous function f on $\partial\Omega$ the solution of the Dirichlet problem in Ω with boundary values f. The notation $LD\tilde{p}$ in (6) is of course an abbreviation for $L(D\tilde{p}|\alpha)|\partial\Omega$.

Suppose $\tilde{p}$ is a continuous function on $\partial\Omega$ which satisfies (6). Then the harmonic functions $D\tilde{p}$ and $LD\tilde{p} + s$, with domains Ω and W' respectively, are equal to $D\tilde{p}$ on α and $\tilde{p}$ on $\partial\Omega$. By the maximum principle they must therefore be identical in $\Omega \cap W'$. Hence $p = D\tilde{p} \cup (LD\tilde{p} + s)$ is a well-defined harmonic function on W. Since p has the representation $LD\tilde{p} + s$ in W' it will evidently serve as the required global representative of the given coset $s + \operatorname{im} L$.

4. An invertible operator. Let T be the linear operator $f \to LDf$ of $C(\partial\Omega)$ into itself. Let I be the identity operator on this space. To solve (6) we must show that $s \in \operatorname{im}(I - T)$.

Recall that if X is a Banach space and $T\colon X \to X$ is a linear operator with $\|T\| < 1$ then $I - T\colon X \to X$ has an inverse $(I - T)^{-1}$ whose norm is no greater than $(1 - \|T\|)^{-1}$. In our case, $C(\partial\Omega)$ becomes a Banach space under the sup norm $\|\cdot\|_{\partial\Omega}$. However, the inequality $\|T\| < 1$ needed to prove that $I - T$ is invertible, and hence that $s \in \operatorname{im}(I - T)$, is not valid. Indeed, $\|T\| = 1$. Clearly this difficulty can be overcome if we exhibit a subspace $X \subset C(\partial\Omega)$ such that (i) X is a Banach subspace, (ii) when restricted to X T has norm < 1, (iii) $T(X) \subset X$, and (iv) $s \in X$. When this has been accomplished the proof of Theorem 1 will be complete, and we will also have the estimate

$$\|(I - T)^{-1}\| \leqq (1 - \|T\|)^{-1} \tag{7}$$

which will be used later (§ 8).

5. Existence of X. Let ω be the harmonic function on $\Omega\cap W'$ with boundary values 0 on α and 1 on $\partial\Omega$. Let X be the kernel of the continuous linear functional $f\to\int_{\partial\Omega}f d\omega^*$ on $C(\partial\Omega)$. Then (i) holds since X is a closed subspace of $C(\partial\Omega)$.

For the proof of (ii) we shall show that if $f\in X$ then Df changes sign on α. For if that were true we would then have

$$\|Tf\|_{\partial\Omega}=\|LDf\|_{\partial\Omega}\leqq\|Df\|_{\alpha}\leqq q\|f\|_{\partial\Omega}\qquad(q<1)$$

where the first inequality follows by property (4) of normal operators and the second follows by the q-lemma applied to the compact set α in the Riemann surface Ω.

Before verifying that $Df|\alpha$ changes sign we pause to note a useful property of normal operators. Let $g\in C(\alpha)$. Green's formula applied to the open set $\Omega\cap W'$, with oriented boundary $\partial\Omega-\alpha$, and the functions Lg and ω yields

$$\int_{\alpha}g\,d\omega^*=\int_{\partial\Omega}(Lg)d\omega^*. \tag{8}$$

This formula implies

$$\int_{\partial\Omega}fd\omega^*=\int_{\alpha}(Df)d\omega^* \tag{8'}$$

since D is also a normal operator.

Along α the measure $d\omega^*$ is positive. Thus the condition $f\in X$, which by (8′) is equivalent to the vanishing of $\int_{\alpha}(Df)d\omega^*$, implies that Df is not of constant sign on α. This proves (ii).

From (8), (8′) we see that $f\in X$ implies $LDf\in X$, and (iii) follows.

Green's formula applied to s and ω, together with the fact that $s=0$ along α, yields $\int_{\partial\Omega}sd\omega^*=\int_{\partial\Omega}ds^*$. Hence the hypothesis that s has vanishing flux is equivalent to the condition $s\in X$. Thus (iv) holds, and the proof of the main existence theorem is complete.

