ON THE EXISTENCE OF MEROMORPHIC FUNCTIONS WITH PREASSIGNED ASYMPTOTIC SPOTS

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In his paper [2], Heins introduced the notion of asymptotic spot of an interior transformation and then in [3], especially, he examined asymptotic spots of entire and meromorphic functions. Let $f(z)$ be meromorphic in $|z|<\infty$, and let w_0 denote a point of the extended w-plane. Then σ is called an asymptotic spot over w_0 when σ is a function (a correspondence from sets to sets) whose domain is the family Φ_{w_0} of simply-connected Jordan regions containing w_0 and which satisfies: (a) for each $\Omega\in\Phi_{w_0}$, $\sigma(\Omega)$ is a component of $f^{-1}(\Omega)$ which is not relatively compact, and (b) if $\Omega_1\subset\Omega_2$, for $\Omega_1, \Omega_2\in\Phi_{w_0}$, then $\sigma(\Omega_1)\subset\sigma(\Omega_2)$. Let $\mathfrak{G}_\Omega(w, w_0)$ denote Green's function for Ω with the pole at w_0. We put

$$u_{\sigma(\Omega)}(z)\equiv \text{G.H.M.}\, \mathfrak{G}_\Omega(f_{\sigma(\Omega)}(z), w_0),$$

where $f_{\sigma(\Omega)}(z)$ is the restriction of $f(z)$ to $\sigma(\Omega)$ and G.H.M. means the greatest hermonic minorant. We associate with the pair (σ, Ω) an index $h(\sigma, \Omega)$ as follows. If $u_{\sigma(\Omega)}(z)\equiv 0$, then $h(\sigma, \Omega)=0$. If $u_{\sigma(\Omega)}(z)>0$ and is represented as a finite sum of n mutually non-proportional minimal positive harmonic functions on $\sigma(\Omega)$, then $h(\sigma, \Omega)=n$. In the remaining case, we set $h(\sigma, \Omega)=+\infty$. The index $h(\sigma, \Omega)$ is monotone in Ω, i.e. if $\Omega_1\subset\Omega_2$, then $h(\sigma, \Omega_1)\leqq h(\sigma, \Omega_2)$. The harminic index $h(\sigma)$ of σ is then defined as

$$\inf_{\Omega\in\Phi_{w_0}} h(\sigma, \Omega).$$

Now Heins proposed the following realization problem: Let $w_1, \cdots, w_n$ denote $n(\geqq 1)$ given points on the extended w-plane and $h_1, \cdots, h_n$ denote n given positive integers. Does there exist a meromorphic function $f(z)$ in $|z|<\infty$ which satisfies: (I) the asymptotic spots of $f(z)$ having positive harmonic indices are n in number, say $\sigma_1, \cdots, \sigma_n$, (II) σ_k lies over w_k and $h(\sigma_k)=h_k$, (III) $f(z)$ is of order $H/2$, where $H=\sum_{k=1}^n h_k$?

The object of the present paper is to give a solution for this problem.

Heins showed an affirmative answer for the special cases: (i) $n=1$, (ii) $n=2$, $h_1=h_2=2$. As a direct consequence of the method which Heins used to construct an example of the case (ii), M. Ozawa has informed to the author an affirmative answer for the case (iii) $n=2$, $h_1=h_2=m$. In fact, it is shown that the argument similar to the case (ii) in [3] (p. 439) remains valid in the case (iii) by considering the starting function $g(z)=e^{-iz}\cos z^m$ in place of $g(z)=e^{-iz}\cos z^2$.

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Here we shall give an answer for the case: $n=2$, h_1, h_2, w_1 and w_2 unrestricted and further solve the general problem affimatively.

To this end we need some preparatory considerations. Suppose that G is a Jordan region in $|z|<\infty$ and that U is a harmonic function non-negative on G which vanishes on the boundary of G. Further suppose that $\{G_k\}$ is a family of Jordan subregions of G satisfying $G_k \cap G_l = \phi$ for $k \neq l$, and that U_k is a harmonic function non-negative on G_k which vanishes continuously on the boundary of G_k and is dominated by U on G_k. Let U_k^* denote the least harmonic majorant of the subharmonic function which is equal to U_k on G_k and to zero on $G-G_k$. Then we get the following lemma.

LEMMA. *Under the above assumption it holds*

$$\sum U_k^* \leqq U;$$

if each U_k is minimal in G_k, then U_1^*, U_2^*, $\cdots$ are minimal and mutually non-proportional in G.

The proof of the lemma is contained in (f) and (c) of [2] (pp. 442–445).

In [3], Heins formulated the Denjoy-Carleman-Ahlfors theorem and gave the following theorem (p. 431).

