ON APPROXIMATIONS TO SOME LIMITING DISTRIBUTIONS WITH APPLICATIONS TO THE THEORY OF SAMPLING INSPECTIONS BY ATTRIBUTES

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§1. Introduction and summary.

In the previous papers [7], [8], [11] and [12], we have discussed on the several approximations to the probability distributions and noted their applications. The purposes of this paper are to continue and extend our discussions, hence this paper which is a continuation of [8] and [11] may be seen as the Part III of them.

Poisson approximations to binomial distribution and to Poisson binomial distribution were treated ([2], [3]), but in [2] the expressions of the evaluations for the error term are not quite simple ones. In §2, using the analogous technique to that in [8], we evaluate the errors of the approximation in term of p under some restrictive conditions, and remark that when binomial distributionis replaced by negative binomial distribution, the similar results hold. Based upon these results, we can deal with some of its applications to the sampling inspection theory.

Binomial approximation to Poisson binomial distribution was treated by LeCam [6]. In §3, we shall show that first approximating term of the above approximation can be expressed as the sum of binomial distribution and its difference of the second order. The error terms of the approximations of the approximation are evaluated in terms of the square sum of Δp_k where $\Delta p_k = p_k - \bar{p}$.[1)]

The evaluation of the error of the normal opproximation to the binomial distribution is given in [3] and [7]. In §4, we shall treat the normal approximation to the Poisson binomial distribution which seems to be not investigated ever. For evaluation of the error of approximation, we utilize the results of our previous paper [7], and obtain the similar expression to the results of [7].

Finally in order to show the applicability of our results in §1, we shall proceed to some problems on sampling inspections by attributes based on prior distribution, and add some remarks and tables.

In [9] and [10], we have also stated some results concerning with those problems from the other view points.

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1) Meaning of the notations of p_k and $\bar{p}$ are described in [8].

§2. Poisson approximations to binomial distribution and negative binomial distribution.

Poisson approximation to Poisson binomial distribution was discussed in [8], and the method we have used there enables us to simplify the results of [11]; the evaluation term of the error of the approximation can be expressed by a polynomial of p. Thus we have the following

THEOREM 1. *Let X, Y and Z are random variables whose probability distributions are given by*

$$P(X=k)=b(k:\ n,\ p)=\binom{n}{k}p^k(1-p)^{n-k} \qquad (0\leqq k\leqq n),$$

$$(2.1)\qquad P(Y=k)=\pi(k:\ h,\ d)=\frac{h(h+d)\cdots(h+\overline{k-1}\,d)}{k\,!}(1+d)^{-d/h-k}$$

$$P(Z=k)=p(k:\ \lambda)=\frac{\lambda^k e^{-\lambda}}{k\,!} \qquad (0\leqq k),$$

respectively, then we have

$$(2.2)\qquad P(l'+1\leqq X\leqq l)=P(l'+1\leqq Z\leqq l)+R_0,$$

$$(2.3)\qquad P(l'+1\leqq X\leqq l)=P(l'+1\leqq Z\leqq l)+\frac{\lambda p}{2}\{\Delta^2P(l)-\Delta^2P(l')\}+R_1,$$

$$(2.4)\qquad \begin{aligned} P(l'+1\leqq X\leqq l)&=P(l'+1\leqq Z\leqq l)+\frac{\lambda p}{2}\{\Delta^2P(l)-\Delta^2P(l')\}\\ &\quad+\frac{\lambda p^2}{3}\{\Delta^3P(l)-\Delta^3P(l')\}+\frac{\lambda^2p^2}{8}\{\Delta^4P(l)-\Delta^4P(l')\}+R_2\end{aligned}$$

and $\Delta^iP(\cdot)$ which are the i-th differences of Poisson distribution term have the same meaning as in [11], *where*

$$(2.5)\qquad |R_0|<p \ \ or \ \ \frac{1}{2}p+5p^2,\ |R_1|<5p^2 \ \ or \ \ 3p^2+20p^3,\ |R_2|<65p^3,$$

provided $p<1/10$.

