ON MARKOV CHAINS WITH REWARDS

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1. Let $X_0, X_1, X_2, \cdots$ be a Markov chain with the state space $\boldsymbol{S}=\{1, 2, \cdots, N\}$ and the transition probability matrix $P=(p_{ij})$, which earns r_{ij} dollars when it makes a transition from state i to state j. We call r_{ij} the "reward" associated with the transition from i to j. Let us define $v_i(n)$ as the expectation of the total earnings $R(n)$ in the next n transition if the system is now in state i. Howard [1] has given the recurrence relation

$$(1) \qquad v_i(n)=\sum_{j=1}^{N} p_{ij}r_{ij}+\sum_{j=1}^{N} p_{ij}v_j(n-1) \qquad (i=1, 2, \cdots, N;\ n=1, 2, \cdots)$$

and the asymptotic form that $v_i(n)$ assumes for large n. For practical applications, it is desirable to find the variance and the asymptotic behavior of $R(n)$. And it is also intersting from the theorical view-point. In this paper, we shall try to find the asymptotic form that $\mathrm{Var}\,(R(n))$ assumes for large n and to give a proof of the limit theorem for $R(n)$ as $n\to\infty$, which is essentially equivalent to the limit theorem for finite regular Markov chains.

2. Since, by the definition of Markov chains, we have

$$\begin{aligned}
&P\{X_1=i_1, X_2=i_2, \cdots, X_n=i_n \mid X_0=i\}\\
&=P\{X_1=i_1 \mid X_0=i\}P\{X_2=i_2 \mid X_1=i_1, X_0=i\}\cdots P\{X_n=i_n \mid X_{n-1}=i_{n-1}, \cdots, X_0=i\}\\
&=p_{ii_1}p_{i_1i_2}\cdots p_{i_{n-1}i_n},
\end{aligned}$$

the moment generating function of the probability distribution of $R(n)$ under the condition $X_0=i$ is equal to

$$(2) \qquad \begin{aligned}
\phi_{in}(\theta)&\overset{\text{def}}{=}E\{e^{\theta R(n)} \mid X_0=i\}\\
&=\sum_{i_1,\cdots,i_n} e^{\theta(r_{ii_1}+r_{i_1i_2}+\cdots+r_{i_{n-1}i_n})}P\{X_1=i_1, \cdots, X_n=i_n \mid X_0=i\}\\
&=\sum_{i_n}\sum_{i_1,\cdots,i_{n-1}} e^{\theta r_{ii_1}}p_{ii_1}e^{\theta r_{i_1i_2}}p_{i_1i_2}\cdots e^{\theta r_{i_{n-1}i_n}}p_{i_{n-1}i_n}.
\end{aligned}$$

Introducing the $N\times N$ matrix $\Pi(\theta)\overset{\text{def}}{=}(e^{\theta r_{ij}}p_{ij})$ with elements $e^{\theta r_{ij}}p_{ij}$ and two N-dimentional vectors

$$\boldsymbol{\phi}_n(\theta)=\begin{bmatrix}\phi_{1n}(\theta)\\ \phi_{2n}(\theta)\\ \vdots\\ \phi_{Nn}(\theta)\end{bmatrix} \quad \text{and} \quad \mathbf{e}=\begin{bmatrix}1\\ 1\\ \vdots\\ 1\end{bmatrix},$$

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we have from (2) that $\boldsymbol{\phi}_n(\theta)=\Pi(\theta)^n\mathbf{e}$ or

$$\boldsymbol{\phi}_n(\theta)=\Pi(\theta)\boldsymbol{\phi}_{n-1}(\theta) \qquad (n=1, 2, 3, \cdots) \tag{3}$$

where $\boldsymbol{\phi}_0(\theta)\equiv\mathbf{e}$, which implies

$$\boldsymbol{\phi}_n'(\theta)=\Pi'(\theta)\boldsymbol{\phi}_{n-1}(\theta)+\Pi(\theta)\boldsymbol{\phi}_{n-1}'(\theta). \tag{4}$$

