TANGENT BUNDLE OF A MANIFOLD WITH A NON-LINEAR CONNECTION

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The concept of a non-linear connection was introduced by Friesecke, and was later studied by Kawaguchi and others [1, 2, 3, 4, 5, 6]. On the other hand, the geometry of tangent bundle of a Riemannian manifold has been studied by Sasaki and that of a Finslerian manifold by Yano and Davies [8, 9, 12].

In this paper, we shall study the geometry of the tangent bundle of a manifold with a non-linear connection. As is well known, a linear connection is by definition a mapping of $\mathfrak{X}\times\mathfrak{X}$ into $\mathfrak{X}$. Then, in § 1 we define a non-linear connection as a mapping ∇ of $\mathfrak{X}\times\mathfrak{X}$ into $\mathfrak{X}$, where $\mathfrak{X}$ is the totality of differentiable vector fields on the manifold. By studying vector fields on the tangent bundle, we shall show in § 2 that there exists an almost complex structure in the tangent bundle of a manifold with a non-linear connection. In § 3 we introduce the so-called adopted frame which is very useful for our discussions. § 4 is devoted to the study of integrability conditions of a non-linear connection and of the almost complex structure determined by a non-linear connection. Since the tangent bundle of a manifold with a non-linear connection admits an almost complex structure, we can define almost analytic vector fields on tangent bundle, which will be discussed in § 5.

§1. Non-linear connection.

Let $\mathfrak{F}(M^n)$ be the set of all differentiable functions of class C^∞ on an n-dimensional differentiable manifold M^n of class C^∞ and $\mathfrak{X}(M^n)$ the set of all differentiable vector fields of class C^∞ on M^n.

Let us suppose that there is given a mapping ∇: $\mathfrak{X}(M^n)\times\mathfrak{X}(M^n)\to\mathfrak{X}(M^n)$ satisfying the conditions:[1)]

$$\text{(a)}\quad \nabla_{Y+Z}X=\nabla_Y X+\nabla_Z X,$$

$$\text{(b)}\quad \nabla_{fY}X=f\nabla_Y X,$$

$$(1.1)\qquad \text{(c)}\quad \nabla_Y(fX)=(Yf)X+f\nabla_Y X,$$

$$\text{(d)}\quad (\nabla_Y X)_p=(\mathring{\nabla}_Y X)_p, \qquad \text{if}\quad X_p=0,$$

$$\text{(e)}\quad (\nabla_Y(X+\bar{X}))_p=(\nabla_Y X)_p+(\nabla_Y \bar{X})_p, \qquad \text{if}\quad X_p+\bar{X}_p=0,$$

Received January 24, 1966.

1) This definition was suggested by Professor S. Ishihara.

X, $\bar{X}$, Y and Z being arbitrary elements of $\mathfrak{X}(M^n)$ and f an arbitrary element of $\mathfrak{F}(M^n)$, where X_p denotes the value of a vector field X at a point p of M^n, and the symbol $\mathring{\nabla}$ appearing in the equation (d) above denotes an arbitrary linear connection in M^n. We shall call such a mapping ∇ a *non-linear connection* in M^n. $(\mathring{\nabla}_Y X)_p$ does not depend on the linear connection $\mathring{\nabla}$ involved if $X_p=0$.

As is well known [7], a linear connection D in M^n is by definition a mapping D: $\mathfrak{X}(M^n)\times\mathfrak{X}(M^n)\to\mathfrak{X}(M^n)$ which satisfies

$$(1.2)\qquad \begin{aligned} &\text{(a)}\quad D_{Y+Z}X=D_YX+D_ZX,\\ &\text{(b)}\quad D_{fY}X=fD_YX,\\ &\text{(c)}\quad D_Y(fX)=(Yf)X+fD_YX,\\ &\text{(d)}\quad D_Y(X+\bar{X})=D_YX+D_Y\bar{X}, \end{aligned}$$

X, $\bar{X}$, Y and Z being arbitrary elements of $\mathfrak{X}(M^n)$ and f an arbitrary element of $\mathfrak{F}(M^n)$. Comparing (1.2) with (1.1), we see easily that a linear connection is a non-linear connection.