If in [11], *X is replaced by Y, we have the similar relations substituting h and d in the place of λ and p respectively and changing the signs before the terms of order p and p^3.*

Proofs are quite similar to those stated in [8] and [11], but need the cumbersome calculations, so we shall omit their proofs. We wish to remark some propositions.

REMARK. We can utilize our approximation formula for negative binomial distribution as cited above, only changing the two signs which is the quite simple procedure. The errors caused by those formulas are expected to be the same as in the case of the Poisson approximation to binomial distribution. For such a circumstance we may quote the paper by Patil [13] which shows the close connections between the binomial distribution and negative binomial distribution.

In this place we wish to remark also that our Poisson approximation formula can be utilized for the calculation of imcomplete beta function.

CALCULATIONS. We have calculated the differences between the value of the first and the second approximation formula and the value of the binomial distribution by making the tables of the first and the second differences as in Fig. 1.

1) We find the following region of (n, p) with the error bounded by 0.001, which means our formula ensures the precision of three decimals under 0. When we use the first approximation formula, the region is $10 \leqq n \leqq 40$, $p \leq 0.15$ and the second approximation gives the region $5 \leqq n \leqq 10$, $p \leq 0.25$ and $10 \leqq n \leqq 40$, $p \leqq 0.30$.

2) In practice, for the sake of convenience of the simple calculations, frequently binomial distribution is replaced by the Poisson distribution. But the error caused by such an approximation seems to be not estimated by simple term.

In this place we show the result of this evaluation which has proceeded by evaluation of the first approximation term and by noticing the fact that the second approximation term is negligible. The evaluation of the first approximation term was done by the fact that the difference of the second order is maximized at $k=\lambda+1/2 \pm \sqrt{\lambda+1/4}$ (see Fig. 2).

$\lambda = 1$

c	Poisson	Δ	Δ^2	Δ^3	Δ^4
⋮	⋮	⋮	⋮	⋮	⋮
3	9810	0613	−0460	0766	−0428
2	9197	1839	−1226	0614	0152
1	7358	3679	−1840	−1840	2454
0	3679	3679	0000	−3679	1839
	0000	0000	3679	3679	−7358
			0000	0000	3679
					0000

Fig. 1.

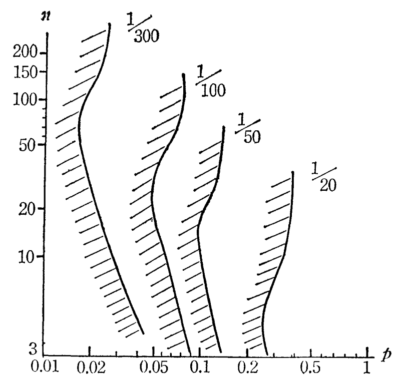

Fig. 2. Areas with error bounded by 1/300∼1/20.

§ 3. Binomial approximations to Poisson binomial distribution.

Let X_k be a random variable such that $P(X_k=1)=p_k$ and $P(X_k=0)=1-p_k=q_k$, then $\sum_{k=1}^{n} X_k = S$ is a Poisson binomial distribution random variable. In this section we shall treat the case when p_k is not small but its variation $\sigma_p^2 = \sum_{k=1}^{n}(p_k-\bar{p})^2$ (where $\bar{p}=(1/n)\sum_{k=1}^{n} p_k$) is very small, we have already solved the case when $\max_k p_k$ is very small. The problem of this section was firstly discussed by LeCam [6], on the other hand our object is to show that the distribution of S can be approximated by use of the difference of the term of the binomial distribution, and to estimate the evaluation of the error associated with this approximation formula.

Relations between binomial distribution and Poisson binomial distribution are also discussed by Feller [2].

We shall set forth our discussion in three steps for the sake of simplicity.