THEOREM A. *Let H denote the grand total of the harmonic indices of all the asymptotic spots of a non-constant meromorphic function f in $|z|<\infty$. Let $T(r; f)$ denote the Nevanlinna characteristic function of f. If $H=+\infty$, then*

$$\lim_{r\to\infty} \frac{\log T(r; f)}{\log r} = +\infty.$$

If $2 \leqq H < \infty$, then

$$\liminf_{r\to\infty} \frac{T(r; f)}{r^{H/2}} > 0.$$

If $H=1$ and the asymptotic spot σ_0 with index one is such that for some Ω of its domain, the complement of $\sigma_0(\Omega)$ intersects all circles $\{|z|=r\}$ with r sufficiently large, then

$$\liminf_{r\to\infty} \frac{T(r; f)}{r^{1/2}} > 0.$$

Now we observe Mittag-Leffler's function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)} \qquad (0<\alpha<2)$$

which is an entire function of order $1/\alpha$ and quote the following theorem (cf. § 3.62 in [1]):

If $0<\alpha<1$ there exists a constant K independent of α such that

$$(1) \qquad \left| E_\alpha(z) - \frac{\exp z^{1/\alpha}}{\alpha} + \frac{1}{z\Gamma(1-\alpha)} \right| \leqq \frac{K}{\alpha^2 |z|^2} \text{ for } |\arg z| \leqq \frac{3}{4}\alpha\pi,\ |z| \geqq 2,$$

$$(2)\qquad \left|E_\alpha(z)+\frac{1}{z\Gamma(1-\alpha)}\right|\leqq\frac{K}{\alpha^2|z|^2}\quad\text{for}\quad \frac{3}{4}\alpha\pi\leqq\arg z\leqq 2\pi-\frac{3}{4}\alpha\pi,\ |z|\geqq 2.$$

By using Mittag-Leffler's function $E_\alpha(z)$, we put

$$f_k(z)=E_{2/H}(z\varepsilon^{k-1})\qquad (k=1,2,\cdots,H(\geqq 3)),$$

where ε is a primitive H-th root of 1: $\varepsilon=\cos(2\pi/H)-i\sin(2\pi/H)$.

PROPOSITION 1. *The function*

$$f(z)=\sum_{j=1}^{h_1}f_j(z)\Big/\sum_{j=h_1+1}^{H}f_j(z)$$

has the desired properties for the case: $n=2$, $w_1=\infty$, $w_2=0$ and $H\geqq 3$.

Proof. We define an asymptotic spot σ_1 over $w_1=\infty$ as follows. For $-\pi/H\leqq\arg z\leqq-\pi/2H$

$$|f_1(z)|\geqq\frac{H}{2}|\exp z^{H/2}|-\frac{1}{|z|\Gamma(1-2/H)}-\frac{H^2K}{4|z|^2},$$

$$|f_H(z)|\leqq\frac{H}{2}|\exp z^{-H/2}|+\frac{1}{|z|\Gamma(1-2/H)}-\frac{H^2K}{4|z|^2}\leqq\frac{H}{2}+\frac{1}{|z|\Gamma(1-2/H)}+\frac{H^2K}{4|z|^2},$$

$$|f_j(z)|\leqq\frac{1}{|z|\Gamma(1-2/H)}+\frac{H^2K}{4|z|^2}\qquad (j=2,\cdots,H-1);$$

$$|f(z)|\geqq\left\{|f_1(z)|-\sum_{j=2}^{h_1}|f_j(z)|\right\}\Big/\sum_{j=h_1+1}^{H}|f_j(z)|$$

$$\geqq\left\{|\exp z^{H/2}|-\frac{2}{|z|\Gamma(1-2/H)}-\frac{H^2K}{2|z|^2}\right\}\Big/\left\{1+\frac{2}{|z|\Gamma(1-2/H)}+\frac{H^2K}{2|z|^2}\right\}.$$

For $|\arg z-2(k-1)\pi/H|\leqq\pi/2H$ $(k=1,\cdots,h_1)$

$$|zf_k(z)|\geqq\frac{H}{2}|\exp(z\varepsilon^{k-1})^{H/2}|-\frac{1}{\Gamma(1-2/H)}-\frac{H^2K}{4|z|},$$

$$|zf_j(z)|\leqq\frac{1}{\Gamma(1-2/H)}+\frac{H^2K}{4|z|}\qquad (j=1,\cdots,k-1,k+1,\cdots,H);$$

$$|f(z)|\geqq\left\{|zf_k(z)|-\sum_{j=1}^{k-1}|zf_j(z)|-\sum_{j=k+1}^{h_1}|zf_j(z)|\right\}\Big/\sum_{j=h_1+1}^{H}|zf_j(z)|$$