Moreover, introducing the $N\times N$ matrix $Q_1=(r_{ij}p_{ij})$ with elements $r_{ij}p_{ij}$ and the vectors

$$\mathbf{v}(n)\overset{\text{def}}{=}\begin{bmatrix} v_1(n) \\ \vdots \\ v_N(n) \end{bmatrix}$$

with the components $v_i(n)\overset{\text{def}}{=}E\{R(n)\,|\,X_0=i\}=\phi_{in}'(0)$, we have by setting $\theta=0$ in (4) that

$$\mathbf{v}(n)=Q_1\mathbf{e}+P\mathbf{v}(n-1), \tag{5}$$

because $\boldsymbol{\phi}_{n-1}(0)=\mathbf{e}$, $\boldsymbol{\phi}_n'(0)=\mathbf{v}(n)$, $\Pi(0)=P$ and $\Pi'(0)=Q_1$. (5) is equivalent to (1). Now, as the preparation for finding the asymptotic formula for $\mathrm{Var}\,(R(n))$ as $n\to\infty$, in what follows, we shall outline Howard's method giving the asymptotic form for $\mathbf{v}(n)$. Since we have $|v_i(n)|\leqq nr$ $(n=1, 2, \cdots)$ where $r=\mathrm{Max}_{i,j=1,2,\cdots,N}|r_{ij}|$, the infinite series $\sum_{n=0}^{\infty} v_i(n)z^n$ is absolutely convergent for $|z|<1$. Hence the so-called z-transform $\boldsymbol{v}(z)=\sum_{n=0}^{\infty}\mathbf{v}(n)z^n$ of the sequence $\{\mathbf{v}(n)\}$ can be defined in $|z|<1$. $\mathbf{v}(0)$ being the zero vector, we get from (5)

$$\frac{1}{z}\,\boldsymbol{v}(z)=\frac{1}{1-z}\,Q_1\mathbf{e}+P\boldsymbol{v}(z)$$

or

$$[I-zP]\boldsymbol{v}(z)=\frac{z}{1-z}\,Q_1\mathbf{e},$$

which implies

$$\boldsymbol{v}(z)=\frac{z}{1-z}[I-zP]^{-1}Q_1\mathbf{e}, \tag{6}$$

because $\det(I-zP)\neq0$ in a neighborhood of $z=0$. We assume that $X_0, X_1, X_2, \cdots$ is regular or mixing, that is, there exists a natural number m such that every element of the matrix P^m is strictly positive. Then, we have $z=1$ as a simple root of the equation $\det(I-zP)=0$. And, when the remaining roots $\alpha_1, \alpha_2, \cdots, \alpha_k$ of $\det(I-zP)=0$ exist, it is known that $|\alpha_l|>1$ $(l=1, 2, \cdots, k)$. $m_1, m_2, \cdots, m_k$ being the multiplicities of $\alpha_1, \alpha_2, \cdots, \alpha_k$ respectively, we have

$$\det(I-zP)=(1-z)\left(1-\frac{z}{\alpha_1}\right)^{m_1}\cdots\left(1-\frac{z}{\alpha_k}\right)^{m_k},$$

so that, by partial-fraction expansion, it is obtained that every element of $[I-zP]^{-1}$ equals the sum of fractions, whose forms are const./$(1-z)$ and const./$(1-z/\alpha_l)^{\nu}$

($l=1, 2, \cdots, k$; $\nu=1, 2, \cdots, m_l$), plus a polynomial of z. Expressing this fact in the matrix-form, we have

$$
\begin{aligned}
[I-zP]^{-1} &= \frac{1}{1-z} S + \sum_{l=1}^{k} \sum_{\nu=1}^{m_l} \frac{1}{(1-z/\alpha_l)^{\nu}} T_{l\nu} + \sum_{n=0}^{\kappa} T_n z^n \\
&= \frac{1}{1-z} S + \mathcal{T}(z), \qquad \text{say,}
\end{aligned}
$$

where S, $T_{l\nu}$, T_n are $N\times N$ matrices independent of z. Therefore, we have that

$$
\begin{aligned}
[I-zP]^{-1} &= \sum_{n=0}^{\infty} \left\{ S + \sum_{l=1}^{k} \sum_{\nu=1}^{m_l} \frac{(n+\nu-1)(n+\nu-2)\cdots(n+1)}{(\nu-1)!\alpha_l^{n}} T_{l\nu} \right\} z^n + \sum_{n=0}^{\kappa} T_n z^n \\
&= \sum_{n=0}^{\infty} (S+H(n)) z^n, \qquad \text{say,}
\end{aligned}
$$

