SURFACES IN THE 4-DIMENSIONAL EUCLIDEAN SPACE ISOMETRIC TO A SPHERE

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In [3], the author introduced some kinds of curvatures and torsion form for surfaces in a higher dimensional Euclidean space. These curvatures are linearly dependent on the Gaussian curvature and carry out the same rôles of the curvature and the torsion of a curve in the 3-dimensional Euclidean space with the torsion form. In the present paper, the author will investigate the isometric immersions of the two dimensional sphere into the 4-dimensional Euclidean space with constant curvatures.

§ 1. Preliminaries.

Let M^2 be a 2-dimensional oriented Riemannian C^∞-manifold with an isometric immersion

$$x: \quad M^2 \to E^4$$

of M^2 into a 4-dimensional Euclidean space E^4. Let $F(M^2)$ and $F(E^4)$ be the bundles of orthonormal frames of M^2 and E^4 respectively. Let B be the set of elements $b=(p, e_1, e_2, e_3, e_4)$ such that $(p, e_1, e_2)\in F(M^2)$ and $(x(p), e_1, e_2, e_3, e_4)\in F(E^4)$ whose orientations is coherent with the one of E^4, identifying e_i with $dx(e_i)$, $i=1, 2$. $B\to M^2$ may be considered as a principal bundle with the fibre $O(2)\times SO(2)$. Let

$$\tilde{x}: \quad B \to F(E^4)$$

be the mapping naturally defined by $\tilde{x}(b)=(x(p), e_1, e_2, e_3, e_4)$. Let B_ν be the set of elements (p, e) such that $p\in M^2$ and e is a unit normal vector to the tangent plane $dx(T_p(M^2))$ at $x(p)$. $B_\nu\to M^2$ is the so-called normal circle bundle of M^2 in E^4 whose fibre at p is denoted by S_p^1. Let S_0^3 be the unit 3-sphere in E^4 with the origin as its center. Let

$$\tilde{\nu}: \quad B \to S_0^3$$

be the mapping defined by $\tilde{\nu}(p, e)=e$.

We have the differential forms $\omega_1, \omega_2, \omega_{12}, \omega_{13}, \omega_{14}, \omega_{23}, \omega_{24}, \omega_{34}$ on B derived from the basic forms and the connection forms on $F(E^4)$ of the Euclidean space E^4 through $\tilde{x}$ as follows:

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$$(1)\qquad \begin{cases} dx=\omega_1 e_1+\omega_2 e_2, \quad de_A=\sum_{B=1}^{4}\omega_{AB}e_B, \qquad A=1, 2, 3, 4, \\ \qquad\qquad \omega_{AB}=-\omega_{BA} \end{cases}$$

$$(2)\qquad \begin{cases} d\omega_1=\omega_2\wedge\omega_{21}, \qquad d\omega_2=\omega_1\wedge\omega_{12} \\ d\omega_{12}=\sum_{r=3}^{4}\omega_{1r}\wedge\omega_{r2}, \qquad d\omega_{34}=\sum_{i=1}^{2}\omega_{3i}\wedge\omega_{i4} \\ d\omega_{ir}=\omega_{ij}\wedge\omega_{jr}+\omega_{it}\wedge\omega_{tr}, \end{cases}$$

$$i, j=1, 2,\ i\neq j; \qquad r, t=3, 4,\ r\neq t$$

and

$$(3)\qquad \omega_{ir}=\sum_{j=1}^{2}A_{rij}\omega_j, \qquad A_{rij}=A_{rji}.$$

ω_1, ω_2, ω_{12} may be considered as the basic forms and the connection form on $F(M^2)$ of M^2 and the Gaussian curvature of M^2 at p is given by

$$(4)\qquad d\omega_{12}=-G(p)\omega_1\wedge\omega_2$$

and

$$(5)\qquad G(p)=\sum_{r=3}^{4}(A_{r11}A_{r22}-A_{r12}A_{r12}).$$

The Lipschitz-Killing curvature at $(p, e)\in B_\nu$ is given by

$$(6)\qquad G(p, e)=\det(A_{3ij}\cos\theta+A_{4ij}\sin\theta),$$

where $e=e_3\cos\theta+e_4\sin\theta$, $(p, e_1, e_2, e_3, e_4)\in B$.

The total curvature at $p\in M^2$ is given by

$$(7)\qquad K^*(p)=\int_0^{2\pi}|G(p, e)|d\theta.$$

Now, for any $e\in S_0^3$, let $m_i(e)$ be the number of critical points of index i for the function

$$x\cdot e:\ M^2\to R, \qquad (x\cdot e)(p)=x(p)\cdot e$$

and put

$$(8)\qquad m(e)=\sum_{i=0}^{2}m_i(e).$$

Let us assume that M^2 is of genus g, then by virtue of the Morse's inequalities we have

$$\text{(9)}\qquad \begin{cases} m_0(e)\geqq 1, & m_1(e)-m_0(e)\geqq 2g-1, \\ m_2(e)-m_1(e)+m_0(e)=2(1-g)=\chi(M^2) & \end{cases}$$

for any $e\in S_0^3$, except a set of measure 0, where $\chi(M^2)$ denotes the Euler characteristic of M^2. Then we have

$$\text{(10)}\qquad \int_{M^2} K^*(p)dV=\int_{S_0^3} m(e)d\Sigma_3,$$

where $dV=\omega_1\wedge\omega_2$ and $d\Sigma_3$ are the volume elements of M^2 and S_0^3.[1)]