$$\geqq\left\{|z\exp(z\varepsilon^{k-1})^{H/2}|-\frac{2}{\Gamma(1-2/H)}-\frac{H^2K}{2|z|}\right\}\Big/\left\{\frac{2}{\Gamma(1-2/H)}+\frac{H^2K}{2|z|}\right\}$$

$$\geqq\left\{|z\cosh(z\varepsilon^{k-1})^{H/2}|-\frac{2}{\Gamma(1-2/H)}-\frac{H^2K}{2|z|}\right\}\Big/\left\{\frac{2}{\Gamma(1-2/H)}+\frac{H^2K}{2|z|}\right\}.$$

For $|\arg z+\pi/H-2k\pi/H|\leqq\pi/2H$ $(k=1,\cdots,h_1-1)$

$$\left|zf_k(z)-\frac{H}{2}z\exp(z\varepsilon^{k-1})^{H/2}\right|\leqq\frac{1}{\Gamma(1-2/H)}+\frac{H^2K}{4|z|},$$

$$\left|zf_{k+1}(z)-\frac{H}{2}\,z\exp(z\varepsilon^{k})^{H/2}\right|\leqq\frac{1}{\Gamma(1-2/H)}+\frac{H^2K}{4|z|},$$

$$|zf_j(z)|\leqq\frac{1}{\Gamma(1-2/H)}+\frac{H^2K}{4|z|}\qquad(j=1,\cdots,k-1,k+2,\cdots,H);$$

$$|f(z)|\geqq\left\{|zf_k(z)+zf_{k+1}(z)|-\sum_{j=1}^{k-1}|zf_j(z)|-\sum_{j=k+2}^{h_1}|zf_j(z)|\right\}\Big/\sum_{j=h_1+1}^{H}|zf_j(z)|$$

$$\geqq\left\{|z\cosh z^{H/2}|-\frac{1}{\Gamma(1-2/H)}-\frac{H^2K}{4|z|}\right\}\Big/\left\{\frac{2}{\Gamma(1-2/H)}+\frac{H^2K}{4|z|}\right\}.$$

And for $\pi/2H+2(h_1-1)\pi/H\leqq\arg z\leqq\pi/H+2(h_1-1)\pi/H$

$$|f_{h_1}(z)|\geqq\frac{H}{2}|\exp(z\varepsilon^{h_1-1})^{H/2}|-\frac{1}{|z|\Gamma(1-2/H)}-\frac{H^2K}{4|z|^2},$$

$$|f_{h_1+1}(z)|\leqq\frac{H}{2}|\exp(z\varepsilon^{h_1})^{H/2}|+\frac{1}{|z|\Gamma(1-2/H)}+\frac{H^2K}{4|z|^2}$$

$$\leqq\frac{H}{2}+\frac{1}{|z|\Gamma(1-2/H)}+\frac{H^2K}{4|z|^2},$$

$$|f_j(z)|\leqq\frac{1}{|z|\Gamma(1-2/H)}+\frac{H^2K}{4|z|^2}\qquad(j=1,\cdots,h_1-1,h_1+2,\cdots,H);$$

$$|f(z)|\geqq\left\{|f_{h_1}(z)|-\sum_{j=1}^{h_1-1}|f_j(z)|\right\}\Big/\sum_{j=h_1+1}^{H}|f_j(z)|$$

$$\geqq\left\{|\exp(z\varepsilon^{h_1-1})^{H/2}|-\frac{2}{|z|\Gamma(1-2/H)}-\frac{H^2K}{2|z|^2}\right\}\Big/\left\{1+\frac{2}{|z|\Gamma(1-2/H)}+\frac{H^2K}{2|z|^2}\right\}.$$

From these inequalities we see that if M is sufficiently large the open set $\{z;\ |f(z)|>M\}$ contains the union G_1 of regions

$$\left\{z;\ |\exp z^{H/2}|>M^2,\ -\frac{\pi}{H}<\arg z\leqq-\frac{\pi}{2H}\right\},$$

$$\left\{z;\ |z\cosh z^{H/2}|>M^2,\ -\frac{\pi}{2H}\leqq\arg z\leqq\frac{\pi}{2H}+\frac{2(h_1-1)\pi}{H}\right\}$$

and

$$\left\{z;\ |\exp(z\varepsilon^{h_1-1})^{H/2}|>M^2,\ \frac{\pi}{2H}+\frac{2(h_1-1)\pi}{H}\leqq\arg z<\frac{\pi}{H}+\frac{2(h_1-1)\pi}{H}\right\}.$$