Now, we shall establish the local representation of a non-linear connection. Let us suppose that a non-linear connection ∇, which satisfies the conditions (1.1), is given in M^n. Let U be a coordinate neighbourhood of M^n with local coordinates (ξ^h) and $e_i=\partial/\partial\xi^i=\partial_i$ the natural frame corresponding to (ξ^h). Taking an arbitrary element X of $\mathfrak{X}(M^n)$, we may represent $\nabla_{e_i}X$ as

$$(1.3)\qquad \nabla_{e_i}X=(\partial_iX^h+\Gamma_i{}^h(\xi, X))e_h$$

where we have put $X=X^he_h$ in U. Then, we get n^2-functions $\Gamma_i{}^h(\xi, X)$ depending on coordinates ξ^h of a point in U and a vector field X. Taking arbitrary elements X and Y of $\mathfrak{X}(M^n)$, we have

$$\nabla_YX=Y^i\nabla_{e_i}X$$

because of (a), (b) of (1.1), where we have put $Y=Y^ie_i$ in U. The equation above then reduces to

$$(1.4)_a\qquad \nabla_YX=Y^i(\partial_iX^h+\Gamma_i{}^h(\xi, X))e_h.$$

Further, $\Gamma_i{}^h(\xi, X)$ are functions depending only on the coordinates (ξ^h) of the point p and the value X_p of the vector field X at p, that is,

$$(1.4)_b\qquad \Gamma_i{}^h(\xi, X)_p=\Gamma_i{}^h(\xi, \tilde{X})_p,\qquad \text{if}\quad X_p=\tilde{X}_p.$$

In fact, if we assume

$$X_p=\tilde{X}_p,$$

then, taking account of (1.3), we get

$$(\nabla_{e_i}(X-\tilde{X}))_p=(\partial_i(X-\tilde{X})^h+\Gamma_i{}^h(\xi, X)-\Gamma_i{}^h(\xi, \tilde{X}))_pe_h$$

because of (c) and (e) of (1. 1). On the other hand, putting $X=X^h e_h$, $\tilde{X}=\tilde{X}^h e_h$ and $X_p=X^h_p e_h$, $\tilde{X}_p=\tilde{X}^h_p e_h$, we have from (d) of (1. 1)

$$(\nabla_{e_i}(X-\tilde{X}))_p=(\partial_i(X^h-\tilde{X}^h))_p e_h, \qquad \text{if} \quad X_p=\tilde{X}_p.$$

Thus the equation above reduces to $(1.\,4)_b$.

Taking an arbitrary constant t, if we put $f=t$ in (c) of (1. 1), we have

$$\nabla_{e_i} tX = t\nabla_{e_i} X,$$

which implies together with (1. 3)

$$(1.\,4)_c \qquad t\Gamma_i{}^h(\xi, X)=\Gamma_i{}^h(\xi, tX).$$

The condition $(1.\,4)_b$ shows that $\Gamma_i{}^h$ are functions defined in the open set $\pi^{-1}(U)$ of the tangent bundle $T(M^n)$ and hence we may represent $\Gamma_i{}^h$ as follows

$$\Gamma_i{}^h(\sigma)=\Gamma_i{}^h(\xi^k, v^k),$$

where (ξ^h) are coordinates of the point $p=\pi(\sigma)$ belonging to U and (v^h) are components of the tangent vector v with respect to the natural frame (e_i), π being the projection $T(M^n)\to M^n$. The condition $(1.\,4)_c$ means that the functions $\Gamma_i{}^h(\xi^k, v^k)$ are homogeneous of degree 1 with respect to n-variables v^k. Hereafter in the present paper, we shall denote $\Gamma_i{}^h(\xi^k, v^k)$ simply by $\Gamma_i{}^h(\xi, v)$, which give the value of $\Gamma_i{}^h$ at an element σ of $T(M^n)$, where $\pi(\sigma)=p$. The functions $\Gamma_i{}^h$ thus defined in $\pi^{-1}(U)$ are called *coefficients of the given non-linear connection* ∇ with respect to local coordinates (ξ^h) in U.

Let $U(\xi^i)$ and $U'(\xi^{i'})$ be two intersecting coordinate neighbourhoods of M^n, and $\Gamma_i{}^h(\xi, v)$ and $\Gamma_{i'}{}^{h'}(\xi', v')$ coefficients of a non-linear connection in U and U' respectively. To arbitrary X and Y belonging to $\mathfrak{X}$ there corresponds an element $\nabla_Y X$ of $\mathfrak{X}$, that is,

$$\nabla_Y X= Y^i(\partial_i X^h+\Gamma_i{}^h(\xi, X))e_h= Y^{i'}(\partial_{i'} X^{h'}+\Gamma_{i'}{}^{h'}(\xi', X'))e_{h'},$$

where we have put $X=X^i e_i=X^{i'} e_{i'}$ and $Y=Y^i e_i= Y^{i'} e_{i'}$. Therefore, we have

$$(1.\,4)_d \qquad \Gamma_{i'}{}^{h'}(\xi', X')=\Gamma_i{}^h \frac{\partial \xi^i}{\partial \xi^{i'}} \frac{\partial \xi^{h'}}{\partial \xi^h} - \frac{\partial \xi^i}{\partial \xi^{i'}} \frac{\partial^2 \xi^{h'}}{\partial \xi^i \partial \xi^j} X^j,$$

which is the so-called *transformation law of coefficients* $\Gamma_i{}^h$ *of a non-linear connection* corresponding to the coordinate transformation

$$\xi^{h'}=\xi^{h'}(\xi^1, \cdots, \xi^h)$$

in $U\cap U'$.