Let $\lambda(p)$ and $\mu(p)$ be the maximum and the minimum of $G(p, e)$ on S_p^1 respectively. $\lambda(p)$ and $\mu(p)$ are continuous on M^2 and differentiable on the open subset of M^2 in which $\lambda\neq\mu$. λ and μ are called the principal curvature and the secondary curvature of M^2 in E^4 respectively. Let $(p, \bar{e}_3)$ be a point of B_ν at which $G(p, \bar{e}_3)=\lambda(p)$. If $\lambda(p)\neq\mu(p)$, there exist two such points that they are two vectors at $x(p)$ with the opposite directions. For any $(p, e_1, e_2)\in F(M^2)$, the element $b=(p, e_1, e_2, \bar{e}_3, \bar{e}_4)\in B$ is uniquely determined from $\bar{e}_3$ and $G(p, \bar{e}_4)=\mu(p)$. $b=(p, e_1, e_2, \bar{e}_3, \bar{e}_4)$ is called a *Frenet frame* of M^2 in E^4. Then

$$\text{(11)}\qquad G(p, e)=\lambda(p)\cos^2\theta+\mu(p)\sin^2\theta,$$

where $e=\bar{e}_3\cos\theta+\bar{e}_4\sin\theta$, and we have

$$\text{(12)}\qquad \lambda(p)+\mu(p)=G(p).$$

Now, let us introduce the open set of M^2 by

$$M_-=\{p\in M^2, \lambda(p)\mu(p)<0\}$$

and the continuous function $\alpha(p)$ on M_- by

$$\text{(13)}\qquad \cos 2\alpha=-\frac{\lambda+\mu}{\lambda-\mu}, \qquad 0<\alpha<\frac{\pi}{2}.$$

Then, we have

$$K^*(p)=\begin{cases} (4\alpha-\pi)G(p)+4\sqrt{-\lambda\mu} & (p\in M_-), \\ \pi|G(p)| & (p\bar{\in} M_-) \end{cases}$$

Making use of (10), (9), the above equations and the Euler's formula:

$$\int_{M^2} G(p)dV=2\pi\chi(M^2)=4\pi(1-g),$$

we get the following formulas

$$\text{(14)}\qquad \int_{S_0^3} m_1(e)d\Sigma_3=-\pi\int_{M_2} G(p)dV+2\int_{M_-}\left\{-\left(\frac{\pi}{2}-\alpha\right)G+\sqrt{-\lambda\mu}\right\}dV,^{2)}$$

1) Where, ω_1 and ω_2 are considered only on the subbundle of $F(M^2)$ whose element (p, e_1, e_2) has the orientation coherent with the one of M^2.

2) See [3], §3.

where $M_2=\{p\in M^2, \lambda(p)\leqq 0\}$. We will be mainly concerned with this formula (14) in this paper.

Now, a local cross-section $b=(p, e_1, e_2, \bar{e}_3(p), \bar{e}_4(p))$ of $B\to F(M^2)$, whose image consists of Frenet frames, is called a Frenet cross-section. Making use of a differentiable Frenet cross-section $b=(p, e_1, e_2, \bar{e}_3, \bar{e}_4)$ of $B\to F(M^2)$, we have

$$(15)\qquad \begin{cases} dx= \omega_1 e_1+\omega_2 e_2, \\ de_1= \omega_{12}e_1+\omega_{13}\bar{e}_3+\omega_{14}\bar{e}_4, \\ de_2=-\omega_{12}e_1+ \omega_{23}\bar{e}_3+\omega_{24}\bar{e}_4, \\ d\bar{e}_3=-\omega_{13}e_1-\omega_{23}e_2 +\bar{\omega}_{34}\bar{e}_4, \\ d\bar{e}_4=-\omega_{14}e_1-\omega_{24}e_2-\bar{\omega}_{34}\bar{e}_3, \end{cases}$$

$$(16)\qquad \omega_{13}\wedge\omega_{23}=\lambda(p)\omega_1\wedge\omega_2,$$

$$(17)\qquad \omega_{14}\wedge\omega_{24}=\mu(p)\omega_1\wedge\omega_2,$$

$$(18)\qquad \omega_{13}\wedge\omega_{24}+\omega_{14}\wedge\omega_{23}=0.$$

$\bar{\omega}_{34}=d\bar{e}_3\cdot\bar{e}_4$ is a 1-form on the domain of the local cross-section in M^2 and it is called the *torsion form* of M^2 in E^4.

§2. M^2 diffeomorphic to S^2.

Let M^2 be diffeomorphic to a two dimensional sphere S^2, then from (9) we have

$$(19)\qquad m_0(e)\geqq 1, \qquad m_2(e)\geqq 1, \qquad m_1(e)=m_0(e)+m_2(e)-2$$

for $e\in S_0^3$, except a set of measure 0. Hence

$$(20)\qquad m(e)=2(m_0(e)+m_2(e)-1)\geqq 2$$

and the equality holds only when $m_0(e)=m_2(e)=1$. If the equality holds for[3] almost $e\in S_0^3$, from (9) we get

$$\int_{M^2} K^*(p)dV=2c_3$$

and so M^2 is a convex surface imbedded in a hyperplane by virtue of Chern-Lashof's theorem [1], where c_3 denotes the volume of the unit 3-sphere S_0^3 and is equal to $2\pi^2$. Hence we have

THEOREM 1. *Let M^2 be a two-dimensional Riemannian manifold diffeomorphic to a sphere and admitting an isometric immersion x: $M^2\to E^4$. If there exists no hyperplane containing $x(M^2)$, then the measure of the set of $e\in S_0^3$ such that $m(e)\geqq 4$ is positive.*

THEOREM 2. *Let M^2 be a two-dimensional Riemannian manifold with non-*

3) In the following, we use simply "almost" in place of "except a set of measure 0".

negative Gaussian curvature, diffeomorphic to a sphere and admitting an isometric immersion $x: M^2 \to E^4$. If there exists no hyperplane containing $x(M^2)$, then the secondary curvature μ is negative at a point.