Clearly the set G_1 is an unbounded region. We define an asymptotic spot σ_1 over $w_1=\infty$ by putting $\sigma_1(|w|>M)\equiv$the component of $f^{-1}(|w|>M)$ containing G_1. Clearly for every $\Omega\in\Phi_{w_1}$, $\sigma(\Omega)$ is well defined suitably. Next we get $h(\sigma_1)\geqq h_1$. In fact, for sufficiently large M the inequality

$$\log\frac{|f(z)|}{M}\geqq U_k\equiv\mathrm{Re}(z\varepsilon^{k-1})^{H/2}-2\log M$$

holds in the region

$$\Delta_k\colon \left\{z;\ U_k(z)>0,\ -\frac{\pi}{H}+\frac{2(k-1)\pi}{H}<\arg z<\frac{\pi}{H}+\frac{2(k-1)\pi}{H}\right\},\ k=1,\ \cdots,\ h_1.$$

Now $\log(|f(z)|/M)$ being superharmonic, $u_{\sigma_1(|w|>M)}(z)\equiv$G.H.M. $\log(|f(z)|/M)$ is non-negative and $u_{\sigma_1(|w|>M)}(z)\geqq U_k(z)$ in Δ_k. Since $U_k(z)$ is minimal in Δ_k, $u_{\sigma_1(|w|>M)}(z)$ dominates at least h_1 mutually non-proportional minimal functions by Lemma. Therefore we get $h(\sigma_1,\ |w|>M)\geqq h_1$ for every large M, and hence $h(\sigma_1)\geqq h_1$.

Similarly we can find an asymptotic spot σ_2 over $w_2=0$ having $h(\sigma_2)\geqq h_2$. In fact, let a set G_2 be the union of regions

$$\left\{z;\ |\exp(z\varepsilon^{h_1-1})^{H/2}|>M^2,\ -\frac{\pi}{H}+\frac{2h_1\pi}{H}<\arg z\leqq-\frac{\pi}{2H}+\frac{2h_1\pi}{H}\right\},$$

$$\left\{z;\ |z\cosh z^{H/2}|>M^2,\ -\frac{\pi}{2H}+\frac{2h_1\pi}{H}\leqq\arg z\leqq\frac{\pi}{2H}+\frac{2(H-1)\pi}{H}\right\}$$

and

$$\left\{z;\ |\exp(z\varepsilon^{H-1})^{H/2}|>M^2,\ \frac{\pi}{2H}+\frac{2(H-1)\pi}{H}\leqq\arg z<\frac{\pi}{H}+\frac{2(H-1)\pi}{H}\right\}.$$

Then the set $\{z;\ |f(z)|<1/M\}$ contains G_2. If an asymptotic spot σ_2 over $w_2=0$ is defined by putting $\sigma_2(|w|<1/M)\equiv$the component of $f^{-1}(|w|<1/M)$ containing G_2, we get $h(\sigma_2)\geqq h_2$ by the argument similar to the case of σ_1.

The order ρ of $f(z)$ is at most $H/2$ since $E_\alpha(z)$ is of order $1/\alpha$. On the other hand, we get, by Theorem A, $\bar{H}\leqq 2\rho$ for the grand total $\bar{H}$ of the harmonic indices of all the asymptotic spots of f. Consequently we have

$$H=h_1+h_2\leqq h(\sigma_1)+h(\sigma_2)\leqq\bar{H}\leqq 2\rho\leqq H,$$

and hence

$$\rho=\frac{H}{2},\ h(\sigma_1)=h_1,\ h(\sigma_2)=h_2\ \text{ and }\ \bar{H}=H.$$

We thus obtain the desired result.

For arbitrary w_1 and w_2, if $w_1\neq w_2$ it suffices to consider a function $L\circ f$ where L is a linear fractional transformation satisfying $L(\infty)=w_1$, $L(0)=w_2$, and if $w_1=w_2$ it suffices to consider a function $f+1/f$ or $1/(f+1/f)+w_1$ according to $w_1=\infty$ or $w_1\neq\infty$. Here we remark that a set $\{z;\ |f+1/f|>M\}$ has two desired unbounded components. For on the half rays $\{z;\ \arg z=-\pi/H\}$ and $\{z;\ \arg z=\pi/H+2(h_1-1)\pi/H\}$ we get $|f+1/f|\leqq 3$ for every large $|z|$.

The assumption $H\geqq 3$ is not essential. For if $H=2$ and $n=2$ we have a particular function $\exp z$ as the above f.