for a neighborhood of $z=0$ and so $H(n)$ tends to the zero matrix as $n\to\infty$. Analogous considerations give

$$
\begin{aligned}
\frac{z}{1-z}[I-zP]^{-1} &= \frac{z}{(1-z)^2} S + \frac{z}{1-z} \mathcal{T}(z) \\
&= \frac{z}{(1-z)^2} S + \frac{1}{1-z} \mathcal{T}(1) + \mathcal{T}_1(z),
\end{aligned}
$$

where $\mathcal{T}_1(z)$ is a matrix-valued expression similar to $\mathcal{T}(z)$, so that we have form (6)

$$
\begin{aligned}
\boldsymbol{v}(z) &= \sum_{n=0}^{\infty} \mathbf{v}(n) z^n \\
&= \frac{z}{(1-z)^2} SQ_1\mathbf{e} + \frac{1}{1-z} \mathcal{T}(1)Q_1\mathbf{e} + \mathcal{T}_1(z)Q_1\mathbf{e} \qquad (7) \\
&= \sum_{n=0}^{\infty} \{ nSQ_1\mathbf{e} + \ \ (1)Q_1\mathbf{e} + H_1(n)Q_1\mathbf{e} \} z^n,
\end{aligned}
$$

where $H_1(n)$ tends to the zero matrix as $n\to\infty$. Comparing the coefficients of z^n in (7), we get

$$
\begin{aligned}
\mathbf{v}(n) &= nSQ_1\mathbf{e} + \mathcal{T}(1)Q_1\mathbf{e} + H_1(n)Q_1\mathbf{e} \\
&\fallingdotseq nSQ_1\mathbf{e} + \mathcal{T}(1)Q_1\mathbf{e} \qquad \text{as } n\to\infty,
\end{aligned}
\tag{8}
$$

which has been shown by Howard [1]. By using the method mentioned above, we shall give an asymptotic formula of Var$(R(n))$ for large n. By differentiating (4), we have

$$
\boldsymbol{\phi}_n''(\theta) = \Pi''(\theta)\boldsymbol{\phi}_{n-1}(\theta) + 2\Pi'(\theta)\boldsymbol{\phi}_{n-1}'(\theta) + \Pi(\theta)\boldsymbol{\phi}_{n-1}''(\theta). \tag{9}
$$

Introducing the $N\times N$ matrix $Q_2 \overset{\text{def}}{=} (r_{ij}^2 p_{ij})$ with elements $r_{ij}^2 p_{ij}$ and N-dimensional vectors

$$
\mathbf{w}(n) \overset{\text{def}}{=} \begin{bmatrix} w_1(n) \\ \vdots \\ w_N(n) \end{bmatrix}
$$

with the components $w_i(n) \overset{\text{def}}{=} E\{R(n)^2 \mid X_0=i\} = \phi_{in}''(0)$, we have, by setting $\theta=0$ in (9),

$$\mathbf{w}(n)=Q_2\mathbf{e}+2Q_1\mathbf{v}(n-1)+P\mathbf{w}(n-1). \tag{10}$$

Since $\mathbf{w}(0)$ is the zero vector and $\boldsymbol{w}(z) \overset{\text{def}}{=} \sum_{n=0}^{\infty} \mathbf{w}(n)z^n$ can be defined for $|z|<1$, we get from (10)

$$\frac{1}{z}\boldsymbol{w}(z)=\frac{1}{1-z}Q_2\mathbf{e}+2Q_1\boldsymbol{v}(z)+P\boldsymbol{w}(z)$$