Proof. If $\mu \geqq 0$ everywhere, it must be $M_- = \phi$. It may be put $M_2 = \phi$ since $G(p) \geqq 0$ everywhere. From (14), we get

$$\int_{S_0^3} m_1(e) d\Sigma_3 = 0,$$

which follows $m_1(e) = m_0(e) + m_2(e) - 2 = 0$, hence $m(e) = 2$ for almost points $e \in S_0^3$. By Theorem 1, there exists a hyperplane containing $x(M^2)$. This contradicts the assumptions.

THEOREM 3. *Let M^2 be a two dimensional Riemannian manifold with constant positive Gaussian curvature $1/a^2$, diffeomorphic to a sphere and admitting an isometric immersion $x: M^2 \to E^4$. If there exists no hyperplane containing $x(M^2)$, the principal curvature λ is constant and $m(e) = 4$ for almost $e \in S_0^3$, then $\lambda a^2 = t$ is a constant such that*

$$\sin \sqrt{t(t-1)} = \frac{1}{2t-1}, \qquad 1.5 < t < 2. \tag{21}$$

Proof. By (12) and Theorem 2, the secondary curvature μ of $x: M^2 \to E^4$ is a negative constant and $\lambda a^2 = t > 1$. Accordingly, α is also constant on $M_- = M^2$. By the assumption, $m_0(e) + m_2(e) = 3$ and $m_1(e) = 1$. Hence from (14) we have

$$\begin{aligned} c_3 = 2\pi^2 &= 2\int_{M^2} \left\{ \sqrt{\lambda\left(\lambda - \frac{1}{a^2}\right)} - \left(\frac{\pi}{2} - \alpha\right)\frac{1}{a^2} \right\} dV \\ &= 8\pi \left\{ \sqrt{t(t-1)} - \left(\frac{\pi}{2} - \alpha\right) \right\}, \end{aligned}$$

hence

$$\frac{3\pi}{4} - \alpha = \sqrt{t(t-1)}. \tag{22}$$

On the other hand, from (13) we get

$$\cos 2\alpha = -\frac{1}{2t-1},$$

hence

$$\sin 2\sqrt{t(t-1)} = \sin\left(\frac{3\pi}{2} - 2\alpha\right) = -\cos 2\alpha = \frac{1}{2t-1}$$

and

$$\frac{\pi}{4} < \alpha < \frac{\pi}{2}.$$

From (22), it must be

$$\frac{1}{2}+\sqrt{\frac{\pi^2}{16}+\frac{1}{4}}=1.43\cdots<t<\frac{1}{2}+\sqrt{\frac{\pi^2}{4}+\frac{1}{4}}=2.14\cdots.$$

There exists a unique value in this interval that satisfies (21) and furthermore we can easily see that $1.5<t<2$.

§ 3. Two examples of analytic isometric immersions and imbeddings of S^2 in E^4.

In this section, we shall give two examples of isometric immersions and imbeddings of a sphere S^2 into E^4 such that the immersion or the imbedding $x: S^2\to E^4$ is analytic and the image $x(S^2)$ is not contained in any hyperplane of E^4.

As in the ordinary method, we represent S^2 by

(23) $$x_1=a\sin u\cos v,\quad x_2=a\sin u\sin v,\quad x_3=a\cos u\qquad 0\leqq u\leqq\pi,\quad 0\leqq v<2\pi$$

in E^3. Its line element is

(24) $$ds^2=a^2du^2+a^2\sin^2 u\, dv^2.$$

EXAMPLE 1. Let $x:\ S^2\to E^4$ be given by

(25) $$\begin{cases} x_1=\dfrac{a}{2}\sin^2 u\cos 2u=\dfrac{a}{8}(-1+2\cos 2u-\cos 4u), \\[2mm] x_2=\dfrac{a}{2}\sin^2 u\sin 2u=\dfrac{a}{8}(\qquad 2\sin 2u-\sin 4u), \\[2mm] x_3=a\sin u\cos v,\qquad x_4=a\sin u\sin v. \end{cases}$$