Now we shall treat the general problem. Let $w_1,\ \cdots,\ w_n$ ($n\geqq 3$) denote n given points in the extended w-plane, and $h_1,\ \cdots,\ h_n$ denotes n given positive integers. We suppose without loss of generality that the set $\{w_k;\ k=1,\ \cdots,\ n\}$ does not contain the point at infinity. For the required properties are invariant under any linear fractional transformation of values of an admissible function.

Again by using Mittag-Leffler's function, we put

$$f_j(z)=E_{2/H}(z\varepsilon^{j-1}) \qquad (j=1,\cdots,H(=h_1+\cdots+h_n\geqq 3)),$$

where ε is a primitive H-th root of 1: $\varepsilon=\cos(2\pi/H)-i\sin(2\pi/H)$. From $f_j(z)$ we construct a function $\tilde{f}_k(z)$ associated with h_k as follows. If $h_k=1$, we put

$$\tilde{f}_k(z)=f_1(z).$$

If $h_k>1$ we put

$$\tilde{f}_k(z)=\left(\sum_{j=1}^{h_k} f_j(z)\right)g_k(z),$$

where $g_k(z)$ is defined by

$$g_k(z)=\begin{cases} E_{2h_k/H}(z\varepsilon^{(h_k-1)/2}) & \text{for } 2h_k<H,\\ E_{2(H-h_k)/H}(z\varepsilon^{H+h_k-1})^{-1} & \text{for } 2h_k>H,\\ \exp z\varepsilon^{(h_k-1)/2} & \text{for } 2h_k=H.\end{cases}$$

PROPOSITION 2. *The function*

$$f(z)=\left\{\sum_{k=1}^{n} w_k\tilde{f}_k(z\varepsilon^{h_1+\cdots+h_{k-1}})+A\right\}\Big/\sum_{k=1}^{n}\tilde{f}_k(z\varepsilon^{h_1+\cdots+h_{k-1}})$$

has the required properties provided A is a sufficiently large number.

Proof. We first examine the properties of $f_k(z)$. From the estimations obtained in (1) and (2) we get

$$\left|\sum_{j=1}^{h_k} f_j(z)\right|\geqq\frac{H}{2}\,|\cosh z^{H/2}|-\frac{H}{|z|\Gamma(1-2/H)}-\frac{H^3K}{4|z|^2}$$

$$\text{for } -\frac{\pi}{H}\leqq\arg z\leqq-\frac{\pi}{H}+\frac{2h_k\pi}{H},$$

$$\left|\sum_{j=1}^{h_k} f_j(z)\right|\leqq\frac{H}{2}\,|\exp(z\varepsilon^{h_k-1})^{H/2}|+\frac{H}{|z|\Gamma(1-2/H)}+\frac{H^3K}{4|z|^2}$$

$$\text{for } -\frac{\pi}{H}+\frac{2h_k\pi}{H}\leqq\arg z\leqq-\frac{\pi}{2H}+\frac{2h_k\pi}{H},$$

$$\left|\sum_{j=1}^{h_k} f_j(z)\right|\leqq\frac{H}{|z|\Gamma(1-2/H)}+\frac{H^3K}{4|z|^2}$$

$$\text{for } -\frac{\pi}{2H}+\frac{2h_k\pi}{H}\leqq\arg z\leqq 2\pi-\frac{3\pi}{2H},$$

$$\left|\sum_{j=1}^{h_k} f_j(z)\right|\leqq\frac{H}{2}\,|\exp z^{H/2}|+\frac{H}{|z|\Gamma(1-2/H)}+\frac{H^3K}{4|z|^2}$$

$$\text{for } -\frac{3\pi}{2H}\leqq\arg z\leqq-\frac{\pi}{H}.$$

Hence for sufficiently large $|z|$ we have

(3) $$\left|\sum_{j=1}^{h_k} f_j(z)\right| \geqq \frac{H}{4}|\cosh z^{H/2}| \quad \text{for } -\frac{\pi}{H} \leqq \arg z \leqq -\frac{\pi}{H}+\frac{2h_k\pi}{H},$$

(4) $$\left|\sum_{j=1}^{h_k} f_j(z)\right| \leqq H \quad \text{for } -\frac{\pi}{H}+\frac{2h_k\pi}{H} \leqq \arg z \leqq 2\pi-\frac{\pi}{H}.$$

