STABILITY AND MIXING IN VON NEUMANN ALGEBRAS

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Summary.

For a sequence of events in a probability space, the concept of stability was introduced by Renyi in [2] and those of mixing and strongly mixing were introduced by Sucheston in [5].

Let $\mathfrak{A}$ be a von Neumann algebra of 'finite class'. Let $\{S_n\}$ be a sequence of elements in the unit sphere of $\mathfrak{A}$. In this paper, we introduce the notion of '*stability*' of $\{S_n\}$ with respect to (w.r.t.) a normal, positive linear functional (state) on $\mathfrak{A}$. We then prove in Theorem 1 that any uniformly bounded sequence of operators in $\mathfrak{A}$, contains a subsequence which is stable w.r.t. any state of $\mathfrak{A}$, and that if the given sequence $\{S_n\}$ is stable w.r.t. a faithful normal trace on $\mathfrak{A}$, then it is stable w.r.t. any state of $\mathfrak{A}$, and that if $\{S_n\}$ is stable w.r.t. a state σ, and ρ any state which is absolutely continuous w.r.t. σ, then $\{S_n\}$ is stable w.r.t. ρ also, i.e., stability is invariant under absolute continuity.

Let $\{P_n\}$ be any sequence of projections in $\mathfrak{A}$. For $\{P_n\}$, we introduce the concepts '*zero-one*', '*mixing*' and '*strongly mixing*' w.r.t. a state of $\mathfrak{A}$. As another application of Theorem 1, we show that these concepts are invariant under absolute continuity. Using the above results and Umegaki's martingale convergence theorem, we prove the main result of this paper (Theorem 4) which runs thus:—Let σ be any state of $\mathfrak{A}$, whose support is central (in particular let σ be a faithful state). Then a sequence $\{P_n\}$ is zero-one w.r.t. σ, if and only if it is strongly mixing w.r.t. σ.

§1. Let $\mathfrak{A}$ be a countably decomposable von Neumann algebra (on some Hilbert space), which is of 'finite class' in the sense of Dixmier [1]. Let I denote the identity operator. A positive normal linear functional m with $m(I)=1$ is called a *state* of $\mathfrak{A}$. A *state* m is said to be a *normal trace*, if, for any two elements S and T in $\mathfrak{A}$, $m(TS)=m(ST)$. A normal trace m is said to be *faithful*, if, for any projection P, $m(P)=0$ implies $P=0$. In what follows m will denote a faithful normal trace on $\mathfrak{A}$.

Definition. A uniformly bounded sequence $\{S_n\}$ of operators in $\mathfrak{A}$, is said to be *stable* w.r.t. a state σ of $\mathfrak{A}$, if, for any projection Q in $\mathfrak{A}$, the sequences $\{\sigma(S_nQ)\}$ and $\{\sigma(QS_n)\}$ both converge. If there exists an operator D such that for any pro-

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jection Q in $\mathfrak{A}$, $\sigma(S_nQ)\to\sigma(DQ)$ and $\sigma(QS_n)\to\sigma(QD)$ then D is called the *local density* of $\{S_n\}$ w.r.t. σ.

The terms '*strongly dense domain*', '*measurable operator*', '*essentially measurable operator*', '*integrable operator*', '*strong-product of measurable operators*' all have the same meaning as in Segal's paper [3]. For any two measurable operators A and B, AB and $A\times B$ will denote respectively the ordinary product and strong product of A and B. For any integrable operator T, $m(T)$ will denote the integral of T. Now we shall state and prove

THEOREM 1. *Any uniformly bounded sequence of operators in $\mathfrak{A}$ contains a subsequence $\{S_n\}$ with the following property·—There exists a unique operator D in $\mathfrak{A}$, such that for any state σ of $\mathfrak{A}$, $\{S_n\}$ is stable w.r.t. σ and D is the local density of $\{S_n\}$ w.r.t. σ. The operator D depends only on the sequence $\{S_n\}$ and is independent of σ.*

Before proving the theorem, we shall state three lemmas, but shall not prove them as the proofs are quite elementary.