so that

$$\begin{aligned}
\boldsymbol{w}(z)&=\frac{z}{1-z}[I-zP]^{-1}Q_2\mathbf{e}+2z[I-zP]^{-1}Q_1\boldsymbol{v}(z)\\
&=\left\{\frac{z}{(1-z)^2}S+\frac{1}{1-z}\mathfrak{T}(1)+\mathfrak{T}_1(z)\right\}Q_2\mathbf{e}\\
&\quad+2z\left\{\frac{1}{1-z}S+\mathfrak{T}(z)\right\}Q_1\left\{\frac{z}{(1-z)^2}SQ_1+\frac{1}{1-z}\mathfrak{T}(1)Q_1+\mathfrak{T}_1(z)Q_1\right\}\mathbf{e}\\
&=\frac{2z^2}{(1-z)^3}SQ_1SQ_1\mathbf{e}+\frac{z}{(1-z)^2}\{SQ_2+2SQ_1\mathfrak{T}(1)Q_1+2\mathfrak{T}(1)Q_1SQ_1\}\mathbf{e}\\
&\quad+\frac{1}{1-z}\{\mathfrak{T}(1)Q_2+2SQ_1\mathfrak{T}_1(1)Q_1-2\mathfrak{T}'(1)Q_1SQ_1-2\mathfrak{T}(1)Q_1SQ_1+2\mathfrak{T}(1)Q_1\mathfrak{T}(1)Q_1\}\mathbf{e}\\
&\quad+\mathfrak{T}_2(z),
\end{aligned}$$

where $\mathfrak{T}_2(z)$ is a matrix-valued expression similar to $\mathfrak{T}(z)$, which implies

$$\mathbf{w}(n) \doteqdot n^2SQ_1SQ_1\mathbf{e}+n\mathbf{w}_1+\mathbf{w}_2 \qquad \text{as } n\to\infty \tag{11}$$

where

$$\mathbf{w}_1=\{SQ_2+2SQ_1\mathfrak{T}(1)Q_1+2\mathfrak{T}(1)Q_1SQ_1-SQ_1SQ_1\}\mathbf{e}$$

and

$$\mathbf{w}_2=\{\mathfrak{T}(1)Q_2+2SQ_1\mathfrak{T}_1(1)Q_1-2\mathfrak{T}'(1)Q_1SQ_1-2\mathfrak{T}(1)Q_1SQ_1+2\mathfrak{T}(1)Q_1\mathfrak{T}(1)Q_1\}\mathbf{e}.$$

Now, the limit distribution of our regular Markov chain being $\{\pi_1, \pi_2, \cdots, \pi_N\}$, it is known that

$$S=\begin{pmatrix}\pi_1 & \pi_2 & \cdots & \pi_N\\ \pi_1 & \pi_2 & \cdots & \pi_N\\ \cdot & \cdot & \cdots & \cdot\\ \pi_1 & \pi_2 & \cdots & \pi_N\end{pmatrix},$$

from which we get that all components of vector $SQ_1\mathbf{e}$ are equal to $g \overset{\text{def}}{=} \sum_{j,k}\pi_j r_{jk} p_{jk}$. Hence we have from (8)

$$v_i(n) \doteqdot ng+v_i, \tag{12}$$

where v_i is the component of the vector $\mathbf{v}=\mathfrak{T}(1)Q_1\mathbf{e}$. Similarly, all components of vector $SQ_1SQ_1\mathbf{e}$ are equal to

$$\sum_{j,k,l,m} \pi_j r_{jk} p_{jk} \pi_l r_{lm} p_{lm} = \Big(\sum_{j,k} \pi_j r_{jk} p_{jk}\Big)^2 = g^2,$$

so that we have from (11)

$$w_i(n) \doteqdot n^2 g^2 + n w_{1i} + w_{2i} \qquad \text{as } n \to \infty, \tag{13}$$

where w_{1i} and w_{2i} are the components of $\mathbf{w}_1$ and $\mathbf{w}_2$ respectively. By using the initial probability distribution $\pi_i(0) \overset{\text{def}}{=} P\{X_0 = i\}$, it follows from (12) and (13) that

$$E\{R(n)\} \doteqdot ng + v \quad \text{and} \quad E\{R(n)^2\} \doteqdot n^2 g^2 + n w_1 + w_2, \tag{14}$$

where

$$v = \sum_{i=1}^{N} \pi_i(0) v_i, \qquad w_1 = \sum_{i=1}^{N} \pi_i(0) w_{1i} \qquad \text{and} \qquad w_2 = \sum_{i=1}^{N} \pi_i(0) w_{2i},$$

so that *we have*

$$\operatorname{Var}(R(n)) \doteqdot n(w_1 - 2gv) + (w_2 - v^2). \tag{15}$$

Our method is applicable to the moments of higher order of $R(n)$.