Concerning $g_k(z)$ we have

$$|g_k(z)| \geqq \frac{H}{2h_k}|\exp(z\varepsilon^{(h_k-1)/2})^{H/2h_k}| - \frac{1}{|z|\Gamma(1-2h_k/H)} - \frac{H^2K}{4h_k^2|z|^2}$$

$$\text{for } -\frac{\pi}{H} \leqq \arg z \leqq -\frac{\pi}{H}+\frac{2h_k\pi}{H},$$

$$|g_k(z)| \leqq \frac{H}{2h_k}|\exp(z\varepsilon^{(h_k-1)2})^{H/2h_k}| + \frac{1}{|z|\Gamma(1-2h_k/H)} + \frac{H^2K}{4h_k^2|z|^2}$$

$$\text{for } -\frac{\pi}{H}-\frac{h_k\pi}{2H} \leqq \arg z \leqq -\frac{\pi}{H} \text{ and for } -\frac{\pi}{H}+\frac{2h_k\pi}{H} \leqq \arg z \leqq -\frac{\pi}{H}+\frac{5h_k\pi}{2H},$$

$$|g_k(z)| \leqq \frac{1}{|z|\Gamma(1-2h_k/H)} + \frac{H^2K}{4h_k^2|z|}$$

$$\text{for } -\frac{\pi}{H}+\frac{5h_k\pi}{2H} \leqq \arg z \leqq 2\pi-\frac{\pi}{H}-\frac{h_k\pi}{2H}$$

if $2h_k<H$. If $2h_k>H$, then we have

$$|g_k(z)| \geqq \left\{\frac{H}{2(H-h_k)}|\exp(z\varepsilon^{(H+h_k-1)})^{H/(2H-2h_k)}| + \frac{1}{|z|\Gamma(2h_k/H-1)} + \frac{H^2K}{4(H-h_k)^2|z|^2}\right\}^{-1}$$

$$\text{for } 2\pi-\frac{\pi}{H} \leqq \arg z \leqq \frac{5\pi}{2}-\frac{\pi}{H}-\frac{h_k\pi}{2H} \text{ and}$$

$$\text{for } -\frac{\pi}{2}-\frac{\pi}{H}-\frac{5h_k\pi}{2H} \leqq \arg z \leqq -\frac{\pi}{H}+\frac{2h_k\pi}{H},$$

$$|g_k(z)| \geqq \left\{\frac{1}{|z|\Gamma(2h_k/H-1)} + \frac{H^2K}{4(H-h_k)^2|z|^2}\right\}^{-1}$$

$$\text{for } \frac{\pi}{2}-\frac{\pi}{H}-\frac{h_k\pi}{2H} \leqq \arg z \leqq -\frac{\pi}{2}-\frac{\pi}{H}+\frac{5h_k\pi}{2H},$$

$$|g_k(z)| \leqq \left\{\frac{H}{2(H-h_k)}|\exp(z\varepsilon^{(H+h_k-1)})^{H/(2H-2h_k)}| - \frac{1}{|z|\Gamma(2h_k/H-1)} - \frac{H^2K}{4(H-h_k)^2|z|^2}\right\}^{-1}$$

$$\text{for } -\frac{\pi}{H}-\frac{2h_k\pi}{H} \leqq \arg z \leqq 2\pi-\frac{\pi}{H}.$$

Further if $2h_k=H$, then we have

$$|g_k(z)| = |\exp z\varepsilon^{(h_k-1)/2}|.$$

Thus, if $2h_k<H$, then we have for sufficiently large $|z|$

$$(5) \qquad |g_k(z)| \geqq \frac{1}{2}\cdot\exp(z\varepsilon^{(h_k-1)/2})^{H/2h_k}| \quad \text{for } -\frac{\pi}{H} \leqq \arg z \leqq -\frac{\pi}{H}+\frac{2h_k\pi}{H},$$

$$(6) \qquad |g_k(z)| \leqq H \qquad \text{for } -\frac{\pi}{H}+\frac{2h_k\pi}{H} \leqq \arg z \leqq 2\pi-\frac{\pi}{H}.$$

If $2h_k > H$, then we have for sufficiently large $|z|$

$$(7) \qquad |g_k(z)| \geqq \left\{H|\exp(z\varepsilon^{(H+h_k-1)})^{H/(2H-2h_k)}|+\frac{1}{|z|}\right\}^{-1}$$

$$\text{for } 2\pi-\frac{\pi}{H} \leqq \arg z \leqq \frac{5\pi}{2}-\frac{\pi}{H}-\frac{h_k\pi}{2H} \text{ and}$$

$$\text{for } -\frac{\pi}{2}-\frac{\pi}{H}+\frac{5h_k\pi}{2H} \leqq \arg z \leqq -\frac{\pi}{H}+\frac{2h_k\pi}{H},$$

$$(8) \qquad |g_k(z)| \geqq |z| \qquad \text{for } \frac{\pi}{2}-\frac{\pi}{H}-\frac{h_k\pi}{2H} \leqq \arg z \leqq -\frac{\pi}{2}-\frac{\pi}{H}+\frac{5h_k\pi}{2H},$$

$$(9) \qquad |g_k(z)| \leqq 1 \qquad \text{for } -\frac{\pi}{H}+\frac{2h_k\pi}{H} \leqq \arg z \leqq 2\pi-\frac{\pi}{H}.$$