We get easily

$$ds^2=\sum_{A=1}^{4}dx_Adx_A=a^2du^2+a^2\sin^2 u\, dv^2,$$

hence (25) is isometric. Except the north pole (0, 0, 1) and the south pole (0, 0, -1), the mapping $x:\ S^2\to E^4$ is one-to-one and the two poles are mapped to the origin (0, 0, 0, 0) of E^4. Hence x is an analytic isometric immersion of S^2 into E^4. Putting

$$e_1^*=\frac{1}{a}\frac{\partial x}{\partial u}=\left(\frac{-\sin 2u+\sin 4u}{2},\ \frac{\cos 2u-\cos 4u}{2},\ \cos u\cos v,\ \cos u\sin v\right),$$

$$e_2^*=\frac{1}{a\sin u}\frac{\partial x}{\partial v}=(0,\ 0,\ -\sin v,\ \cos v),$$

$$e_3^*=(-\sin 3u,\ \cos 3u,\ 0,\ 0),$$

$$e_4^*=\left(-\frac{\cos 2u+\cos 4u}{2},\ -\frac{\sin 2u+\sin 4u}{2},\ \sin u\cos v,\ \sin u\sin v\right),$$

$(p,\ e_1^*,\ e_2^*,\ e_3^*,\ e_4^*)\in B$ and $dx=e_1^*\omega_1^*+e_2^*\omega_2^*$, $\omega_1^*=adu$, $\omega_2^*=a\sin u\, dv$. Putting

$$de_A^*=\sum_B \omega_{AB}^* e_B^*, \qquad \omega_{ir}^*=\sum_j A_{rij}^*\omega_j^*,$$

we have

$$(A_{3ij}^*)=\begin{pmatrix} \dfrac{3\sin u}{a} & 0 \\ 0 & 0 \end{pmatrix}, \qquad (A_{4ij}^*)=\begin{pmatrix} -\dfrac{1}{a} & 0 \\ 0 & -\dfrac{1}{a} \end{pmatrix}.$$

For $e=e_3^*\cos\theta+e_4^*\sin\theta$, the Lipschitz-Killing curvature is given by

$$G(p, e)=\det(A_{3ij}^*\cos\theta+A_{4ij}^*\sin\theta)$$
$$=\frac{1}{2a^2}(1-\cos 2\theta-3\sin u\sin 2\theta),$$

hence

(26) $$\lambda(p)=\frac{1}{2a^2}(1+\sqrt{1+9\sin^2 u}), \quad \mu(p)=\frac{1}{2a^2}(1-\sqrt{1+9\sin^2 u}),$$

$\mu\leqq 0$ and $\mu=0$ only at the poles.

Putting $\bar{e}_i=e_i^*$, $i=1, 2$, $\bar{e}_3=e_3^*\cos\theta_0+e_4^*\sin\theta_0$, $\bar{e}_4=-e_3^*\sin\theta_0+e_4^*\cos\theta_0$, where

$$\theta_0=\frac{3\pi}{4}-\frac{\alpha_0}{2}, \qquad \cos\alpha_0=\frac{3\sin u}{\sqrt{1+9\sin^2 u}}, \qquad 0<\alpha_0\leqq\frac{\pi}{2},$$

then $(p, \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4)$ is a Frenet frame, from which the torsion form of x: $S^2\to E^4$ is

(27) $$\bar{\omega}_{34}=\omega_{34}^*+d\theta_0=\frac{9\cos u\,(1+6\sin^2 u)}{2(1+9\sin^2 u)}\,du.$$

Since we can not choose θ so that

$$A_{3ij}^*\cos\theta+A_{4ij}^*\sin\theta=0,$$

there exists no hyperplane containing $x(S^2)$.

EXAMPLE 2. Let x: $S^2\to E^4$ be given by

(28) $$\begin{cases} x_1=\dfrac{4a}{3}\cos^3\dfrac{u}{2}=\dfrac{a}{3}\left(\cos\dfrac{3u}{2}+3\cos\dfrac{u}{2}\right), \\ x_2=\dfrac{4a}{3}\sin^3\dfrac{u}{2}=\dfrac{a}{3}\left(-\sin\dfrac{3u}{2}+3\sin\dfrac{u}{2}\right), \\ x_3=a\sin u\cos v, \quad x_4=a\sin u\sin v. \end{cases}$$

This is an analytic isometric imbedding of S^2 into E^4. Putting

$$e_1^*=\frac{1}{a}\frac{\partial x}{\partial u}=\left(-\sin u\cos\frac{u}{2},\ \sin u\sin\frac{u}{2},\ \cos u\cos v,\ \cos u\sin v\right),$$

$$e_2^* = \frac{1}{a \sin u} \frac{\partial x}{\partial v} = (0, 0, -\sin v, \cos v),$$

$$e_3^* = \left(\cos u \cos\frac{u}{2},\ -\cos u \sin\frac{u}{2},\ \sin u \cos v,\ \sin u \sin v\right),$$

$$e_4^* = \left(\sin\frac{u}{2},\ \cos\frac{u}{2},\ 0,\ 0\right),$$

$(p, e_1^*, e_2^*, e_3^*, e_4^*) \in B$ and $dx = e_1^* \omega_1^* + e_2^* \omega_2^*$, $\omega_1^* = a\,du$, $\omega_2^* = a \sin u\, dv$. Putting

$$de_A^* = \sum_B \omega_{AB}^* e_B^*, \quad \omega_{ir}^* = \sum_j A_{rij}^* \omega_j^*,$$

we have

$$\omega_{12}^* = \cos u\, dv, \quad \omega_{13}^* = -du, \quad \omega_{14}^* = \frac{1}{2} \sin u\, du, \quad \omega_{23}^* = -\sin u\, dv,$$

$$\omega_{24}^* = 0,$$

$$(A_{3ij}^*) = \begin{pmatrix} -\frac{1}{a} & 0 \\ 0 & -\frac{1}{a} \end{pmatrix}, \qquad (A_{4ij}^*) = \begin{pmatrix} \frac{\sin u}{2a} & 0 \\ 0 & 0 \end{pmatrix}.$$