LEMMA 1. *A uniformly bounded sequence $\{S_n\}$ in $\mathfrak{A}$ is stable w.r.t. m, if and only if, for any integrable operator A, the sequence $\{m(AS_n)\}$ converges.*

LEMMA 2. *If $\{S_n\}$ is stable w.r.t. m, then there exists a unique element D in $\mathfrak{A}$, such that for any integrable operator A,*

$$m(AD)=\lim_{n\to\infty} m(AS_n).$$

LEMMA 3. *Any uniformly bounded sequence in $\mathfrak{A}$, contains a subsequence which is stable w.r.t. m.*

Lemma 3 is an immediate consequence of Lemmas 1 and 2, and elementary properties of abstract Hilbert spaces.

Proof of the Theorem. By Lemma 3, the sequence given in the theorem has a stable subsequence w.r.t. m which also will be denoted by $\{S_n\}$. Let an arbitrary state σ of $\mathfrak{A}$ be given. By the non-commutative version of the Radon-Nikodym theorem, there exists an integrable operator R (the Radon-Nikodym derivative of σ w.r.t. m) such that for any B in $\mathfrak{A}$, $\sigma(B)=m(RB)$. Hence for any projection Q in $\mathfrak{A}$, $\sigma(S_nQ)=m(RS_nQ)=m((S_nQ)\times R)$. Now the two measurable operators $(S_nQ)\times R$ and $S_n\times(Q\times R)$ both agree with the essentially measurable operator S_nQR on the domain of R which is strongly dense. Hence by a lemma of Segal [3, Corollary 5·1, page 413], they both are identical with the closure of S_nQR and hence are themselves identical. So $\sigma(S_nQ)=m((S_nQ)\times R)=m(S_n\times(Q\times R))$. Hence

$$\lim_{n\to\infty} m(S_n\times(Q\times R))=m(D\times(Q\times R))=m(RDQ)=\sigma(DQ),$$

i.e. $\sigma(S_nQ)\to\sigma(DQ)$. Similarly one can show that $\sigma(QS_n)\to\sigma(QD)$ as $n\to\infty$. Hence $\{S_n\}$ is stable w.r.t. σ, and has local density D w.r.t. σ. Hence the theorem.

REMARK. From the proof of Theorem 1, we have immediately the following proposition:—a sequence $\{S_n\}$ stable w.r.t. m and having local density D w.r.t. m is stable w.r.t. an arbitrary state σ, and has local density D w.r.t. σ.

In the following, we shall prove a more general case of the remark, in which a state will take the place of the trace m. In what follows $\{S_n\}$ will denote any uniformly bounded sequence in $\mathfrak{A}$.

THEOREM 2. *Let σ and ρ be any two states of $\mathfrak{A}$. Let $\{S_n\}$ be stable w.r.t. σ and have local density D w.r.t. σ. If ρ is absolutely continuous w.r.t. σ, then $\{S_n\}$ is stable w.r.t. ρ, and has local density D w.r.t. ρ. In other words, stability and local density are invariant under absolute continuity.*

This theorem will not follow by a straightforward application of the Radon-Nikodym theorem, since the latter theorem has a simple form only when either σ or both σ and ρ are traces. However we shall prove the theorem by applying Theorem 1.

Proof. Case 1. Let σ be faithful. By Theorem 1, $\{S_n\}$ has a subsequence $\{S_n'\}$ which is stable w.r.t. m and has local density say D', w.r.t. m. Again, by Theorem 1, $\{S_n'\}$ has local density D' w.r.t. σ. Hence for any projection Q, $\sigma(DQ)=\lim_{n\to\infty}\sigma(S_n'Q)=\sigma(D'Q)$, i.e., $\sigma((D-D')Q)=0$. As Q is arbitrary and σ is faithful, this implies that $D=D'$. Hence by the assumption on the original sequence $\{S_n\}$ it is stable w.r.t. m and has local density D w.r.t. m. Since ρ is absolutely continuous w.r.t. m, the desired result follows from Theorem 1.