If $2h_k = H$, then we have

$$(10) \qquad |g_k(z)| = |\exp z\varepsilon^{(h_k-1)/2}| \quad \text{for} -\frac{\pi}{H} \leqq \arg z \leqq -\frac{\pi}{H}+\frac{2h_k\pi}{H},$$

$$(11) \qquad |g_k(z)| \leqq 1 \qquad \text{for } -\frac{\pi}{H}+\frac{2h_k\pi}{H} \leqq \arg z \leqq 2\pi-\frac{\pi}{H}.$$

Therefore from (3) and (5), for sufficiently large M, the set $\{z;\ |\tilde{f}_k(z)|>M\}$ contains an unbounded region

$$\tilde{G}_k(M) \equiv \left\{z;\ |\cosh z^{H/2}\exp(z\varepsilon^{(h_k-1)})^{H/2h_k}|>M^2,\ -\frac{\pi}{H} \leqq \arg z \leqq -\frac{2h_k\pi}{H}\right\}$$

when $2h_k<H$. Or from (3), (7) and (8), for sufficiently M, the set $\{z;\ |\tilde{f}_k(z)|>M\}$ contains an unbounded region $\tilde{G}_k(M)$ which is a union of regions

$$\left\{z;\ |\cosh z^{H/2}|\left\{H|\exp(z\varepsilon^{(H+h_k-1)})^{H/(2H-2h_k)}|+\frac{1}{|z|}\right\}^{-1}>M^2,\right.$$

$$\left.2\pi-\frac{\pi}{H} \leqq \arg z \leqq \frac{5\pi}{2}-\frac{\pi}{H}-\frac{h_k\pi}{2H} \text{ and } -\frac{\pi}{2}-\frac{\pi}{H}+\frac{5h_k\pi}{2H} \leqq \arg z \leqq -\frac{\pi}{H}-\frac{2h_k\pi}{H}\right\}$$

and

$$\left\{z;\ |z\cosh z^{H/2}|>M^2,\ \frac{\pi}{2}-\frac{\pi}{H}-\frac{h_k\pi}{2H} \leqq \arg z \leqq -\frac{\pi}{2}-\frac{\pi}{H}+\frac{5h_k\pi}{2H}\right\}$$

when $2h_k>H$. Or from (3) and (10), for sufficiently large M, the set $\{z;\ |\tilde{f}_k(z)|>M\}$ contains an unbounded region

$$\tilde{G}_k(M) \equiv \left\{z;\ |\cosh z^{H/2} \exp(z\varepsilon^{(h_k-1)/2})| > M^2,\ -\frac{\pi}{H} \leqq \arg z \leqq -\frac{\pi}{H} + \frac{2h_k\pi}{H}\right\}$$

when $2h_k = H$. Moreover we have

$$(12) \qquad |\tilde{f}_k(z)| \geqq |\cosh z^{H/2}|$$

in the set $\{z;\ |\cosh z^{H/2}| > M^2,\ -\pi/H < \arg z < -\pi/H + 2h_k\pi/H\}$ which is contained in $\tilde{G}_k(M)$ for every k. Further from (4), (6), (9) and (11) we have

$$(13) \qquad |\tilde{f}_k(z)| \leqq H^2 \qquad \text{for } -\frac{\pi}{H} + \frac{2h_k\pi}{H} \leqq \arg z \leqq 2\pi - \frac{\pi}{H}$$

for every k.

Now we define an asymptotic spot σ_k over w_k as follows. Let G_k be the obtained from $\tilde{G}_k$ by the rotation $z \to z\varepsilon^{h_1+\cdots+h_{k-1}}$. Then for $z \in G_k(M^2)$ we have

$$|\tilde{f}_k(z\varepsilon^{h_1+\cdots+h_{k-1}})| > M^2$$

and

$$|\tilde{f}_j(z\varepsilon^{h_1+\cdots+h_{j-1}})| < H^2 \qquad (j=1, \cdots, k-1, k+1, \cdots, n),$$

and hence for a sufficiently large M

$$\begin{aligned}
&|f(z) - w_k| \\
&\leqq \left\{\sum_{j \neq k} |w_j - w_k|\,|\tilde{f}_j(z\varepsilon^{h_1+\cdots+h_{j-1}})| + |A|\right\} \Big/ \left\{|\tilde{f}_k(z\varepsilon^{h_1+\cdots+h_{k-1}})| - \sum_{j \neq k} |\tilde{f}_j(z\varepsilon^{h_1+\cdots+h_{j-1}})|\right\} \\
&\leqq \left\{\sum_{j \neq k} |w_j - w_k| H^2 + |A|\right\} \Big/ (M^2 - H^3) \\
&\leqq \frac{1}{M}.
\end{aligned}$$