For $e = e_3^* \cos\theta + e_4^* \sin\theta$, the Lipschitz-Killing curvature is given by

$$G(p, e) = \frac{1}{2a^2}\left(1 + \cos 2\theta - \frac{\sin u}{2} \sin 2\theta\right),$$

hence

$$\lambda(p) = \frac{1}{2a^2}\left(1 + \sqrt{1 + \frac{\sin^2 u}{4}}\right), \qquad \mu(p) = \frac{1}{2a^2}\left(1 - \sqrt{1 + \frac{\sin^2 u}{4}}\right). \tag{29}$$

Putting $\bar{e}_i = e_i^*$, $i = 1, 2$, $\bar{e}_3 = \bar{e}_3^* \cos\theta_0 + e_4^* \sin\theta_0$, $\bar{e}_4 = -e_3^* \sin\theta_0 + e_4^* \cos\theta_0$, where

$$\theta_0 = \pi - \frac{\alpha_0}{2}, \qquad \cos\alpha_0 = \frac{1}{\sqrt{1 + \frac{\sin^2 u}{4}}}, \qquad 0 \leqq \alpha_0 < \frac{\pi}{2},$$

then $(p, \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4)$ is a Frenet frame, from which the torsion form of this imbedding is

$$\bar{\omega}_{34} = \omega_{34}^* + d\theta_0 = -\frac{\cos u\,(6 + \sin^2 u)}{2\,(4 + \sin^2 u)}\, du. \tag{30}$$

Since we can not choose θ so that

$$A_{3ij}^* \cos\theta + A_{4ij}^* \sin\theta = 0,$$

there exists no hyperplane containing $x(S^2)$.

Now, for the two isometric mappings we have

$$\lambda(p)=\frac{1}{2a^2}(1+\sqrt{1+h^2\sin^2 u})\geqq\frac{1}{a^2}, \qquad \mu(p)=\frac{1}{2a^2}(1-\sqrt{1+h^2\sin^2 u})\leqq 0,$$

where $h=3$ or $1/2$, hence (14) becomes

$$\begin{aligned}\int_{S_0^3} m_1(e)d\Sigma_3 &= \int_{M^2}\left\{-(\pi-2\alpha)\frac{1}{a^2}+2\sqrt{-\lambda\mu}\right\}dV\\ &=\int_0^{2\pi}\int_0^{\pi}\left(-\sin u\,\mathrm{Cos}^{-1}\frac{1}{\sqrt{1+h^2\sin^2 u}}+h\sin^2 u\right)du\,dv\\ &=2\pi\left\{\left[\cos u\,\mathrm{Cos}^{-1}\frac{1}{\sqrt{1+h^2\sin^2 u}}\right]_0^{\pi}-h\int_0^{\pi}\frac{\cos^2 u}{1+h^2\sin^2 u}du\right.\\ &\qquad\left.+h\int_0^{\pi}\sin^2 u\,du\right\}=\frac{2-2\sqrt{1+h^2}+h^2}{h}\pi^2.\end{aligned}$$

Accordingly, we have

$$\frac{\int_{S_0^3} m_1(e)\,d\Sigma_3}{c_3}=\frac{2-2\sqrt{1+h^2}+h^2}{2h}=\begin{cases}\dfrac{11-2\sqrt{10}}{6}\doteqdot 0.779 & (h=3),\\[2ex] 2+\dfrac{1}{4}-\sqrt{5}\doteqdot 0.014 & \left(h=\dfrac{1}{2}\right)\end{cases}$$

This shows that for the isometric mappings (25) and (28), $m_0(e)=m_2(e)=1$ and $m_1(e)=0$ hold good for $e\in S_0^3$, at least about 22.1% and 98.6% of the point of S_0^3 respectively.

§ 4. An example of isometric imbedding of S^2 in E^4 with constant curvatures.

The two examples in § 3 are constructed by the method that taking the plane curves:

$$x_1=\frac{a}{2}\sin^2 u\cos 2u, \qquad x_2=\frac{a}{2}\sin^2 u\sin 2u$$

and

$$x_1=\frac{4a}{3}\cos^3\frac{u}{2}, \qquad x_2=\frac{4a}{3}\sin^3\frac{u}{2} \qquad \text{(asteroid)}$$

corresponding to the segment in E^3 joining the two poles of S^2, the parallel circles of S^2 are transformed to the circles in E^4 with their centers on these curves that the planes containing these circles are parallel to the x_3x_4-coordinate plane. By means of the same method, let x: $S^2\rightarrow E^4$ be given by

(31) $\qquad x_1=af(u),\quad x_2=a\,g(u),\quad x_3=a\sin u\cos v,\quad x_4=a\sin u\sin v,$

where $f(u)$ and $g(u)$ are indetermined functions. In order that x is isometric, it must be

(32) $\qquad f'^2+g'^2=\sin^2 u.$

Putting

$$e_1^* = \frac{1}{a}\frac{\partial x}{\partial u} = (f'(u),\ g'(u),\ \cos u \cos v,\ \cos u \sin v),$$

$$e_2^* = \frac{1}{a \sin u}\frac{\partial x}{\partial u} = (0,\ 0,\ -\sin v,\ \cos v),$$