1. *Taylor's expansion.*

As we have done in [8] and [11], we start from the following characteristic function:

$$E(e^{ist})=f_n(t)=\prod_{k=1}^{n}\{1+p_k(e^{it}-1)\}. \tag{3. 1}$$

From (3. 1), we have

$$\log f_n(t)=\sum_{k=1}^{n}\log\{1+p_k(e^{it}-1)\}=\sum_{k=1}^{n}\log\{(\bar{p}e^{it}+\bar{q})+\Delta p_k(e^{it}-1)\} \tag{3. 2}$$

where $\bar{p}=(1/n)\sum_k p_k$ and $\Delta p_k=p_k-\bar{p}$, $\bar{q}=1-\bar{p}$. Restricting $\bar{p}<1/2$ which does not lose its generality, we further obtain

$$\log f_n(t)=n\log(\bar{p}e^{it}+\bar{q})+\sum_{k=1}^{n}\log\left\{1+\frac{\Delta p_k(e^{it}-1)}{\bar{p}e^{it}+\bar{q}}\right\}. \tag{3. 3}$$

Assuming Δp_k $(k=1, 2, \cdots, n)$ are all so small that $2|\Delta p_k|/(\bar{q}-\bar{p})<1/2$ we can expand the second term of (3. 3):

$$\begin{aligned}&\log\left\{1+\frac{\Delta p_k(e^{it}-1)}{\bar{p}e^{it}+\bar{q}}\right\}\\ &=\frac{\Delta p_k(e^{it}-1)}{\bar{p}e^{it}+\bar{q}}-\frac{1}{2}(\Delta p_k)^2\left(\frac{e^{it}-1}{\bar{p}e^{it}+\bar{q}}\right)^2+\frac{1}{3}(\Delta p_k)^3\left(\frac{e^{it}-1}{\bar{p}e^{it}+\bar{q}}\right)^3+\Theta_3^{(k)}\end{aligned} \tag{3. 4}$$

or

$$=\frac{\Delta p_k(e^{it}-1)}{\bar{p}e^{it}+\bar{q}}-\frac{1}{2}(\Delta p_k)^2\left(\frac{e^{it}-1}{\bar{p}e^{it}+\bar{q}}\right)^2+\Theta_2^{(k)} \tag{3. 5}$$

or

$$=\frac{\Delta p_k(e^{it}-1)}{\bar{p}e^{it}+\bar{q}}+\Theta_1^{(k)} \tag{3. 6}$$

where

$$\Theta_j^{(k)}=(-1)^j\int_0^{\Delta p_k(e^{it}-1)/(\bar{p}e^{it}+\bar{q})}\frac{z^j}{1+z}dz \tag{3. 7}$$

and

$$\begin{aligned}|\Theta_j^{(k)}|&\leqq\frac{1}{1-2\Delta p_k/(1-2\bar{p})}\cdot\frac{1}{j+1}\cdot\frac{|\Delta p_k|^{j+1}\cdot|e^{it}-1|^{j+1}}{(\bar{q}-\bar{p})^{j+1}}\\ &\leqq\frac{1}{j+1}\cdot\frac{1}{1-2\alpha/(1-2\bar{p})}\frac{1}{(1-2\bar{p})^{j+1}}\cdot|\Delta p_k|^{j+1}\cdot|e^{it}-1|^{j+1}\end{aligned}$$

and $\alpha=\max_k|\Delta p_k|$. Operating the exponential calculus to the above both sides, we have

$$f_n(t)=(\bar{p}e^{it}+\bar{q})^n\cdot\exp\left[-\frac{1}{2}\left(\frac{e^{it}-1}{\bar{p}e^{it}+\bar{q}}\right)^2\left\{\sum_{k=1}^{n}(\Delta p_k)^2\right\}\right]$$