Case 2. Let C be the maximal null projection in $\mathfrak{A}$ of σ. Set $P=I-C$. Then P is the support of σ. Let $l=1/(1+m(C))$. For any A in $\mathfrak{A}$, let $\sigma_1(A)=l(\sigma(A)+m(CA))$. Then σ_1 is also a state on $\mathfrak{A}$, which is faithful. For any B in $\mathfrak{A}$, $l\cdot\sigma(S_nB)=\sigma_1(PS_nB)$. Let $A_n=PS_n$. Since for any projection Q, $\sigma(S_nQ)\to\sigma(DQ)$, it follows that $\sigma_1(A_nQ)\to\sigma_1(PDQ)$. By case 1, $\{A_n\}$ is stable w.r.t. ρ and has local density PD w.r.t. ρ. But for any projection Q, $\rho((A_n-S_n)Q)=0$ and $\rho(PDQ)=\rho(DQ)$. Hence $\{S_n\}$ is stable w.r.t. ρ and has local density D w.r.t. ρ.

§ **2.** Before proceeding further, we shall define a few more concepts:—

DEFINITION. A sequence $\{P_n\}$ of projections in $\mathfrak{A}$ is said to be *mixing* with *density* γ $(0<\gamma<1)$ w.r.t. a state σ, if $\sigma(P_n)\to\gamma$ and for any projection Q in $\mathfrak{A}$, $\lim_{n\to\infty}(\sigma(P_nQ)-\sigma(P_n)\cdot\sigma(Q))=0$. Let $\mathfrak{B}_n$ be the von Neumann algebra generated by the sequence $\{P_{n+r}, r=0,1,2,\cdots\}$. Put $\mathfrak{B}=\cap_{n=1}^{\infty}\mathfrak{B}_n$. $\{P_n\}$ is said to be *zero-one* w.r.t. σ, if, for any projection Q in $\mathfrak{B}$, $\sigma(Q)=0$ or 1. $\{P_n\}$ is said to be *strongly mixing,* if every sequence following $\{P_n\}$ is *mixing*, where a sequence $\{Q_n\}$ of projections is said to *follow* $\{P_n\}$ if, for each n, Q_n belongs to $\mathfrak{B}_n$.

Clearly strongly mixing implies mixing. For every sequence of projections trivially follows itself. An example of a mixing sequence will be given in Theorem 4.

THEOREM 3. *The properties of mixing (and density) and strongly mixing are invariant under absolute continuity.*

Here again, the proof is not straightforward. For, if $\{P_n\}$ is mixing with density γ w.r.t. a state σ, and ρ another state absolutely continuous w.r.t. σ, it is not obvious that $\rho(P_n)\to\gamma$ and $\{P_n\}$ is mixing w.r.t. ρ. We shall state a lemma first.

LEMMA 4. *A sequence $\{P_n\}$ of projections is mixing w.r.t. a state σ, if and only if $\{P_n\}$ is stable w.r.t. σ and its local density w.r.t. σ is a scalar multiple of the support projection S_σ of σ. This non-negative scalar will in fact be the density of the mixing sequence.*

Proof of the lemma is elementary and hence omitted.

Now to prove the theorem. Let $\{P_n\}$ be mixing (with density γ) w.r.t. a state σ. By Lemma 4 and Theorem 2 it follows that $\{P_n\}$ is stable w.r.t. ρ and has local density γS_σ w.r.t. ρ. Applying Lemma 4 again, it follows that mixing is invariant under absolute continuity.

§ **3.** We now state and prove the main result of this paper:—

THEOREM 4. *Let σ be any state of $\mathfrak{A}$, whose support is central (i.e., a projection belonging to the center of $\mathfrak{A}$). Let $\{P_n\}$ be any arbitrary sequence of projections in $\mathfrak{A}$, with $\sigma(P_n)\to\gamma$ $(0<\gamma<1)$. Then $\{P_n\}$ is zero-one w.r.t. σ, if and only if it is strongly mixing w.r.t. σ.*

Proof. Let R be the support of σ and let $k=1/m(R)$. For any T in $\mathfrak{A}$, set $\tau(T)=km(RT)$. As R is central, it follows that τ is a normal trace on $\mathfrak{A}$ with $\tau(R)=1=\tau(I)$. Since τ and σ are absolutely continuous w.r.t. each other (both having the same support) and since by Theorem 3, the properties of zero-one and strongly mixing are invariant under absolute continuity, it suffices to prove the theorem for τ in place of σ. Firstly, we shall state some definitions and prove some lemmas.