3. To simplify the explanation of our method in this section, we assume in addition to the conditions of the preceding section that the equation $\det(I - zP) = 0$ has the simple roots $1, \alpha_1, \cdots, \alpha_{N-1}$, that is, $k = N-1$ and $m_1 = \cdots = m_{N-1} = 1$. Since we have for any fixed θ that

$$0 \leqq \phi_{in}(\theta) \leqq e^{nr\theta},$$

where $r = \operatorname{Max}_{i,j=1,2,\cdots,N} |r_{ij}|$, $\boldsymbol{\Phi}(\theta, z) \overset{\text{def}}{=} \sum_{n=0}^{\infty} \boldsymbol{\phi}_n(\theta) z^n$ can be defined for $|z| < e^{-r\theta}$. From (3), we get

$$\frac{1}{z}(\boldsymbol{\Phi}(\theta, z) - \mathbf{e}) = \Pi(\theta) \boldsymbol{\Phi}(\theta, z) \tag{16}$$

or

$$\boldsymbol{\Phi}(\theta, z) = [I - z\Pi(\theta)]^{-1} \mathbf{e}. \tag{17}$$

The equation $\det(I - z\Pi(\theta)) = 0$ has N roots $\zeta_0(\theta), \zeta_1(\theta), \cdots, \zeta_{N-1}(\theta)$, which are analytic in a neighborhood of $\theta = 0$ and satisfy that

$$\zeta_0(\theta) \to 1, \ \zeta_1(\theta) \to \alpha_1, \ \cdots, \ \zeta_{N-1}(\theta) \to \alpha_{N-1}$$

as $\theta \to 0$. Then, there exist positive numbers ε and θ_0 such that

$$|\zeta_0(\theta)| < 1 + \varepsilon < 1 + 2\varepsilon < |\zeta_l(\theta)| \quad \text{for } |\theta| < \theta_0 \qquad (l = 1, 2, \cdots, N-1), \tag{18}$$

because $|\alpha_l| > 1$ $(l = 1, 2, \cdots, N-1)$. The consideration similar to the one in the preceding section gives

$$\boldsymbol{\Phi}(\theta, z) = \frac{\boldsymbol{\sigma}(\theta)}{1 - z/\zeta_0(\theta)} + \frac{\boldsymbol{\tau}_1(\theta)}{1 - z/\zeta_1(\theta)} + \cdots + \frac{\boldsymbol{\tau}_{N-1}(\theta)}{1 - z/\zeta_{N-1}(\theta)}, \tag{19}$$

where $\boldsymbol{\sigma}(\theta), \boldsymbol{\tau}_1(\theta), \cdots, \boldsymbol{\tau}_{N-1}(\theta)$ are N-dimensional vectors analytic on θ for $|\theta| < \theta_0$, and

$$(20)\quad \begin{aligned}\boldsymbol{\phi}_n(\theta)&=\frac{1}{\zeta_0(\theta)^n}\boldsymbol{\sigma}(\theta)+\frac{1}{\zeta_1(\theta)^n}\boldsymbol{\tau}_1(\theta)+\cdots+\frac{1}{\zeta_{N-1}(\theta)^n}\boldsymbol{\tau}_{N-1}(\theta)\\ &=\frac{1}{\zeta_0(\theta)^n}\left[\boldsymbol{\sigma}(\theta)+\left(\frac{\zeta_0(\theta)}{\zeta_1(\theta)}\right)^n\boldsymbol{\tau}_1(\theta)+\cdots+\left(\frac{\zeta_0(\theta)}{\zeta_{N-1}(\theta)}\right)^n\boldsymbol{\tau}_{N-1}(\theta)\right],\end{aligned}$$

which implies

$$(21)\quad \boldsymbol{\phi}_n'(\theta)=\frac{-n\zeta_0'(\theta)\boldsymbol{\sigma}(\theta)}{\zeta_0(\theta)^{n+1}}+\frac{\boldsymbol{\sigma}'(\theta)}{\zeta_0(\theta)^n}+\sum_{l=1}^{N-1}\left\{\frac{-n\zeta_l'(\theta)\boldsymbol{\tau}_l(\theta)}{\zeta_l(\theta)^{n+1}}+\frac{\boldsymbol{\tau}_l'(\theta)}{\zeta_l(\theta)^n}\right\}$$