(3. 8)

$$\cdot\exp\left[\frac{1}{3}\left(\frac{e^{it}-1}{\bar{p}e^{it}+\bar{q}}\right)^3\cdot\left\{\sum_{k=1}^{n}(\Delta p_k)^3\right\}\right]\cdot e^{\Theta_3}$$

or

(3. 9) $$f_n(t)=(\bar{p}e^{it}+\bar{q})^n\cdot\exp\left[\left\{-\frac{1}{2}\left(\frac{e^{it}-1}{\bar{p}e^{it}+\bar{q}}\right)^2\right\}\cdot\left\{\sum_{k=1}^{n}(\Delta p_k)^2\right\}\right]\cdot e^{\Theta_2}$$

or

(3. 10) $$f_n(t)=(\bar{p}e^{it}+\bar{q})^n\cdot e^{\Theta_1}$$

where we must note that $\sum_{k=1}^{n}\Delta p_k=0$ and $\Theta_j=\sum_{k=1}^{n}\Theta_j^{(k)}$.

In the above expression, if we pay our attention to (3. 9), for example, we have

$$f_n(t)=(\bar{p}e^{it}+\bar{q})^n\cdot\left[1-\frac{1}{2}\left\{\sum_{k=1}^{n}(\Delta p_k)^2\right\}\left(\frac{e^{it}-1}{\bar{p}e^{it}+\bar{q}}\right)^2\right.$$

(3. 11)

$$\left.+\vartheta\cdot\frac{1}{2}\left\{\frac{1}{2}\sum_{k=1}^{n}(\Delta p_k)^2\right\}^2\cdot\left|\frac{e^{it}-1}{\bar{p}e^{it}+\bar{q}}\right|^4\cdot\exp\left\{\frac{1}{2}\sum_{k=1}^{n}(\Delta p_k)^2\cdot\left|\frac{e^{it}-1}{\bar{p}e^{it}+\bar{q}}\right|^2\right\}\right]\cdot\{1+\vartheta\cdot|\Theta_2|e^{|\Theta_2|}\}$$

$$=(\bar{p}e^{it}+\bar{q})^n-\frac{1}{2}\left\{\sum_{k=1}^{n}(\Delta p_k)^2\right\}(\bar{p}e^{it}+\bar{q})^{n-2}\cdot(e^{2it}-2e^{it}+1)+J_1+J_2$$

where ϑ's where unspecified complex valued quantities such that $|\vartheta|\leqq 1$ and

$$|J_1|\leqq\frac{1}{8}\sigma_p^4\frac{t^4}{(1-2\bar{p})^4}\cdot\exp\left\{\frac{1}{2}\sigma_p^2\frac{t^2}{(1-2\bar{p})^2}\right\},$$

$$|J_2|\leqq\frac{1}{3}\cdot\frac{1}{1-2\alpha/(1-2\bar{p})}\cdot\frac{1}{(1-2\bar{p})^3}\cdot\left(\sum_{k=1}^{n}|\Delta p_k|^3\right)\cdot 2|t|^2$$

(3. 12)

$$\cdot\exp\left\{\frac{1}{2}\frac{t^2}{2(1-2\bar{p})^2}\sigma_p^2+\frac{1}{3}\frac{1}{1-2\alpha/(1-2\bar{p})}\frac{2|t|^2}{(1-2\bar{p})^3}\left(\sum_{k=1}^{n}|\Delta p_k|^3\right)\right\}$$

$$\leqq\frac{1}{3}\cdot\frac{1}{1-2(\bar{p}+\alpha)}\frac{\alpha\sigma_p^2t^2}{(1-2\bar{p})^2}\cdot\exp\left\{\frac{1}{2}\frac{\sigma_p^2t^2}{(1-2\bar{p})^2}+\frac{1}{3}\frac{1}{1-2(\bar{p}+\alpha)}\cdot\frac{2\alpha\sigma_p^2t^2}{(1-2\bar{p})^2}\right\}$$

where $\sigma_p^2=\sum_{k=1}^{n}|\Delta p_k|^2$.

2. *Theorems.*

We shall state our theorems in the following two steps.