For any von Neumann subalgebra $\mathfrak{B}$ ($\mathfrak{B}\subset\mathfrak{A}$), let $R\mathfrak{B}=\{RS\colon S \text{ in } \mathfrak{B}\}$. As R is central easy to check that $R\mathfrak{B}$ is also a von Neumann algebra acting on the range of the projection R. Let us denote the conditional expectation of T w.r.t. $\mathfrak{B}$ under the trace τ by $E_\tau(T/\mathfrak{B})$. However as τ is not faithful, this has to be defined in an indirect way.

For any T in $\mathfrak{A}$, RT is an element of $R\mathfrak{A}$. Since τ is faithful normal trace on $R\mathfrak{A}$, the conditional expectation (in the sense of Umegaki [6]) of RT w.r.t. $R\mathfrak{B}$ (under the trace τ) is well-defined and unique. Let it be denoted by $\tilde{S}$, i.e., $E_\tau(RT/R\mathfrak{B})=\tilde{S}$. And, by the definition of $R\mathfrak{B}$, there exists at least one (there may exist more than one) element S in $\mathfrak{B}$ with $RS=\tilde{S}$. If S_1 is another element of $\mathfrak{B}$ such that $RS_1=\tilde{S}$, then easy to verify that for any projection Q in $\mathfrak{A}$, $\tau((S-S_1)Q)=0$, i.e., S is unique to within the support of τ. We define $E_\tau(T/\mathfrak{B})=S$.

We shall now justify our definition of conditional expectation.

Note that for any A in $\mathfrak{B}$, $\tau((RT)A)=\tau(\tilde{S}A)$ (by definition of $\tilde{S}$). For any B in $\mathfrak{B}$,

$$\tau(SB)=\tau(RSB)=\tau((RS)\cdot(RB))=\tau(\tilde{S}RB)$$
$$=\tau((RT)\cdot(RB))=\tau(RTB)=\tau(TB).$$

Also S is an element of $\mathfrak{B}$. Hence we are justified in writing $E_\tau(T/\mathfrak{B})=S$.

Let T be any element of $\mathfrak{A}$. $|T|$ will denote $(T^*T)^{1/2}$ and $||T||$ will denote the operator-norm of T. The L_1-norm of T under τ is the number $\tau(|T|)$, and is denoted by $||T||_1$. The L_2-norm of T is the number $(\tau(T^*T))^{1/2}$ and is denoted by $||T||_2$. A sequence $\{T_n\}$ in $\mathfrak{A}$ *converges* to an element T in $\mathfrak{A}$ *in the L_p-mean,* if $||T_n-T||_p\to 0, p=1, 2$, and *converges* to T *in measure* [4] if given any $\varepsilon>0$, there exists a sequence $\{R_n\}$ of projections in $\mathfrak{A}$ such that $||(T_n-T)R_n||\leqq\varepsilon$ for all n and $\tau(R_n)\to 1$. Convergence in the L_p-mean implies convergence in measure. We shall prove this for $p=2$. The proof for $p=1$ is similar. Let $S_n=(T_n-T)^*(T_n-T)$ $=|T_n-T|^2$. Let P_n be the spectral projection of $|T_n-T|$ corresponding to the interval (ε, ∞) and $R_n=I-P_n$. Then $\tau(S_n)=\tau(S_nP_n)+\tau(S_nR_n)\geqq\tau(S_nP_n)\geqq\varepsilon^2\cdot\tau(P_n)$. Hence $\tau(P_n)\leq(1/\varepsilon^2)\cdot\tau(S_n)\to 0$ as $n\to\infty$. Hence $\tau(R_n)\to 1$ and $||(T_n-T)R_n||\leqq\varepsilon$ for all n which proves that $\{T_n\}$ converges in measure to T.

In what follows, we shall prove several lemmas.

LEMMA 5. *If $\{P_n\}$ is a sequence of projections with $\tau(P_n)\to\gamma$ $(0<\gamma<1)$, then the following conditions are equivalent:*

1. *The sequence $\{P_n\}$ is mixing w.r.t. τ with density γ.*
2. *Let $\mathfrak{A}_n$ be the von Neumann subalgebra generated by the single projection P_n. For each projection Q in $\mathfrak{A}$, the sequence $\{E_\tau(Q/\mathfrak{A}_n)\}$ converges in measure to $\tau(Q)I$.*