The function p is called an *L-principal function* with singularity s.

6. Convergence of operators. We recall the operator $L_{0\Omega}$ defined by vanishing normal derivatives on $\partial\Omega$, and the operator $L_{1\Omega}$ characterized by constant values on sets of components of $\partial\Omega$ corresponding to a given partition [2]. These operators are examples of how normal operators L_Ω on subregions can be used to give, in the limit $\Omega\to W$, normal operators L for open surfaces W'. Suppose a singularity function s_Ω with vanishing flux is given on each $\Omega\cap W'$. The main existence theorem gives functions p_Ω on Ω such that $p_\Omega-s_\Omega\in\operatorname{im}L_\Omega$. Suppose further that s_Ω tends uniformly on compacta to a function s harmonic on W'. The existence theorem yields also a function p harmonic on W such that $p-s\in\operatorname{im}L$. We wish to find conditions on $\{L_\Omega\}$ which will insure that $p_\Omega\to p$ uniformly on compacta.

We shall write $L_\Omega\Rrightarrow L$ provided to each compact set K in W' and each $\varepsilon>0$ there is an Ω_0 such that

$$\|Lf-L_\Omega f\|_K<\varepsilon$$

for all $f\in C(\alpha)$ satisfying $\|f\|_\alpha=1$ and all $\Omega\supset\Omega_0$.

7. Convergence of p_Ω. We are considering a family $\{L_\Omega\}$ of normal operators for $W'\cap\Omega$, where as usual W' is a regular neighborhood of the ideal boundary of a Riemann surface W. In $W'\cap\Omega$ we are given harmonic functions s_Ω and principal functions p_Ω in Ω with singularity s_Ω.

THEOREM 2. *Suppose $L_\Omega\Rrightarrow L$ and s_Ω tends to a limit function s uniformly on compacta. Then for a suitably normalized family $\{p_\Omega\}$ of principal functions with singularity s_Ω the limit*

$$p=\lim_\Omega p_\Omega$$

exists uniformly on compacta of W, where p is an L-principal function with singularity s.

Proof. Suppose given a compact set $K\subset W'$. Let ψ be a regular subregion of W which contains $\overline{W-W'}$ and K. (ψ takes the place of Ω used in the proof of Theorem 1.) We normalize p_Ω, p by an additive constant so that when restricted to $\partial\psi$ they are in the space X defined in § 5. Thus on $\partial\psi$ we have $p=(I-T)^{-1}s$ and $p_\Omega=(I-T_\Omega)^{-1}s_\Omega$ where $T=LD$, $T_\Omega=L_\Omega D$, and D is the Dirichlet operator D: $X\to H(\psi)$. We must prove that $\lim_{\Omega\to W}\|p-p_\Omega\|_{\partial\psi}=0$.

The hypothesis $L_\Omega\Rrightarrow L$ means that $\|T-T_\Omega\|\to 0$ as $\Omega\to W$. Indeed, if $g\in C(\partial\psi)$ then given $\varepsilon>0$ there is an Ω_0 such that

$$\|LDg-L_\Omega Dg\|_{\partial\psi}\leqq\varepsilon\|Dg\|_\alpha$$

if $\Omega\supset\Omega_0$. Since $\|Dg\|_\alpha\leqq\|g\|_{\partial\psi}$ this gives

$$\|Tg-T_\Omega g\|_{\partial\psi}\leqq\varepsilon\|g\|_{\partial\psi}.$$

We have the estimate

$$\begin{aligned}\|p_\Omega-p\|_{\partial\psi}&=\|(I-T_\Omega)^{-1}s_\Omega-(I-T)^{-1}s\|_{\partial\psi}\\ &\leqq\|(I-T_\Omega)^{-1}-(I-T)^{-1}\|\cdot\|s_\Omega\|_{\partial\psi}+\|(I-T)^{-1}\|\cdot\|s_\Omega-s\|_{\partial\psi},\end{aligned}$$

and this upper bound tends to 0 as $\Omega\to W$. In fact, this is obviously the case for the term $\|(I-T)^{-1}\|\cdot\|s_\Omega-s\|_{\partial\psi}$, and the term $\|(I-T_\Omega)^{-1}-(I-T)^{-1}\|$ is bounded by $\|T-T_\Omega\|\cdot(1-\|T\|)^{-1}(1-\|T_\Omega\|^{-1})$ since $\|T\|<1$ and $\|T_\Omega\|<1$. The convergence to 0 now follows from the earlier remark that $\|T-T_\Omega\|\to 0$. This completes the proof.