THEOREM 2. *Let* $\max_k \Delta p_k=\alpha$, $2\alpha/(1-2\bar{p})<1/2$ *and let* Y *be a random variable such that*

$$P(Y=k)=b(k:\ n,\ \bar{p})=\binom{n}{k}\bar{p}^k(1-\bar{p})^{n-k},$$

then we have

$$P(l'+1\leqq S\leqq l)=P(l'+1\leqq Y\leqq l)-\frac{1}{2}\sigma_p^2\cdot\Delta^2 P(l'+1\leqq Y'\leqq l)+R \tag{3.13}$$

where Y' *is a random variable such that* $P(Y'=k)=b(k:\ n-2,\ \bar{p})$ *and* $\Delta^2 P$ *has the same meaning as in Theorem* 1, *so that we have*

$$\begin{aligned}\Delta^2 P(l'+1\leqq Y'\leqq l)&=\{b(l+1;\ n-2,\ \bar{p})-b(l:\ n-2,\ \bar{p})\}-\{b(l'+1:\ n-2,\ \bar{p})-b(l':\ n-2,\ \bar{p})\}\\ &=\Delta b'(l)-\Delta b'(l')\end{aligned}$$

say, and

$$\begin{aligned}|R|&\leqq R_1+R_2,\\ R_1&=\frac{1}{4}\left\{\exp\frac{\pi^2\sigma_p^2}{2(1-2\bar{p})^2}\cdot\left(\frac{\pi^2\sigma_p^2-1}{2(1-2\bar{p})^2}-1\right)+1\right\}\fallingdotseq 3\left(\frac{\sigma_p}{1-2\bar{p}}\right)^4,\\ R_2&=\frac{1}{3}\ \frac{\alpha}{1-2(\bar{p}+\alpha)}\cdot\left\{\exp\frac{\sigma_p^2\pi^2}{(1-2\bar{p})^2}\left(\frac{1}{2}+\frac{2\alpha}{3(1-2(\bar{p}+\alpha))}\right)-1\right\}\Big/\left(\frac{1}{2}\right.\\ &\qquad\left.+\frac{1}{3}\ \frac{2\alpha}{1-2(\bar{p}+\alpha)}\right)\\ &=\frac{1}{3}\ \frac{\alpha}{1-2\beta}\cdot\left\{\exp\frac{\sigma_p^2\pi^2}{(1-2\bar{p})^2}\left(\frac{1}{2}+\frac{1}{3}\ \frac{2\alpha}{1-2\beta}\right)-1\right\}\Big/\left(\frac{1}{2}+\frac{1}{3}\ \frac{2\alpha}{1-2\beta}\right)\\ &\fallingdotseq 3.2\frac{\alpha}{1-2\beta}\left(\frac{\sigma_p}{1-2\bar{p}}\right)^2\quad(\beta=\bar{p}+\alpha).\end{aligned}$$

Proof. For the proof of our theorem, we need only to follow the way as we showed in [8]. Hence by the formula (2. 8) in [8], we have from (3. 11) and (3. 12)

$$\begin{aligned}\sum_{k=l'+1}^{l} P(S=k)&=\frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(t)\cdot\left(\sum_{k=l'+1}^{l} e^{-ikt}\right)dt\\ &=\frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(t)\cdot e^{-i(l+l'+1)t/2}\ \frac{\sin(l'-l)t/2}{\sin(t/2)}\,dt\\ &=\frac{1}{2\pi}\int_{-\pi}^{\pi}(\bar{p}e^{it}+\bar{q})^n\cdot\sum_{k=l'+1}^{l} e^{-ikt}\,dt \qquad (3.15)\\ &\quad-\frac{1}{2}\sigma_p^2\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}(\bar{p}e^{it}+\bar{q})^{n-2}\cdot(e^{2it}-2e^{it}+1)\sum_{k=l'+1}^{l} e^{-ikt}\,dt\end{aligned}$$

$$+\vartheta\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}(J_1+J_2)e^{-i(l+l'+1)t/2}\frac{\sin(l'-l)t/2}{\sin(t/2)}dt,$$

where J_1, J_2 are defined by (3. 11).