Proof. As $\tau(P_n)\to\gamma$, without loss of generality, one can assume that $0<\tau(P_n)<1$ for all n. Firstly, we shall show condition 1 implies condition 2. Easy to see that one version of conditional expectation of Q w.r.t. $\mathfrak{A}_n$ can be taken to be $(\tau(QP_n)/\tau(P_n))P_n+(\tau(QP_n^\perp)/\tau(P_n^\perp))P_n^\perp$, where $P_n^\perp=I-P_n$. In fact in this case the conditional expectation is even unique. Then condition 1 is equivalent to saying that $\tau(QP_n)/\tau(P_n)\to\tau(Q)$, whence it also follows that $\tau(QP_n^\perp)/\tau(P_n^\perp)\to\tau(Q)$. Therefore, putting

$$a_n=(\tau(QP_n)/\tau(P_n))-\tau(Q) \qquad \text{and} \qquad b_n=(\tau(QP_n^{\perp})/\tau(P_n^{\perp}))-\tau(Q),$$

the condition 1 is equivalent to saying that $a_n \to 0$ whence it also follows that $b_n \to 0$. Hence $\|E_\tau(Q/\mathfrak{A}_n)-\tau(Q)I\| \leqq |a_n|+|b_n| \to 0$ and $\{E_\tau(Q/\mathfrak{A}_n)\}$ converges in measure to $\tau(Q)I$.

For the converse, let a_n and b_n be the same as above for a fixed Q. Then $E_\tau(Q/\mathfrak{A}_n)-\tau(Q)I=a_nP_n+b_nP_n^{\perp}$. We shall now show that given any $\varepsilon>0$, one can find an integer $N>0$ such that for any $n \geqq N$, $|a_n|<\varepsilon$ and $|b_n|<\varepsilon$ so that the sequences $\{a_n\}$ and $\{b_n\}$ both converge to zero. Let $S_n=a_nP_n+b_nP_n^{\perp}$. Since by our assumption $\{S_n\}$ converges in measure to zero, there exists a sequence $\{R_n\}$ of projections with $\|S_nR_n\|<\varepsilon$ for all n and $\tau(R_n) \to 1$. By assumption $\tau(P_n) \to \gamma$ and $\tau(P_n^{\perp}) \to 1-\gamma$. Let $\delta=\min(\gamma, 1-\gamma)$. We may assume $\varepsilon<\delta$. Hence one can find integers N_1 and N_2 (both>0) such that $\tau(P_n)>\delta-\varepsilon/2$ and $\tau(P_n^{\perp})>\delta-\varepsilon/2$ for all $n \geqq N_1$ and $\tau(R_n)>1-\delta-\varepsilon/4$ for all $n \geqq N_2$. Let $N=\max(N_1, N_2)$. Then for any $n \geqq N$, $P_n \wedge R_n$ and $P_n^{\perp} \wedge R_n$ are non-null. Let x_n be any unit vector in $P_n \wedge R_n$. $\|S_nR_n\|<\varepsilon$ implies $\|S_nR_nx_n\|=\|S_nx_n\|=\|a_nx_n\|=|a_n|<\varepsilon$. Similarly for any $n \geqq N$, $|b_n|<\varepsilon$. Hence the sequences $\{a_n\}$ and $\{b_n\}$ both converge to zero, i.e., $(\tau(QP_n)-\tau(Q)\cdot\tau(P_n)) \to 0$ as $n \to \infty$, which shows that $\{P_n\}$ is mixing. Hence the lemma.

LEMMA 6. *Let $\mathfrak{B}$ and $\mathfrak{C}$ be von Neumann subalgebras with $\mathfrak{C} \subset \mathfrak{B} \subset \mathfrak{A}$. Then $\tau(|E_\tau(T/\mathfrak{C})|) \leqq \tau(|E_\tau(T/\mathfrak{B})|)$ for any T in $\mathfrak{A}$.*

Proof. If τ is faithful, the required inequality is known by Umegaki [6]. If τ is not faithful, then the support R of τ belongs to the center of $\mathfrak{A}$, and $R|T|=|RT|$ for any T in $\mathfrak{A}$. Hence,

$$\tau(|E_\tau(T/\mathfrak{C})|)=\tau(R|E_\tau(T/\mathfrak{C})|)=\tau(|E_\tau(RT/R\mathfrak{C})|)$$

and

$$\tau(|E_\tau(T/\mathfrak{B})|)=\tau(|E_\tau(RT/R\mathfrak{B})|).$$

Since τ is faithful regarded as a trace on $R\mathfrak{A}$, the inequality follows immediately from that.