and so

$$(22)\quad \begin{aligned}\mathbf{v}(n)=\boldsymbol{\phi}_n'(\theta)&=-n\zeta_0'(0)\boldsymbol{\sigma}(0)+\boldsymbol{\sigma}'(0)+\sum_{l=1}^{N-1}\left\{-\frac{n\zeta_l'(0)}{\alpha_l^{n+1}}\boldsymbol{\tau}_l(0)+\frac{1}{\alpha_l^n}\boldsymbol{\tau}_l'(0)\right\}\\ &\fallingdotseq-n\zeta_0'(0)\boldsymbol{\sigma}(0)+\boldsymbol{\sigma}'(0).\end{aligned}$$

This result shows with (8)

$$(23)\quad -\zeta_0'(0)\boldsymbol{\sigma}(0)=SQ_1\mathbf{e}=g\mathbf{e}.$$

Similarly, we get

$$\begin{aligned}\mathbf{w}(n)=\boldsymbol{\phi}_n''(0)&\fallingdotseq n(n+1)\zeta_0'(0)^2\boldsymbol{\sigma}(0)-2n\zeta_0'(0)\boldsymbol{\sigma}'(0)\\ &\quad-n\zeta_0''(0)\boldsymbol{\sigma}(0)+\boldsymbol{\sigma}''(0).\end{aligned}$$

Since $\boldsymbol{\Phi}(0,z)=(1-z)^{-1}\mathbf{e}$, we have

$$(24)\quad \boldsymbol{\sigma}(0)=\mathbf{e},$$

which implies with (23) that

$$(25)\quad -\zeta_0'(0)=g.$$

Now, we shall consider the asymptotic behavior of $R(n)$ as $n\to\infty$. The moment generating function of the randon variable $[R(n)-ng]/\sqrt{n}$ under the condition $X_0=i$ is

$$\psi_{in}(\theta)\overset{\text{def}}{=}E\{e^{\theta[R(n)-ng]/\sqrt{n}}|X_0=i\}=e^{-\sqrt{n}g\theta}\phi_{in}\left(\frac{\theta}{\sqrt{n}}\right).$$

Hence, introducing the N-dimensional vector

$$\boldsymbol{\psi}_n(\theta)\overset{\text{def}}{=}\begin{bmatrix}\psi_{1n}(\theta)\\ \vdots\\ \psi_{Nn}(\theta)\end{bmatrix},$$

we have by (20)

$$(26)\quad \begin{aligned}\boldsymbol{\psi}_n(\theta)&=e^{-\sqrt{n}g\theta}\boldsymbol{\phi}_n\left(\frac{\theta}{\sqrt{n}}\right)\\ &=\frac{1}{\{e^{(g/\sqrt{n})\theta}\zeta_0(\theta/\sqrt{n})\}^n}\left\{\boldsymbol{\sigma}\left(\frac{\theta}{\sqrt{n}}\right)+\left(\frac{\zeta_0(\theta/\sqrt{n})}{\zeta_1(\theta/\sqrt{n})}\right)^n\boldsymbol{\tau}_1\left(\frac{\theta}{\sqrt{n}}\right)\right.\\ &\qquad\left.+\cdots+\left(\frac{\zeta_0(\theta/\sqrt{n})}{\zeta_{N-1}(\theta/\sqrt{n})}\right)^n\boldsymbol{\tau}_{N-1}\left(\frac{\theta}{\sqrt{n}}\right)\right\}.\end{aligned}$$

For any fixed θ, we have $|\theta/\sqrt{n}|<\theta_0$ for all sufficiantly large n so that by (18)

$$\left|\frac{\zeta_0(\theta/\sqrt{n})}{\zeta_l(\theta/\sqrt{n})}\right|<\frac{1+\varepsilon}{1+2\varepsilon}<1 \qquad (l=1,2,\cdots,N-1)$$

and

$$\left(\frac{\zeta_0(\theta/\sqrt{n})}{\zeta_l(\theta/\sqrt{n})}\right)^n\to 0 \quad \text{as } n\to\infty \qquad (l=1,2,\cdots,N-1). \tag{27}$$