Conversely we can show that the functions $\Gamma_i{}^h$ defined in each open set $\pi^{-1}(U)$ determine a non-linear connection ∇ globally in M^n, if $\Gamma_i{}^h$ satisfy the conditions $(1.\,4)_a$, $(1.\,4)_b$, $(1.\,4)_c$ and $(1.\,4)_d$.

We shall conclude this section by showing that a manifold with a non-linear connection in our sense is a general affine space of paths and that any general affine space of paths admits naturally a non-linear connection. Let M^n be an n-dimensional differentiable manifold in which a system of curves called a *system of paths* is given by a system of ordinary differential equations of the form

$$\frac{d^2\xi^h}{dt^2}+\Gamma^h(\xi,\dot{\xi})=0;\qquad \dot{\xi}^h=\frac{d\xi^h}{dt},\tag{1.5}$$

where $\Gamma^h(\xi,\dot{\xi})$ are functions of the $2n$ independent variables ξ^h and $\dot{\xi}^h$, homogeneous of degree 2 with respect to $\dot{\xi}^h$ and t is a scalar parameter determined up to an affine transformation. Such a space is called a *general affine space* of paths [11].

In our manifold with a non-linear connection, we have

$$\nabla_{\dot{\xi}}\dot{\xi}=\left(\frac{d^2\xi^h}{dt^2}+\Gamma_j{}^h(\xi,\dot{\xi})\dot{\xi}^j\right)e_h,$$

where we have put $\dot{\xi}=\dot{\xi}^h e_h$ which is the tangent vector of a curve $\xi^h=\xi^h(t)$ in the manifold. If we put

$$\Gamma^h=\Gamma_j{}^h\dot{\xi}^j,\tag{1.6}$$

our manifold becomes a general affine space of paths defined by differential equation (1.5) above, because the functions Γ^h defined above are homogeneous functions of degree 2 with respect to $\dot{\xi}^h$.

Conversely, if there is given such a system of paths, we put

$$\Gamma_i{}^h(\xi,\eta)=\frac{1}{2}\frac{\partial\Gamma^h}{\partial\eta^i}(\xi,\eta);\qquad \dot{\xi}^j=\eta^j,\tag{1.7}$$

then these functions $\Gamma_i{}^h$ are homogeneous functions of degree 1 with respect to $\dot{\xi}^h$. We see easily that such functions define a non-linear connection ∇ globally in M^n.

§ 2. Vectors and almost complex structure in $T(M^n)$.

Let M^n be an n-dimensional differentiable manifold with a non-linear connection, and $T(M^n)$ its tangent bundle. Let U be a coordinate neighbourhood of M^n and (ξ^h) local coordinates defined in U. Then, the open set $\pi^{-1}(U)$ is a coordinate neighbourhood of $T(M^n)$ and (ξ^h,η^h) are local coordinates in $\pi^{-1}(U)$, π being the bnndle projection: $T(M^n)\to M^n$ where, for a point σ having local coordinates (ξ^h,η^h) in $\pi^{-1}(U)$, the point $p=\pi(\sigma)$ has local coordinates (ξ^h) in U and (η^h) are linear coordinates in the fibre $F_p=\pi^{-1}(p)$ with respect to the natural frame $\partial/\partial\xi^h$.

To the transformation of local coordinates in $U(\xi)\cap U'(\xi')\neq\phi$

$$\xi^{h'}=\xi^{h'}(\xi^1,\cdots,\xi^n),\tag{2.1}$$

there corresponds a transformation of local coordinates in $\pi^{-1}(U)\cap\pi^{-1}(U')\neq\phi$

$$\xi^{h'}=\xi^{h'}(\xi^1, \cdots, \xi^n), \qquad \eta^{h'}=\eta^h \frac{\partial \xi^{h'}}{\partial \xi^h}. \tag{2. 2}$$

If we put

$$\eta^{h'}=\xi^{h^{*'}}=\xi^{(n+h)'}, \qquad \eta^h=\xi^{h^*}=\xi^{n+h},$$

then we may rewrite (2. 2) as

$$\xi^{A'}=\xi^{A'}(\xi^B)=\xi^{A'}(\xi, \xi^*), \tag{2. 3}$$

where $A, B, C=1, 2, \cdots, 2n$. The Jacobian matrix of the transformation (2. 3) is given by

$$\left(\frac{\partial \xi^{A'}}{\partial \xi^A}\right)=\begin{pmatrix} \dfrac{\partial \xi^{h'}}{\partial \xi^h} & 0 \\ \dfrac{\partial^2 \xi^{h'}}{\partial \xi^h \partial \xi^j}\eta^j & \dfrac{\partial \xi^{h'}}{\partial \xi^h} \end{pmatrix}. \tag{2. 4}$$