Therefore the set $\{z;\ |f(z) - w_k| < 1/M\}$ contains the region $G_k(M^2)$. We then define $\sigma_k(|w - w_k| < 1/M)$ as the compoment of $\{z;\ |f(z) - w_k| < 1/M\}$ containing $G_k(M^2)$. Further we see that all the spots σ_k, $k=1, \cdots, n$, are different each other. In fact, by (13) we have

$$\begin{aligned}
|f(z) - w_k| &\geqq \left\{|A| - \sum_{j \neq k} |w_j - w_k|\,|\tilde{f}_j(z\varepsilon^{h_1+\cdots+h_{j-1}})/\right\} \Big/ \sum_{j=1}^{n} |\tilde{f}_j(z\varepsilon^{h_1+\cdots+h_{j-1}})| \\
&\geqq \left\{|A| - \sum_{j \neq k} |w_j - w_k| H^2\right\} \Big/ nH^2
\end{aligned}$$

on the half rays $\{z;\ \arg z = -\pi/H + 2(h_1 + \cdots + h_{k-1})\pi/H\}$ and $\{z;\ \arg z = -\pi/H + 2(h_1 + \cdots + h_k)\pi/H\}$ and if A is a sufficiently large constant there exists a positive number d such that $|f(z) - w_k| > d > 0$.

We next show that $h(\sigma_k) \geqq h_k$. In $G_k(M^2)$ we have

$$\frac{1}{M|f(z) - w_k|}$$

$$\geqq\left\{|\tilde{f}_k(z\varepsilon^{h_1+\cdots+h_{k-1}})|-\sum_{j\neq k}|\tilde{f}_j(z\varepsilon^{h_1+\cdots+h_{j-1}})|\right\}\Big/M\left(|A|+\sum_{j\neq k}|w_j-w_k||\tilde{f}_j(z\varepsilon^{h_1+\cdots+h_{j-1}})\right)$$

$$\geqq\left\{|\tilde{f}_k(z\varepsilon^{h_1+\cdots+h_{k-1}})|-nH^2\right\}\Big/M\left(|A|+\sum_{j\neq k}|w_j-w_k|H^2\right)$$

$$\geqq|\tilde{f}_k(z\varepsilon^{h_1+\cdots+h_{k-1}})|/M^2$$

for a sufficiently large M. By (12) and by Lemma, the function

$$u_{\sigma_k(|w-w_k|<1/M)}(z)\equiv\text{G.H.M.}\frac{1}{M|f(z)-w_k|}$$

dominates at least h_k mutually non-proportional minimal functions. Therefore $h(\sigma_k, |w-w_k|<1/M)\geqq h_k$ and hence $h(\sigma_k)\geqq h_k$.

The order ρ of $f(z)$ is at most $H/2$ since $f_j(z)$ is at most of order $H/2$ and $g_k(z)$ is of order $H/2h_k$ or $H/(2H-2h_k)$. On the other hand, we get, by Theorem A, $\bar{H}\leqq 2\rho$ for the grand total $\bar{H}$ of the harmonic indices of all the asymptotic spots of f. Consequently we have

$$H=h_1+\cdots+h_n\leqq h(\sigma_1)+\cdots+h(\sigma_n)\leqq\bar{H}\leqq 2\rho\leqq H,$$

and hence

$$\rho=\frac{H}{2},\ h(\sigma_1)=h_1,\ \cdots,\ h(\sigma_n)=h_n,\ \text{and}\ \bar{H}=H.$$

We thus have the desired result.

Finally as a direct consequence of Propositions 1 and 2, we have the following theorem:

THEOREM. *Let* $w_1, \cdots, w_n$ *denote* $n(\geqq 1)$ *given points on the extended plane and* $h_1, \cdots, h_n$ *denote* n *given positive integers. Then there exists a meromorphic function* $f(z)$ *in* $|z|<\infty$ *which satisfies* (I) *the asymptotic spots of* $f(z)$ *with positive harmonic indices are* n *in number, say* $\sigma_1, \cdots, \sigma_n$, (II) σ_k *lies over* w_k *and* $h(\sigma_k)=h_k$, (III) $f(z)$ *is of order* $H/2$, *where* $H=\sum_{k=1}^n h_k$.

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