On the other hand, we have

$$\begin{aligned} e^{(g/\sqrt{n})\theta}\zeta_0\left(\frac{\theta}{\sqrt{n}}\right)&=\left(1+\frac{g}{\sqrt{n}}\theta+\frac{g^2}{2n}\theta^2+\cdots\right)\left(1-\frac{g}{\sqrt{n}}\theta+\frac{\zeta_0''(0)}{2n}\theta^2+\cdots\right)\\ &=1-\frac{1}{2n}(g^2-\zeta_0''(0))\theta^2+\cdots \end{aligned}$$

and so

$$\left\{e^{(g/\sqrt{n})\theta}\zeta_0\left(\frac{\theta}{\sqrt{n}}\right)\right\}^n\to e^{-[g^2-\zeta_0''(0)]\theta^2/2} \qquad \text{as } n\to\infty. \tag{28}$$

From (26), (27) and (28), we have

$$\boldsymbol{\phi}_n(\theta)\to e^{[g^2-\zeta_0''(0)]\theta^2/2}\,\boldsymbol{\sigma}(0)=e^{[g^2-\zeta_0''(0)]\theta^2/2}\,\mathbf{e}$$

or

$$\begin{bmatrix}\phi_{1n}(\theta)\\ \vdots\\ \phi_{Nn}(\theta)\end{bmatrix}\to e^{[g^2-\zeta_0''(0)]\theta^2/2}\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix} \qquad \text{as } n\to\infty.$$

Therefore, we get

$$\begin{aligned} \phi_n(\theta)&\overset{\text{def}}{=}E\{e^{\theta[R(n)-ng]/\sqrt{n}}\}\\ &=\sum_{i=1}^{N}\pi_i(0)\phi_{in}(\theta)\to e^{[g^2-\zeta_0''(0)]\theta^2/2} \qquad \text{as } n\to\infty, \end{aligned}$$

which implies that $[R(n)-ng]/\sqrt{n}$ *converges in distribution to the normal distribution* $\mathrm{N}(0, g^2-\zeta_0''(0))$.

REMARK 1. We shall try to find $\zeta_0'(0)=-g$ and $\zeta_0''(0)$ from the equation

$$\det(I-z\Pi(\theta))\equiv a(\theta)z^N+b(\theta)z^{N-1}+\cdots+l(\theta)=0.$$

Since $\zeta_0(\theta)$ is a root of $a(\theta)z^N+b(\theta)z^{N-1}+\cdots+l(\theta)=0$ for a neighborhood of $\theta=0$, we have

$$\begin{aligned} &(a(0)+a'(0)\theta+\cdots)(1+\zeta_0'(0)\theta+\cdots)^N\\ &\quad+(b(0)+b'(0)\theta+\cdots)(1+\zeta_0'(0)\theta+\cdots)^{N-1}\\ &\quad+\cdots\\ &\quad+(l(0)+l'(0)\theta+\cdots)=0 \end{aligned}$$

so that we get

$$\zeta_0'(0) = -\frac{a'(0)+b'(0)+\cdots+l'(0)}{Na(0)+(N-1)b(0)+\cdots+0\cdot l(0)}, \tag{29}$$

whose denominator is never equal to zero, because $z=1$ is a simple root of the equation

$$\det(I-zP) = a(0)z^N + b(0)z^{N-1} + \cdots + l(0) = 0.$$

In the similar way, we have

$$\zeta_0''(0) = -\frac{\{N(N-1)a(0)+(N-1)(N-2)b(0)+\cdots\}\cdot\zeta_0'(0)^2 + 2\{Na'(0) + (N-1)b'(0)+\cdots\}\cdot\zeta_0'(0) + \{a''(0)+b''(0)+\cdots\}}{Na(0)+(N-1)b(0)+\cdots}. \tag{30}$$