Because of (2. 4) the transformation of components of an arbitrary vector V at a point σ belonging to $\pi^{-1}(U)\cap\pi^{-1}(U')$ is given by

$$\begin{pmatrix} V^{h'} \\ V^{h^{*'}} \end{pmatrix}=\begin{pmatrix} \dfrac{\partial \xi^{h'}}{\partial \xi^h} V^h \\ \dfrac{\partial^2 \xi^{h'}}{\partial \xi^h \partial \xi^j}\eta^j V^h+\dfrac{\partial \xi^{h'}}{\partial \xi^h} V^{h^*} \end{pmatrix},$$

where we have put $V=V^A\partial_A=V^{A'}\partial_{A'}$ at $\sigma\in\pi^{-1}(U)\cap\pi^{-1}(U')$.

Let v be a vector field on M^n. We may consider the following three vector fields $'X$, $''X$ and $\bar{X}$ of $T(M^n)$:

(2. 5)

(a) $'X$ has the components $('X^A)$ at $\sigma(\xi, \eta)$ of $T(M^n)$, where

$$('X^A)=\begin{pmatrix} v^h(\xi) \\ -\Gamma_j{}^h(\xi, \eta)v^j(\xi) \end{pmatrix},$$

(b) $''X$ has the components $(''X^A)$ at $\sigma(\xi, \eta)$ of $T(M^n)$, where

$$(''X^A)=\begin{pmatrix} 0 \\ v^h(\xi) \end{pmatrix},$$

(c) $\bar{X}$ has the components $(\bar{X}^A)$ at $\sigma(\xi, \eta)$ of $T(M^n)$, where

$$(\bar{X}^A)=\begin{pmatrix} v^h(\xi) \\ \eta^j\partial_j v^h(\xi) \end{pmatrix}.$$

We shall call (a), (b) and (c) the horizontal, vertical and complete lifts of v [12].

Consider the set of values, at a point $\sigma\in T(M^n)$, of horizontal lifts of all vector fields in M^n. Obviously, such a set of vectors determines an n-dimensional subspace H_σ of tangent space at each point σ of $T(M^n)$. Such a subspace H_σ is spanned by a basis (B_i):

$$(B_i{}^A)=\begin{pmatrix}\delta_i^h\\ -\Gamma_i{}^h\end{pmatrix}, \tag{2. 6}$$

where B_i is the horizontal lift of local vector field e_i on M^n. We shall call H_σ the *horizontal plane* at σ.

Consequently we get a distribution

$$H\colon \sigma\rightarrow H_\sigma,$$

and we shall call H the *horizontal plane field* or the *horizontal distribution.*

It is evident that the set of values, at a point $\sigma\in T(M^n)$, of vertical lifts of all vector fields in M^n determines the tangent space of fibre F_p and such subspace $T_\sigma(F_p)$ is spanned by a basis (C_{i*}):

$$(C_{i*}{}^A)=\begin{pmatrix}0\\ \delta_i^h\end{pmatrix}, \tag{2. 7}$$

where C_{i*} is the vertical lift of local vector field e_i on M^n.

Consequently we get a integral distribution $\sigma\rightarrow T_\sigma(F_p)$ which is complementary to the horizontal distribution H.

Thus the tangent space $T_\sigma(T(M^n))$ of $T(M^n)$ at σ is a direct sum:

$$T_\sigma(T(M^n))=H_\sigma+T_\sigma(F_p), \tag{2. 8}$$

where $p=\pi(\sigma)$.

Equation (2. 8) shows that an arbitrary vector field X of $T(M^n)$ can be written uniquely as follows

$$X='X+''X, \tag{2. 9}$$

where $'X_\sigma\in H_\sigma$ and $''X_\sigma\in T_\sigma(F_p)$. A vector field X is said to be horizontal if

$$X='X, \tag{2. 10}$$

and to be vertical if

$$X=''X. \tag{2. 11}$$

By making use of the functions $\Gamma_i{}^h(\xi,\eta)$ defined in each $\pi^{-1}(U)\subset T(M^n)$, which are the coefficients of the given non-linear connection, we may show that such $T(M^n)$ admits an almost complex structure [10]. If we define $(2n)^2$-functions $F_B{}^A(\xi,\eta)$ on $T(M^n)$ as follows:

(2. 12) $$(F_B{}^A(\xi,\eta))=\begin{pmatrix} \Gamma_i{}^h(\xi,\eta) & \delta_i^h \\ -\delta_i^h-\Gamma_i{}^a(\xi,\eta)\Gamma_a{}^h(\xi,\eta) & -\Gamma_i{}^h(\xi,\eta)\end{pmatrix},$$

then we can easily verify that

(2. 13) $$F_B{}^A F_C{}^B=-\delta_C^A.$$

Since we have directly from the above definitions

(2. 14) $$F_B{}^A B_i{}^B=-C_{i^*}{}^A \quad \text{and} \quad F_B{}^A C_{i^*}{}^B=B_i{}^A,$$

we can easily verify that the matrix $(F_B{}^A)$ defined in each neighbourhood $\pi^{-1}(U)$ determines globally a tensor field of type (1, 1) in $T(M^n)$. The tensor field F thus determined by $(F_B{}^A)$ is an almost complex structure in $T(M^n)$, because of (2. 13). Thus we have

THEOREM 2. 1 [10]. *If there is given a non-linear connection in M^n, then there exists an almost complex structure F in the tangent bundle $T(M^n)$ of M^n.*

§ 3. Adapted frame [12].

Let us denote $2n$-vector fields on $\pi^{-1}(U)$

(3. 1) $$(A_\alpha)=(B_i, C_{i^*}) \qquad (\alpha,\beta,\gamma=1,\cdots,2n),$$

where

(3. 2) $$(B_i{}^A)=\begin{pmatrix}\delta_i^h \\ -\Gamma_i{}^h\end{pmatrix}, \qquad (C_{i^*}{}^A)=\begin{pmatrix}0 \\ \delta_i^h\end{pmatrix}.$$

Such a system (A_α) is called the *adapted frame* associated with coordinates (ξ^h) defined in U.

We also denote by

(3. 3) $$(A^\alpha{}_A)=(B^h{}_A, C^{h^*}{}_A)$$

the matrix inverse to the matrix $(B_i{}^A, C_{i^*}{}^A)$. Then we have

(3. 4) $$A_\alpha{}^A A^\alpha{}_B=\delta_B^A, \qquad A_\alpha{}^A A^\beta{}_A=\delta_\alpha^\beta$$

or

(3. 5) $$\begin{cases} B_j{}^A B^j{}_B+C_{j^*}{}^A C^{j^*}{}_B=\delta_B^A, \\ B_j{}^A C^{h^*}{}_A=C_{j^*}{}^A B^h{}_A=0, \\ B_j{}^A B^h{}_A=C_{j^*}{}^A C^{h^*}{}_A=\delta_j^h. \end{cases}$$

Because of (3. 2) and (3. 5) we have

$$(B^h{}_A)=(\delta_i^h, 0), \qquad (C^{h^*}{}_A)=(\Gamma_i{}^h, \delta_i^h) \tag{3. 6}$$

If components of a tangent vector V of $T(M^n)$ at a point σ are V^A, then the components with respect to the adapted frame are $V^\alpha = V^A A^\alpha{}_A$, that is:

$$V=V^A\partial_A = V^\alpha A_\alpha.$$

We see that the components of any horizontal vector field X are given by

$$(X^\alpha)=\begin{pmatrix} X^h \\ 0 \end{pmatrix} \tag{3. 7}$$

with respect to this frame, and those of any vertical vector field X are given by

$$(X^\alpha)=\begin{pmatrix} 0 \\ X^{h^*} \end{pmatrix}. \tag{3. 8}$$

We also see that the components of any complete lift $\bar{X}$ of a vector field v on M^n are given by

$$(\bar{X}^\alpha)=\begin{pmatrix} v^h \\ \eta^j\hat{\nabla}_j v^h \end{pmatrix}, \tag{3. 9}$$

where we have put

$$\hat{\nabla}_j v^h = \partial_j v^h + (\partial_{j^*}\Gamma_a{}^h)v^a.$$

Since the components of the almost complex structure F with respect to the adapted frame is given by

$$F_\beta{}^\alpha = A^\alpha{}_A F_B{}^A A_\beta{}^B,$$

we have from (2. 12), (3. 2) and (3. 6)

$$(F_\beta{}^\alpha)=\begin{pmatrix} 0 & \delta_j^h \\ -\delta_j^h & 0 \end{pmatrix}. \tag{3. 10}$$

Similarly we can find that the so-called Nijenhuis tensor N of the almost complex structure F which is defined in every $\pi^{-1}(U(\xi, \eta))$ by

$$N_{CB}{}^A = F_C{}^D(\partial_D F_B{}^A - \partial_B F_D{}^A) - F_B{}^D(\partial_D F_C{}^A - \partial_C F_D{}^A) \tag{3. 11}$$

has the following components $N_{\gamma\beta}{}^\alpha$ with respect to the adapted frame:

$$\begin{aligned} N_{\gamma\beta}{}^\alpha = {} & F_\gamma{}^\varepsilon(A_\varepsilon F_\beta{}^\alpha - A_\beta F_\varepsilon{}^\alpha) - F_\beta{}^\varepsilon(A_\varepsilon F_\gamma{}^\alpha - A_\gamma F_\varepsilon{}^\alpha) \\ & -\Omega_{\gamma\beta}{}^\alpha - F_\beta{}^\delta F_\nu{}^\alpha \Omega_{\gamma\delta}{}^\nu - F_\gamma{}^\delta F_\nu{}^\alpha \Omega_{\delta\beta}{}^\nu + F_\gamma{}^\delta F_\beta{}^\sigma \Omega_{\delta\sigma}{}^\alpha, \end{aligned} \tag{3. 12}$$

where we have put

(3. 13) $$\Omega_{\gamma\beta}{}^{\alpha}A_{\alpha}=A_{\gamma}A_{\beta}-A_{\beta}A_{\gamma}$$

or

(3. 14) $$\Omega_{\gamma\beta}{}^{\alpha}=A^{\alpha}{}_{A}(A_{\gamma}A_{\beta}{}^{A}-A_{\beta}A_{\gamma}{}^{A}).$$

Finally, if X is a tangent vector field on $T(M^{n})$, then the components of $\underset{X}{\mathcal{L}}F_{C}{}^{B}$ with respect to the adapted frame are given by

(3. 15) $$(\underset{X}{\mathcal{L}}F)_{\beta}{}^{\alpha}=-F_{\beta}{}^{\varepsilon}A_{\varepsilon}X^{\alpha}+F_{\varepsilon}{}^{\alpha}A_{\beta}X^{\varepsilon}+X^{\varepsilon}(\Omega_{\varepsilon\delta}{}^{\alpha}F_{\beta}{}^{\delta}-\Omega_{\varepsilon\beta}{}^{\delta}F_{\delta}{}^{\alpha}).$$

§ 4. Integrability conditions [10, 12].

We shall consider a condition for the horizontal distribution H in $T(M^{n})$ to be integrable. As is well known [7], a necessary and sufficient condition for H to be integrable is that

(4. 1) $$B_{j}B_{i}-B_{i}B_{j}=\Omega_{ji}{}^{h}B_{h}$$

or

(4. 2) $$\Omega_{ji}{}^{h^{*}}=0$$

for the local basis $B_{i}=B_{i}{}^{A}\partial_{A}$ of H. On the other hand, because of (3. 2), (3. 6) and (3. 14) we can show that the non-vanishing components of $\Omega_{\gamma\beta}{}^{\alpha}$ are written as

(4. 3) $$\begin{cases}\Omega_{ji}{}^{h^{*}}=-\Omega_{ij}{}^{h^{*}}=-R_{ji}{}^{h},\\ \Omega_{j^{*}i}{}^{h^{*}}=-\Omega_{ij^{*}}{}^{h^{*}}=-\partial_{j^{*}}\Gamma_{i}{}^{h},\end{cases}$$

where we have put

(4. 4) $$\begin{cases}R_{ji}{}^{h}=\partial_{j}\Gamma_{i}{}^{h}-\partial_{i}\Gamma_{j}{}^{h}-\Gamma_{j}{}^{k}\partial_{k^{*}}\Gamma_{i}{}^{h}+\Gamma_{i}{}^{k}\partial_{k^{*}}\Gamma_{j}{}^{h},\\ \partial_{j}=\partial/\partial\xi^{j} \quad\text{and}\quad \partial_{j^{*}}=\partial/\partial\eta^{j}.\end{cases}$$

Thus we have

THEOREM 4. 1. *A necessary and sufficient condition for the horizontal distribution H in $T(M^{n})$ to be integrable is*

$$R_{ji}{}^{h}=0.$$

An almost complex structure is said to be integrable if the Nijenhuis tensor vanishes identically. Making use of (3. 2), (3. 6), (3. 10), (3. 12) and (4. 3), the components of $N_{\gamma\beta}{}^{\alpha}$ may be written as

(4. 5) $$\begin{cases}N_{ji}{}^{h}=-N_{j^{*}i}{}^{h^{*}}=-N_{ji^{*}}{}^{h^{*}}=-N_{j^{*}i^{*}}{}^{h}=T_{ji}{}^{h},\\ N_{j^{*}i}{}^{h}=N_{ji^{*}}{}^{h}=N_{ji}{}^{h^{*}}=-N_{j^{*}i^{*}}{}^{h^{*}}=R_{ji}{}^{h},\end{cases}$$

where we have put

$$T_{ji}{}^{h}=\partial_{i*}\Gamma_{j}{}^{h}-\partial_{j*}\Gamma_{i}{}^{h}. \tag{4.6}$$