$dx = e_1^*\omega_1^* + e_2^*\omega_2^*$, $\omega_1^* = a\,du$, $\omega_2^* = a \sin u\,dv$. Let $e = (\xi_1, \xi_2, \rho\cos v, \rho \sin v)$ be a normal unit vector at $x(p)$, then

$$\xi_1^2 + \xi_2^2 + \rho^2 = 1, \qquad \xi_1 f' + \xi_2 g' + \rho \cos u = 0,$$

from which putting

$$e_3^* = \left(-\frac{\cos u}{\sin u} f'(u),\ -\frac{\cos u}{\sin u} g'(u),\ \sin u \cos v,\ \sin u \sin v\right),$$

$$e_4^* = \left(\frac{1}{\sin u} g'(u),\ -\frac{1}{\sin u} f'(u),\ 0,\ 0\right), \qquad 0 < u < \pi,$$

$(x(p), e_1^*, e_2^*, e_3^*, e_4^*) \in F(E^4)$. Assuming $(p, e_1^*, e_2^*, e_3^*, e_4^*) \in B$ and putting

$$de_A^* = \sum_B \omega_{AB}^* e_B^*, \quad \omega_{ir}^* = \sum_j A_{rij}^* \omega_j^*,$$

we have

$$\omega_{12}^* = \cos u\,dv, \quad \omega_{13}^* = -du, \quad \omega_{14}^* = \frac{f''g' - f'g''}{\sin u}\,du,$$

$$\omega_{23}^* = -\sin u\,dv, \quad \omega_{24}^* = 0,$$

$$(A_{3ij}^*) = \begin{pmatrix} -\frac{1}{a} & 0 \\ 0 & -\frac{1}{a} \end{pmatrix}, \qquad (A_{4ij}^*) = \begin{pmatrix} \frac{f''g' - f'g''}{a \sin u} & 0 \\ 0 & 0 \end{pmatrix}.$$

For $e = e_3^* \cos\theta + e_4^* \sin\theta$, the Lipschitz-Killing curvature is given by

$$G(p, e) = \frac{1}{2a^2}\left(1 + \cos 2\theta - \frac{f''g' - f'g''}{\sin u} \sin 2\theta\right),$$

hence

$$(33) \qquad \begin{cases} \lambda(p) = \dfrac{1}{2a^2}\left(1 + \sqrt{1 + \dfrac{(f''g' - f'g'')^2}{\sin^2 u}}\right), \\ \mu(p) = \dfrac{1}{2a^2}\left(1 - \sqrt{1 + \dfrac{(f''g' - f'g'')^2}{\sin^2 u}}\right) \end{cases} \qquad (0 < u < \pi).$$

Therefore, in order that the principal curvature λ is constant, it must be

$$(34) \qquad f''g' - f'g'' = c \sin u, \quad c = \text{constant}.$$

By means of (32), we have

$$g'=\varepsilon\sqrt{\sin^2 u-f'^2},\qquad g''=\frac{\varepsilon(\sin u\cos u-f'f'')}{\sqrt{\sin^2 u-f'^2}}\qquad (\varepsilon=\pm 1)$$

and, putting these into (34), we get

$$f''\sin u-f'\cos u=\varepsilon c\sqrt{\sin^2 u-f'^2}. \tag{35}$$

$f'=\sin u$ is a special solution of (35) which gives an isometric imbedding equivalent to $S^2\subset E^3$. Now, putting $f'=\varphi\sin u$, $|\varphi|\leqq 1$, we get from (35) the equation with respect to φ

$$\frac{\varphi'}{\sqrt{1-\varphi^2}}=\frac{\varepsilon c}{\sin u},$$

from which we have

$$\varphi=\sin\left(\varepsilon c\log\tan\frac{u}{2}+c_1\right),\qquad 0<u<\pi,$$

where c_1 is a constant. Accordingly, we have

$$f'=\sin u\sin\left(\varepsilon c\log\tan\frac{u}{2}+c_1\right),$$

$$g'=\varepsilon\sin u\left|\cos\left(\varepsilon c\log\tan\frac{u}{2}+c_1\right)\right|.$$

Making use of the continuity of f' and g' and changing suitably the constants c and c_1, we may put

$$f'=\sin u\sin\left(c\log\tan\frac{u}{2}+c_1\right),$$

$$g'=\sin u\cos\left(c\log\tan\frac{u}{2}+c_1\right),\qquad 0<u<\pi,$$

which satisfy clearly (34) and $(p, e_1^*, e_2^*, e_3^*, e_4^*)\in B$, since

$$\lim_{u\to 0}\det(e_1^*\, e_2^*\, e_3^*\, e_4^*)=1.$$

Accordingly, we have

$$\begin{aligned} f(u)&=\int_0^u\sin u\sin\left(c\log\tan\frac{u}{2}+c_1\right)du+c_2,\\ g(u)&=\int_0^u\sin u\cos\left(c\log\tan\frac{u}{2}+c_1\right)du+c_3,\end{aligned} \tag{36}$$

where c_2 and c_3 are constants. f and g are analytic in the interval $0<u<\pi$, of class C^1 but not of class C^2 on the interval $0\leqq u\leqq\pi$. Let $(p, \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4)$, $\bar{e}_1=e_1^*$, $\bar{e}_2=e_2^*$, $\bar{e}_3=e_3^*\cos\theta_0+e_4^*\sin\theta_0$, $\bar{e}_4=-e_3^*\sin\theta_0+e_4^*\cos\theta_0$, be a Frenet frame, then θ_0 is a constant by means of (34). And so, the torsion form of $x:\ S^2\to E^4$ is

$$\bar{\omega}_{34}=\omega_{34}^{*}=-e_3^{*}\cdot de_4^{*}=-\frac{c\cos u}{\sin u}du, \qquad (0<u<\pi),$$

hence the torsion form is singular at the poles.