LEMMA 7. *If a sequence $\{P_n\}$ of projections is zero-one w.r.t. τ then it is mixing w.r.t. τ.*

Proof. Firstly note that for any T in $\mathfrak{A}$, $\|T\|_2=\|RT\|_2$, where R is the support of τ. For $\|T\|_2^2=\tau(T^*T)=\tau(RT^*T)=\tau((RT)^*(RT))=\|RT\|_2^2$. Let Q be any projection in $\mathfrak{A}$. Let $\mathfrak{F}_k$ be the von Neumann algebra generated by $\{P_{k+r}, r=0, 1, 2, \cdots\}$. Let $R\mathfrak{F}_k$ be the von Neumann algebra consisting of elements RS where S belongs to $\mathfrak{F}_k$. As $R\mathfrak{F}_k$ is a subalgebra of $R\mathfrak{A}$ and τ a faithful normal trace on $R\mathfrak{A}$, the martingale convergence theorem of Umegaki [7] applies. By that theorem

$\{E_\tau(RQ/R\mathfrak{F}_k)\}$ converges in the L_2-mean. Hence $\{E_\tau(Q/\mathfrak{F}_k)\}$ converges in the L_2-mean. And since this sequence is uniformly bounded within the support of τ, convergence in measure, and convergence in the L_p-mean, $(p=1,2)$ are all equivalent. Hence $\{E_\tau(Q/\mathfrak{F}_k)\}$ convergences in the L_1-mean also. Let the limit be denoted by Q_0. As $\{\mathfrak{F}_k\}$ is a decreasing sequence of von Neumann algebras, Q_0 is measurable w.r.t. each $\mathfrak{F}_k$ and being bounded, belongs to $\mathfrak{F}_k$ for each k and hence to $\mathfrak{F}=\bigcap_{k=1}^\infty\mathfrak{F}_k$. For any projection P in $\mathfrak{F}$, $\tau(E_\tau(Q/\mathfrak{F}_k)P)=\tau(QP)$. In view of convergence in the L_2-mean,

$$\tau(Q_0P)=\lim_{k\to\infty}\tau(E_\tau(Q/\mathfrak{F}_k)P)=\tau(QP).$$

Thus $Q_0=E_\tau(Q/\mathfrak{F})$. Note that, for any projection Q, $E_\tau(Q/\mathfrak{F})$ is a scalar multiple of R, if and only if $R\mathfrak{F}$ is trivial, i.e., for any projection Q in $\mathfrak{F}$, $\tau(Q)=0$ or 1. Hence, if a sequence $\{P_n\}$ satisfies the conditions of the lemma, i.e. if $\{P_n\}$ is zero-one w.r.t. τ, then $E_\tau(Q-\tau(Q)I/\mathfrak{F}_n)\to 0$ in the L_1-mean. Let $\mathfrak{A}_n$ be as in Lemma 5. By Lemma 6, $E_\tau(Q-\tau(Q)I/\mathfrak{A}_n)\to 0$ in the L_1-mean, i.e. $E_\tau(Q/\mathfrak{A}_n)\to\tau(Q)I$ in the L_1-mean. Now an application of Lemma 5 yields the desired result.

We shall now return to the proof of Theorem 4.

Let $\{P_n\}$ be zero-one w.r.t. τ. Then it is mixing w.r.t. τ by Lemma 7. Let $\{Q_n\}$ be any sequence of projections following $\{P_n\}$. Then obviously $\{Q_n\}$ is also zero-one w.r.t. τ, and hence mixing w.r.t. τ. For the converse, let $\{P_n\}$ be strongly mixing w.r.t. τ. Let $\mathfrak{F}$ and $\mathfrak{F}_n$ be as in Lemma 7, and P any projection in $\mathfrak{F}$. Then the sequence all whose elements are P, follows the sequence $\{P_n\}$ and is hence mixing, so that $(\tau(P^2)-\tau(P)\cdot\tau(P))=0$, i.e., $\tau(P)=(\tau(P))^2$, and hence $\tau(P)=0$ or 1. Hence the theorem.

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