Thus we get

THEOREM 4.2. *A necessary and sufficient condition for the almost complex structure defined by* (2.12) *to be integrable is that*

$$T_{ji}{}^{h}=0 \quad and \quad R_{ji}{}^{h}=0. \tag{4.7}$$

If we put

$$\frac{\partial\Gamma_{j}{}^{h}}{\partial\eta^{i}}=\Gamma_{j}{}^{h}{}_{i} \tag{4.8}$$

then the conditions (4.7) can be replaced by

$$\Gamma_{j}{}^{h}{}_{i}=\Gamma_{i}{}^{h}{}_{j} \quad \text{and} \quad R_{jia}{}^{h}\eta^{a}=0, \tag{4.9}$$

where we have put

$$R_{kji}{}^{h}=(\partial_{k}\Gamma_{j}{}^{h}{}_{i}-\Gamma_{k}{}^{m}\partial_{m*}\Gamma_{j}{}^{h}{}_{i})-(\partial_{j}\Gamma_{k}{}^{h}{}_{i}-\Gamma_{j}{}^{m}\partial_{m*}\Gamma_{k}{}^{h}{}_{i})+\Gamma_{k}{}^{h}{}_{m}\Gamma_{j}{}^{m}{}_{i}-\Gamma_{j}{}^{h}{}_{m}\Gamma_{k}{}^{m}{}_{i}. \tag{4.10}$$

§ 5. Almost analytic vector fields [10, 12].

Since $T(M^{n})$ with a non-linear connection admits an almost complx structure, we may discuss almost analytic vector fields on $T(M^{n})$.

Let X be an arbitrary vector field on $T(M^{n})$. By the use of (3.15), (3.2), (3.6) and (4.3), we have, for various types of components $(\underset{X}{\mathcal{L}}F)_{\beta}{}^{\alpha}$

$$\text{(5.1)} \quad \begin{cases} \text{(a)} & (\underset{X}{\mathcal{L}}F)_{j}{}^{h}=-(\underset{X}{\mathcal{L}}F)_{j*}{}^{h*}=\partial_{j*}X^{h}+\partial_{j}X^{h*}-\Gamma_{j}{}^{a}\partial_{a*}X^{h*}+X^{a*}\partial_{a*}\Gamma_{j}{}^{h}+X^{a}R_{aj}{}^{h}, \\ \text{(b)} & (\underset{X}{\mathcal{L}}F)_{j*}{}^{h}=(\underset{X}{\mathcal{L}}F)_{j}{}^{h*}=-\partial_{j}X^{h}+\Gamma_{j}{}^{a}\partial_{a*}X^{h}-X^{a}\partial_{j*}\Gamma_{a}{}^{h}+\partial_{j*}X^{h*}, \end{cases}$$

or

$$\text{(5.2)} \quad \begin{cases} \text{(a)} & (\underset{X}{\mathcal{L}}F)_{j}{}^{h}=-(\underset{X}{\mathcal{L}}F)_{j*}{}^{h*}=\nabla_{j*}X^{h}+\nabla_{j}X^{h*}+X^{a}R_{aj}{}^{h}, \\ \text{(b)} & (\underset{X}{\mathcal{L}}F)_{j*}{}^{h}=(\underset{X}{\mathcal{L}}F)_{j}{}^{h*}=-\hat{\nabla}_{j}X^{h}+\nabla_{j*}X^{h*}, \end{cases}$$

where we have put

$$\text{(5.3)} \quad \begin{cases} \nabla_{j*}X^{\alpha}=\partial_{j*}X^{\alpha}, \\ \nabla_{j}X^{\alpha}=\partial_{j}X^{\alpha}-\Gamma_{j}{}^{a}\partial_{a*}X^{\alpha}+X^{a}\partial_{a*}\Gamma_{j}{}^{\alpha}, \\ \hat{\nabla}_{j}X^{\alpha}=\partial_{j}X^{\alpha}-\Gamma_{j}{}^{a}\partial_{a*}X^{\alpha}+X^{a}\partial_{j*}\Gamma_{a}{}^{\alpha}. \end{cases}$$

Let X be an arbitrary horizontal vector field on $T(M^{n})$. The components X^{h*} being zero, we have

$$(5.4)\quad \begin{cases}(\underset{X}{\mathcal{L}}F)_j{}^h=-(\underset{X}{\mathcal{L}}F)_{j^*}{}^{h^*}=\partial_{j^*}X^h+X^aR_{aj}{}^h=\nabla_{j^*}X^h+X^aR_{aj}{}^h,\\ (\underset{X}{\mathcal{L}}F)_{j^*}{}^h=(\underset{X}{\mathcal{L}}F)_j{}^{h^*}=-(\partial_jX^h-\Gamma_j{}^a\partial_{a^*}X^h+X^a\partial_{j^*}\Gamma_a{}^h)=-\hat{\nabla}_jX^h.\end{cases}$$