Hence we have

$$|R|\leqq\frac{1}{2\pi}\cdot2\int_0^{\pi}|\bar{p}e^{it}+\bar{q}|^n\cdot\frac{|J_1|+|J_2|}{t/\pi}dt$$

(3. 16)

$$\leqq\int_0^{\pi}\frac{|J_1|}{t}dt+\int_0^{\pi}\frac{|J_2|}{t}dt\leqq R_1+R_2 \qquad \text{say,}$$

and that

$$R_1=\int_0^{\pi}\frac{1}{8}\sigma_p^4\frac{t^3}{(1-2\bar{p})^4}\cdot\exp\left\{\frac{1}{2}\sigma_p^2\frac{t^2}{(1-2\bar{p})^2}\right\}dt$$

(3. 17(

$$=\frac{1}{4}\left\{\frac{\pi^2\sigma_p^2}{2(1-2\bar{p})^2}\cdot\left(\exp\frac{\pi^2\sigma_p^2}{2(1-2\bar{p})^2}-1\right)+1\right\}=3\left(\frac{\sigma_p}{1-2\bar{p}}\right)^4+O(\sigma_p^5)$$

and

$$R_2=\int_0^{\pi}\frac{1}{3}\frac{\alpha}{1-2(\bar{p}+\alpha)}\cdot\frac{\sigma_p^2\cdot2t}{(1-2\bar{p})^2}\cdot\exp\left\{\frac{1}{2}\frac{\sigma_p^2t^2}{(1-2\bar{p})^2}\right.$$

$$\left.+\frac{1}{3}\frac{1}{1-2(\bar{p}+\alpha)}\cdot\frac{2\alpha\sigma_p^2t^2}{(1-2\bar{p})^2}\right\}dt$$

(3. 18)

$$=\frac{1}{3}\frac{\alpha}{1-2(\bar{p}+\alpha)}\left\{\exp\frac{\sigma_p^2\pi^2}{(1-2\bar{p})^2}\cdot\left(\frac{1}{2}+\frac{1}{3}\frac{2\alpha}{1-2(\bar{p}+\alpha)}\right)-1\right\}\Big/\left(\frac{1}{2}\right.$$

$$\left.+\frac{1}{3}\frac{2\alpha}{1-2(\bar{p}+\alpha)}\right)$$

$$=3.2\frac{\alpha}{1-2\beta}\left(\frac{\sigma_p}{1-2\bar{p}}\right)^2+O(\sigma_p^5). \qquad \text{Q.E.D.}$$

We state the following theorem also, of which proof is omitted since it can be done as the proof of Theorem 2.

THEOREM 3. *Using the same notations as in Theorem 2, we have*

(3. 19) $$P(l'+1\leqq S\leqq l)=P(l'+l\leqq Y\leqq l)+R_1$$

where

$$|R_1|\leqq\sigma_p^2+O(\sigma_p^3)$$

and

$$P(l'+1\leqq S\leqq l)=P(l'+1\leqq Y\leqq l)-\frac{1}{2}\sigma_p^2\Delta^2\left\{\sum_{l'+1}^{l}P(Y'=k)\right\}$$

(3. 20)

$$+\frac{1}{3}\left\{\sum_1^n(\Delta p_k)^3\cdot\Delta^3\left\{\sum_{l'+1}^{l}P(Y''=k)\right\}+R_2,\right.$$

where $R_2=O(\sigma_p^4)$ *and* Y'' *is the random variable distributed as*

$$P(Y''=k)=b(k:\ n-3,\ \bar{p}).$$

§4. Normal approximation to the Poisson binomial distribution.

In this section, we shall show the normal approximation to Poisson binomial distribution using the results of our previous paper [7].

In fact, detailed evaluations which we noted in [7] are quoted and some of them are modified so as to apply to the present studies.