REMARK 2. Let f be any real-valued function difined on $\boldsymbol{S}$. In the case where $r_{ij}=f(i)$, we have that

$$R(n) = f(X_0)+f(X_1)+\cdots+f(X_{n-1}),$$
$$g = \sum_{i,j} \pi_i f(i) p_{ij} = \sum_i \pi_i f(i)$$

and the random variable $[f(X_0)+f(X_1)+\cdots+f(X_{n-1})-ng]/\sqrt{n}$ converges in distribution to a normal distribution as $n\to\infty$. Therefore we have the central limit theorem for finite regular Markov chain $X_0, X_1, \cdots$. Conversely, we can derive our result in this section from the central limit theorem. Introducing the function f defined on $\boldsymbol{S}\times\boldsymbol{S}$ by $f(i,j)=r_{ij}$, we have

$$R(n) = f(X_0, X_1)+f(X_1, X_2)+\cdots+f(X_{n-1}, X_n).$$

Since (X_0, X_1), (X_1, X_2), $\cdots$ is a finite regular Markov chain by taking away the unessential states, we can conclude that from the central limit theorem for Markov chain, $[R(n)-E\{R(n)\}]/\sqrt{n}$ converges in distribution to normal distribution with mean zero, which is essentially equivalent to our result in this section. Therefore, we can state that an alternative proof of the central limit theorem for finite regular Markov chains has been given in this section.

4. In this section, we shall try to apply our method in section 2 to the case with discounting. β being the discount factor, we have

$$\begin{aligned}\phi_{in}(\theta) &\overset{\text{def}}{=} E\{e^{\theta R(n)} \mid X_0=i\} \\ &= \sum_{i_1,\cdots,i_n} e^{\theta(r_{ii_1}+\beta r_{i_1i_2}+\cdots+\beta^{n-1}r_{i_{n-1}i_n})} p_{ii_1}p_{i_1i_2}\cdots p_{i_{n-1}i_n},\end{aligned} \tag{31}$$

where $R(n)$ is the present value of the total reward for a system in state i with n transitions. This shows

$$\boldsymbol{\phi}_n(\theta) = \Pi(\theta)\Pi(\beta\theta)\cdots\Pi(\beta^{n-1}\theta)\mathbf{e}$$

or

$$\boldsymbol{\phi}_n(\theta) = \Pi(\theta)\boldsymbol{\phi}_{n-1}(\beta\theta), \tag{32}$$

which implies

$$\mathbf{v}(n)=\boldsymbol{\phi}_n'(0)=\Pi'(0)\boldsymbol{\phi}_{n-1}(0)+\beta\Pi(0)\boldsymbol{\phi}_{n-1}'(0)=Q_1\mathbf{e}+\beta P\mathbf{v}(n-1) \tag{33}$$

and

$$\begin{aligned} \boldsymbol{v}(z)&=\frac{z}{1-z}[I-\beta zP]^{-1}Q_1\mathbf{e} \\ &=\frac{z}{1-z}\left\{\frac{1}{1-\beta z}S+\mathfrak{T}(\beta z)\right\}Q_1\mathbf{e} \\ &=\frac{1}{1-z}\left\{\frac{1}{1-\beta}S+\mathfrak{T}(\beta)\right\}Q_1\mathbf{e}+\mathfrak{T}_3(z)\mathbf{e}, \end{aligned} \tag{34}$$

where $\mathfrak{T}_3(z)$ is a matrix-valued expression similar to $\mathfrak{T}(z)$ in section 2. Therefore we get

$$\begin{aligned} \mathbf{v}(n)&\fallingdotseq\left\{\frac{1}{1-\beta}S+\mathfrak{T}(\beta)\right\}Q_1\mathbf{e} \\ &=[I-\beta P]^{-1}Q_1\mathbf{e}, \end{aligned} \tag{35}$$

which has been given by Howard [1].

In the similar way, we have

$$\begin{aligned} \mathbf{w}(n)&\fallingdotseq\left(\frac{1}{1-\beta^2}S+\mathfrak{T}(\beta^2)\right)Q_2\mathbf{e}+2\beta\left(\frac{1}{1-\beta^2}S+\mathfrak{T}(\beta^2)\right)Q_1\left(\frac{1}{1-\beta}S+\mathfrak{T}(\beta)\right)Q_1\mathbf{e} \\ &=[I-\beta^2P]^{-1}Q_2\mathbf{e}+2\beta[I-\beta^2P]^{-1}Q_1[I-\beta P]^{-1}Q_1\mathbf{e} \qquad \text{as } n\to\infty. \end{aligned}$$

The skewness of $R(n)$ tends not always to zero so that the distribution of the random variable $\lim_{n\to\infty} R(n)$ is not in general equal to a normal distribution.

Reference

[1] Howard, R. A., Dynamic Programming and Markov Processes. The M. I. T. Press (1960).

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