Especially, if X is an arbitrary horizontal lift of a vector field v on M^n, then (5. 1) are replaced by

$$(5.5)\quad \begin{cases}\text{(a)}\quad (\underset{X}{\mathcal{L}}F)_j{}^h=-(\underset{X}{\mathcal{L}}F)_{j^*}{}^{h^*}=v^aR_{aj}{}^h,\\ \text{(b)}\quad (\underset{X}{\mathcal{L}}F)_j{}^{h^*}=(\underset{X}{\mathcal{L}}F)_{j^*}{}^h=-(\partial_jv^h+v^a\partial_{j^*}\Gamma_a{}^h).\end{cases}$$

Let X be an arbitrary vertical vector field on $T(M^n)$. The components X^h being zero, we have

$$(5.6)\quad \begin{cases}\text{(a)}\quad (\underset{X}{\mathcal{L}}F)_j{}^h=-(\underset{X}{\mathcal{L}}F)_{j^*}{}^{h^*}=\partial_jX^{h^*}-\Gamma_j{}^a\partial_{a^*}X^{h^*}+X^{a^*}\partial_{a^*}\Gamma_j{}^h=\nabla_jX^{h^*},\\ \text{(b)}\quad (\underset{X}{\mathcal{L}}F)_{j^*}{}^h=(\underset{X}{\mathcal{L}}F)_j{}^{h^*}=\partial_{j^*}X^{h^*}=\nabla_{j^*}X^{h^*}.\end{cases}$$

Especially, if X is an arbitrary vertical lift of a vector field v on M^n, then (5. 6) are replaced by

$$(5.7)\quad \begin{cases}\text{(a)}\quad (\underset{X}{\mathcal{L}}F)_j{}^h=-(\underset{X}{\mathcal{L}}F)_{j^*}{}^{h^*}=\partial_jv^h+v^a\partial_{a^*}\Gamma_j{}^h,\\ \text{(b)}\quad (\underset{X}{\mathcal{L}}F)_j{}^{h^*}=(\underset{X}{\mathcal{L}}F)_{j^*}{}^h=0.\end{cases}$$

Thus, we have from (5. 4) and (5. 5)

THEOREM 5. 1. *In a tangent bundle with a non-linear connection a horizontal vector field X is almost analytic if and only if*

$$(5.8)\quad \hat{\nabla}_jX^h=0 \quad \textit{and} \quad \nabla_{j^*}X^h+X^kR_{kj}{}^h=0.$$

Especially, if X is a horizontal lift of a vector field v on M^n, then (5. 8) *is replaced by*

$$(5.9)\quad \hat{\nabla}_jv^h=\partial_jv^h+\Gamma_a{}^h{}_jv^a=0 \quad \textit{and} \quad v^kR_{kj}{}^h=0,$$

where

$$\Gamma_j{}^h{}_i=\frac{\partial\Gamma_j{}^h}{\partial\eta^i}.$$

We have from (5. 6) and (5. 7)

THEOREM 5. 2. *In a tangent bundle with a non-linear connection a vertical vector field X is almost analytic if and only if*

$$(5.10)\quad \nabla_jX^{h^*}=0 \quad \textit{and} \quad \nabla_{j^*}X^{h^*}=0.$$

Especially, if X is a vertical lift of a vector field v on M^n, then (5. 8) *is replaced by*

(5. 11) $$\nabla_j v^h = \partial_j v^h + \Gamma_j{}^h{}_a v^a = 0.$$

Let X be an arbitrary complete lift of a vector field v on M^n, then we have from (3. 9) and (5. 1)

(5. 12) $$\begin{cases} \text{(a)} \quad (\underset{X}{\mathcal{L}} F)_j{}^h = -(\underset{X}{\mathcal{L}} F)_{j^*}{}^{h^*} = (\underset{v}{\mathcal{L}} \Gamma_j{}^h{}_i)\eta^i, \\ \text{(b)} \quad (\underset{X}{\mathcal{L}} F)_{j^*}{}^h = (\underset{X}{\mathcal{L}} F)_j{}^{h^*} = 0, \end{cases}$$

where we have put

(5. 13) $$\underset{v}{\mathcal{L}} \Gamma_j{}^h{}_i = \nabla_j \hat{\nabla}_i v^h + v^a R_{aji}{}^h + \eta^a (\nabla_a v^k) \partial_{k^*} \Gamma_j{}^h{}_i,$$
$$\nabla_j v^h = \partial_j v^h + \Gamma_j{}^h{}_a v^a, \qquad \hat{\nabla}_j v^h = \partial_j v^h + \Gamma_a{}^h{}_j v^a \quad \text{and} \quad \Gamma_j{}^h{}_i = \frac{\partial \Gamma_j{}^h}{\partial \eta^i} \quad [11].$$

Thus, we get from (5. 12)

THEOREM 5. 3. *In a tangent bundle $T(M^n)$ with a non-linear connection, a complete lift X of a vector field v on M^n is almost analytic if and only if*

$$(\underset{v}{\mathcal{L}} \Gamma_j{}^h{}_i)\eta^i = 0.$$

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