Essentially we may put $c_1=c_2=c_3=0$, but regarding the constant c we have

$$t=\lambda a^2=\frac{1}{2}(1+\sqrt{1+c^2}),$$

hence

$$c=2\sqrt{t(t-1)}.$$

Thus we see that by this method we can not construct an isometric imbedding x: $S^2\to E^4$ of class C^2 with constant curvatures and $x(S^2)$ is not contained in any hyperplane in E^4.

§5. Tubular isometric immersions of S^2 in E^4 with constant curvatures.

We say a mapping x of S^2 into E^4 is a tubular isometric immersion, if x is an isometric immersion, the parallel circles of S^2 are transformed to circles in E^4 and the locus of the centers of these circles is orthogonal to the planes containing them.

Let x: $S^2\to E^4$ be a tubular isometric immersion and y: $[0,\pi]\to E^4$ be the mapping which represents the locus of the centers of the image circles of the parallel circles of S^2. Put

$$y=a\boldsymbol{f}, \qquad \boldsymbol{f}=(f_1,f_2,f_3,f_4) \tag{37}$$

and let $(y,\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3,\boldsymbol{u}_4)$ be its Frenet frame, that is

$$\begin{cases} dy=\boldsymbol{u}_1\,d\sigma, \\ d\boldsymbol{u}_1= \boldsymbol{u}_2k_1d\sigma, \\ d\boldsymbol{u}_2=-\boldsymbol{u}_1k_1d\sigma+\boldsymbol{u}_3k_2d\sigma, \\ d\boldsymbol{u}_3=-\boldsymbol{u}_2k_2d\sigma+\boldsymbol{u}_4k_3d\sigma, \\ d\boldsymbol{u}_4=-\boldsymbol{u}_3k_3d\sigma, \end{cases} \tag{38}$$

where σ denotes its arclength,

$$d\sigma=a\sqrt{\boldsymbol{f}'\cdot\boldsymbol{f}'}\,du \tag{39}$$

and k_1, k_2, k_3 are its curvatures. Corresponding to $v=0$ and $v=\pi/2$, let us introduce two orthogonal unit vectors

$$\boldsymbol{p}=\sum_{\beta=2}^{4}\boldsymbol{u}_\beta p_\beta, \qquad \boldsymbol{q}=\sum_{\beta=2}^{4}\boldsymbol{u}_\beta q_\beta$$

such that

$$\boldsymbol{p}\cdot\boldsymbol{p}=\boldsymbol{q}\cdot\boldsymbol{q}=1, \qquad \boldsymbol{p}\cdot\boldsymbol{q}=0. \tag{40}$$

Then, x can be written as

$$x=x(u,v)=y(u)+\boldsymbol{p}a\sin u\cos v+\boldsymbol{q}a\sin u\sin v. \tag{41}$$

Since

$$\begin{aligned} dx=\boldsymbol{u}_1 d\sigma&+\boldsymbol{p}a(\cos u\cos v\,du-\sin u\sin v\,dv)\\ &+\boldsymbol{q}a(\cos u\sin v\,du+\sin u\cos v\,dv)\\ &+\frac{d\boldsymbol{p}}{du}a\sin u\cos v\,du+\frac{d\boldsymbol{q}}{du}a\sin u\sin v\,du, \end{aligned}$$

the line element of $x:\ S^2\rightarrow E^4$ can be written as

$$\begin{aligned} ds^2=a^2\Big\{&(\boldsymbol{f}'\cdot\boldsymbol{f}')+\cos^2 u-2a(\boldsymbol{f}'\cdot\boldsymbol{f}')k_1\sin u(p_2\cos v+q_2\sin v)\\ &+\sin^2 u\left(\frac{d\boldsymbol{p}}{du}\cdot\frac{d\boldsymbol{p}}{du}\cos^2 v+\frac{d\boldsymbol{q}}{du}\cdot\frac{d\boldsymbol{q}}{du}\sin^2 v+2\frac{d\boldsymbol{p}}{du}\cdot\frac{d\boldsymbol{q}}{du}\cos v\sin v\right)\Big\}du^2\\ &+2a^2\sin^2 u\left(\boldsymbol{q}\cdot\frac{d\boldsymbol{p}}{du}\right)du\,dv+a^2\sin^2 u\,dv^2. \end{aligned}$$

Hence, it must be

$$\boldsymbol{q}\cdot\frac{d\boldsymbol{p}}{du}=0, \tag{42}$$

$$\begin{aligned} &\boldsymbol{f}'\cdot\boldsymbol{f}'-2a(\boldsymbol{f}'\cdot\boldsymbol{f}')k_1\sin u\,(p_2\cos v+q_2\sin v)\\ &\qquad+\sin^2 u\left\|\frac{d\boldsymbol{p}}{du}\cos v+\frac{d\boldsymbol{q}}{du}\sin v\right\|^2=\sin^2 u. \end{aligned} \tag{43}$$