8. The main theorem with estimates. Let s be a singularity function as in the hypotheses of Theorem 1. In many applications it is necessary to have bounds for the principal function p with this singularity. To obtain them we again recall the proof of Theorem 1.

For a regular region Ω containing $\overline{W-W'}$ the function p, suitably normalized, satisfies $(I-T)^{-1}s=p$ along $\partial\Omega$, where $T=LD$ is restricted to X. From the estimate (7) we have $\|p\|_{\partial\Omega}\leqq(1-q)^{-1}\|s\|_{\partial\Omega}$. This proves the following result [22]:

THEOREM 3. *In addition to the hypotheses of Theorem* 1 *assume* $s=0$ *on* $\partial W'$. *Then the principal function* p *may be normalized by an additive constant so that*

$$\|p\|_{\partial\Omega}\leqq \frac{1}{1-q}\|s\|_{\partial\Omega}$$

for any region $\Omega\supset\overline{W-W'}$. *The constant* q *is determined by Lemma* 1 *applied to the surface* Ω *and the compact set* $\partial W'$.

9. Auxiliary functions. Given an Ω containing $\overline{W-W'}$, we shall recall the construction [22] of functions $g_{0\Omega}$ and $g_{1\Omega}$ which are harmonic in $W'\cap\Omega$ except for a logarithmic pole and have the reproducing property $L_{i\Omega}f=\int_\alpha f\,dg_{i\Omega}^*$ $(i=0,1)$ for all continuous functions f on α. Such an integral representation will then be used to prove the strong convergence $L_{i\Omega}\Rrightarrow L_i$.

LEMMA 2. *Let* z *be a parameter for a neighborhood of a point* B *in* $\Omega\cap W'$. *There exist functions* $g_{i\Omega}(\cdot,\mathrm{B})$ *harmonic in* $\Omega\cap W'-\{\mathrm{B}\}$ *with* L_i *behavior along* $\partial\Omega$, *vanishing boundary values on* $\partial W'$, *and with the singularity* $(2\pi)^{-1}\log|z-z(\mathrm{B})|$ *at* B $(i=0,1)$.

More precisely, $g_{0\Omega}(\cdot,\mathrm{B})$ has vanishing normal derivative along $\partial\Omega$, $g_{1\Omega}(\cdot,\mathrm{B})$ has constant values along the parts of $\partial\Omega$ associated with an unspecified partition, and the flux of $g_{1\Omega}(\cdot,\mathrm{B})$ vanishes along each part separately. The function

$$h_{i\Omega}(z)=g_{i\Omega}(z,z(\mathrm{B}))-\frac{1}{2\pi}\log|z-z(\mathrm{B})|$$

is supposed to have a harmonic extension to $z=z(\mathrm{B})$.

For the proof, we fix disjoint neighborhoods in W' of $\alpha=-\partial W'$, B, and $\partial\Omega$, and denote them by $\mathcal{A}$, $\mathcal{B}$, and C respectively. We may assume that $\bar{\mathcal{A}}\cup\bar{\mathcal{B}}\cup\bar{C}$ is a regular boundary neighborhood on the surface $\mathring{W}'\cap\Omega-\{\mathrm{B}\}=R$ and that $\mathcal{B}$ corresponds to $|z|<1$. Let v be any harmonic function in $\bar{\mathcal{A}}$ with constant values on α and satisfying $\int_\alpha dv^*=-1$. Define a function s in $\bar{\mathcal{A}}\cup\bar{\mathcal{B}}\cup\bar{C}$ by

$$s=\begin{cases} v & \text{in } \bar{\mathcal{A}},\\ \dfrac{1}{2\pi}\log|z-z(\mathrm{B})| & \text{in } \bar{\mathcal{B}},\\ 0 & \text{in } \bar{C}.\end{cases}$$

Note that s has total flux zero along the ideal boundary of the surface R.

Let L_1 be the L_1-normal operator for $\bar{\mathcal{A}}-\alpha$, D the Dirichlet operator for $\bar{\mathcal{B}}-\{\mathrm{B}\}$, and L_i the L_i-operator for $\bar{C}-\partial\Omega$ $(i=0,1)$. We now apply the main existence theorem to the surface R, the singularity function s, and the normal operator $L=L_1\oplus D\oplus L_i$. This gives a function p_i on R satisfying $p_i-s=L(p_i-s)$. This p_i

has the required properties except that its values on α may be some constant other than 0. We merely subtract such a constant from p_i to obtain $g_{i\Omega}(\cdot, \mathrm{B})$.

10. Integral representations. Let u be harmonic in $W' \cap \bar{\Omega}$. Green's formula may be applied to u and $g_{i\Omega}(\cdot, \mathrm{B})$ on the bordered surface $W' \cap \bar{\Omega} - \mathfrak{B}_r$, where $\mathfrak{B}_r$ corresponds to $\{|z|<r\}$. If we note that

$$\lim_{\Omega\to 0}\int_{\partial\mathfrak{B}_r} u\,dg_{i\Omega}^* = u(\mathrm{B}),$$

$$\lim_{\Omega\to 0}\int_{\partial\mathfrak{B}_r} g_{i\Omega}du^* = 0,$$

we obtain the reproducing formulas

$$(9)\qquad u(\mathrm{B}) = \int_{-\alpha} u\,dg_0^* - \int_{\partial\Omega} g_0 du^*,$$

$$u(\mathrm{B}) = \int_{-\alpha} u dg_1^* + \int_{\partial\Omega} u\,dg_1^* - g_1 du^*.$$

In case u has L_0 or L_1 behavior on $\partial\Omega$ these formulas simplify. In particular, we have

$$(10)\qquad L_{i\Omega}f(\mathrm{A}) = \int_{-\alpha} f\,dg_{i\Omega}^* \qquad (i=0, 1).$$

For a fixed $\mathrm{A} \in W'$ we know [22] that the family $\{g_{i\Omega}\}_\Omega$ converges uniformly on compacta, in the sense that for any compact subset K of W' $\|g_{i\Omega} - g_{i\Omega'}\|_K \to 0$ as Ω and Ω' tend to W. In the limit we obtain

$$L_i f = \int_{-\alpha} f\,dg_i^* \qquad (i=0, 1).$$

Therefore, for fixed $\mathrm{A} \in W'$, $L_{i\Omega}f(\mathrm{A})$ converges to $L_i f(\mathrm{A})$ as $\Omega \to W$, and the convergence is uniform for all $f \in C(\alpha)$ with $\|f\|_\alpha \leqq 1$.

11. Convergence of p_0 and p_1. We are now able to prove the strong convergence $L_{i\Omega} \Rrightarrow L_i$ $(i=0, 1)$. Since $\|L_{i\Omega}f\|_{W'} \leqq \|f\|_\alpha$, the family $\{L_{i\Omega}f\}_{\Omega, f}$ is normal if $\|f\|_\alpha \leqq 1$. From this fact, together with the pointwise convergence $L_{i\Omega}f(\mathrm{A}) \to L_i f(\mathrm{A})$ which is uniform in f when $\|f\| \leqq 1$, it follows that $L_{i\Omega} \Rrightarrow L_i$.

THEOREM 4. *Let W' be a regular boundary neighborhood of the Riemann surface W, $\{\Omega\}$ an exhaustion of W, and $L_{i\Omega}$ $(i=0, 1)$ the normal operator for $W' \cap \Omega$. Then $L_{i\Omega} \Rrightarrow L_i$, where L_i is the normal operator for W'.*

As a consequence we have the following result:

COROLLARY. *For an exhaustion $\{\Omega\}$ let s_Ω be harmonic in $W' \cap \Omega$, satisfy*

$\int_\alpha ds_\Omega^* = 0$, *and converge uniformly on compacta to* s. *Then there are* $L_{i\Omega}$*-principal functions* p_Ω *with singularities* s_Ω, *and an* L_i*-principal function* p *with singularity* s *such that* $p_\Omega \to p$ *uniformly on compacta.*

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UNIVERSITY OF CALIFORNIA, SAN DIEGO,
STANFORD UNIVERSITY;
UNIVERSITY OF CALIFORNIA, LOS ANGELES.