From (43), it must be

$$p_2=q_2=0 \qquad \text{or} \qquad k_1=0.$$

Case: $k_1=0$. (43) becomes

$$\begin{aligned} \boldsymbol{f}'\cdot\boldsymbol{f}'+\sin^2 u\Big\{&\frac{1}{2}\left(\left\|\frac{d\boldsymbol{p}}{du}\right\|^2+\left\|\frac{d\boldsymbol{q}}{du}\right\|^2\right)+\frac{1}{2}\left(\left\|\frac{d\boldsymbol{p}}{du}\right\|^2-\left\|\frac{d\boldsymbol{q}}{du}\right\|^2\right)\cos 2v\\ &+\left(\frac{d\boldsymbol{p}}{du}\cdot\frac{d\boldsymbol{q}}{du}\right)\sin 2v\Big\}=\sin^2 u, \end{aligned}$$

which is equivalent to

$$\left\|\frac{d\boldsymbol{p}}{du}\right\|=\left\|\frac{d\boldsymbol{q}}{du}\right\|, \qquad \frac{d\boldsymbol{p}}{du}\cdot\frac{d\boldsymbol{q}}{du}=0, \tag{44}$$

$$\boldsymbol{f}'\cdot\boldsymbol{f}'=\sin^2 u\left(1-\left\|\frac{d\boldsymbol{p}}{du}\right\|^2\right). \tag{45}$$

In this case, we may consider $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$, $\boldsymbol{u}_4$ being constant unit vectors and $\boldsymbol{f}=(f(u),0,0,0)$. If $\boldsymbol{p}$ is constant, then $\boldsymbol{q}$ is also constant. If $d\boldsymbol{p}/du\neq 0$, then $d\boldsymbol{q}/du$

has the same direction as $\boldsymbol{q}$ by (40), (42) and (44). Hence $\boldsymbol{q}$ is a constant unit vector, hence $\boldsymbol{p}$ must be a constant vector by (44), this contradicts the assumption. Hence, in the case, $\boldsymbol{p}$ and $\boldsymbol{q}$ are constant vectors and from (45) we have $f(u)=\pm\cos u$, thus the mapping x is equivalent to $S^2\to S^2\subset E^3$.

Case: $p_2=q_2=0$. We rewrite (41) as

$$x=x(u,v)=y+\boldsymbol{u}_3 a\sin u\cos\bar{v}+\boldsymbol{u}_4 a\sin u\sin\bar{v},\quad \bar{v}=v-\varphi,\quad \varphi=\varphi(u). \tag{46}$$

Then we have

$$\begin{aligned} dx=&\{\boldsymbol{u}_1+(-\boldsymbol{u}_2k_2+\boldsymbol{u}_4k_3)\,a\sin u\cos\bar{v}-\boldsymbol{u}_3ak_3\sin u\sin\bar{v}\}d\sigma\\ &+\boldsymbol{u}_3a(\cos u\cos\bar{v}\,du-\sin u\sin\bar{v}\,d\bar{v})\\ &+\boldsymbol{u}_4a(\cos u\sin\bar{v}\,du+\sin u\cos\bar{v}\,d\bar{v}), \end{aligned}$$

from which

$$ds^2=(1+a^2k_2^2\sin^2 u\cos^2\bar{v})d\sigma^2+a^2\cos^2 u\,du^2+a^2\sin^2 u(d\bar{v}+k_3d\sigma)^2.$$

In order that x is an isometric immersion, it must be

$$\varphi=\int_0^u k_3\frac{d\sigma}{du}du+c,\qquad c=\text{constant} \tag{47}$$

and

$$\{1+a^2k_2^2\sin^2 u\cos^2(v-\varphi)\}(\boldsymbol{f}'\cdot\boldsymbol{f}')=\sin^2 u.$$

Since u and v are independent variables, it must be $k_2=0$. Hence, the curve $y\colon\ [0,\pi]\to E^4$ is a plane curve. Furthermore,

$$\boldsymbol{u}_3\cos\bar{v}+\boldsymbol{u}_4\sin\bar{v}=(\boldsymbol{u}_3\cos\varphi-\boldsymbol{u}_4\sin\varphi)\cos v+(\boldsymbol{u}_3\sin\varphi+\boldsymbol{u}_4\cos\varphi)\sin v$$

and from (38) and (47)

$$d(\boldsymbol{u}_3\cos\varphi-\boldsymbol{u}_4\sin\varphi)=d(\boldsymbol{u}_3\sin\varphi+\boldsymbol{u}_4\cos\varphi)=0.$$

Therefore, if x is not the trivial imbedding $S^2\to S^2\subset E^3$, then x must be equivalent to the one given in §4. Thus we get

THEOREM 4. *Any tubular isometric immersion of S^2 into E^4 with constant curvatures which is not equivalent to $S^2\to S^2\subset E^3$, is equivalent to the isometric immersion*

$$\begin{cases} x_1=a\displaystyle\int_0^u\sin u\sin\left(c\log\tan\frac{u}{2}\right)du,\\ x_2=a\displaystyle\int_0^u\sin u\cos\left(c\log\tan\frac{u}{2}\right)du,\\ x_3=a\sin u\cos v,\quad x_4=a\sin u\sin v,\qquad a,\ c\neq 0,\ \textit{constants}, \end{cases}$$

and it is of class C^1 and not of class C^2 on S^2 but analytic on the subset excluded the two poles from S^2.

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