Let $\mu=E(S)$ and $\sigma^2=V(S)$, then we have $\mu=\sum_1^n p_k=n\bar{p}$, $\alpha^2=\sum_1^n p_kq_k$ $(q_k=1-p_k)$ and normalized random variable $S'=(S-\mu)/\sigma$ and the characteristic function of S' is

(4. 1)
$$f_{S'}(t)=\prod_{k=1}^{n}(p_ke^{it/\sigma}+q_k)\cdot e^{-i\mu t/\sigma}$$

We shall devide our discussions in some steps for the sake of simplicity

1. *Taylor's expansion of* $f_{S'}(t)$.

Taking logarithm of the both sides of (4. 1), we have under restrictions $p_k<1/2$,

$$\log f_{S'}(t)=-i\frac{\mu}{\sigma}t+\sum_{k=1}^{n}\log(p_ke^{it/\sigma}+q_k)$$

$$=-i\frac{\mu}{\sigma}t+\sum_{k=1}^{n}\log\{1+p_k(e^{it/\sigma}-1)\}$$

(4. 2)

$$=-i\frac{\mu}{\sigma}t+\sum_{k=1}^{n}p_k(e^{it/\sigma}-1)-\frac{1}{2}\sum_{k=1}^{n}p_k^2(e^{it/\sigma}-1)^2$$

$$+\frac{1}{3}\sum_{k=1}^{n}p_k^3(e^{it/\sigma}-1)^3-+\cdots-\frac{1}{6}\sum_{k=1}^{n}p_k^6(e^{it/\sigma}-1)^6+R$$

where

$$R=\sum_{k=1}^{n}\int_0^{p_k(e^{it/\sigma}-1)}\frac{z^6}{1+z}dz.$$

From (4. 2), we have further

$$\log f_{S'}(t) = -\frac{t^2}{2} + \frac{1}{6}\sum_1^n p_k q_k (q_k - p_k)\left(\frac{it}{\sigma}\right)^3 + \frac{1}{24}\sum_1^n p_k q_k (1 - 6p_k q_k)\left(\frac{it}{\sigma}\right)^4$$

$$+ \frac{1}{120}\sum_1^n p_k q_k (q_k - p_k)\{3(q_k - p_k)^2 - 2\}\left(\frac{it}{\sigma}\right)^5 + \frac{1}{720}\sum_1^n p_k A_k \left(\frac{it}{\sigma}\right)^6$$

$$(4.\ 3) \quad + \frac{1}{6}\sum_1^n p_k^6 \left(\frac{it}{\sigma}\right)^6 \cdot \vartheta + \frac{1}{5040}\sum_1^n p_k B_k \left(\frac{it}{\sigma}\right)^7 + \frac{1}{40320}\sum_1^n p_k C_k \cdot \left(\frac{it}{\sigma}\right)^8$$

$$+ \frac{1}{362880}\sum_1^n p_k D_k \left(\frac{it}{\sigma}\right)^9 + v + R$$

where

$$(4.\ 4) \qquad |v| \leqq \sum_1^n p_k E_k \left(\frac{t}{\sigma}\right)^{10} + \sum_1^n p_k F_k \left(\frac{t}{\sigma}\right)^{11},$$

and A_k, B_k, $\cdots$, E_k are the polynomials of forth order in p_k and F_k is a constant having the same meaning as in (1, 3) of [7] p. 48.

Hence we have

$$f_{S'}(t) = e^{-t^2/2}(X + Y)$$

where

$$X = 1 + \frac{1}{6}\sum_{k=1}^n p_k q_k (q_k - p_k)\left(\frac{it}{\sigma}\right)^3$$

$$+ \frac{1}{72}\left[\left\{\sum_{k=1}^n p_k q_k (q_k - p_k)\right\}^2 \left(\frac{it}{\sigma}\right)^6 + 3\sum_{k=1}^n p_k q_k \cdot (1 - 6p_k q_k)\left(\frac{it}{\sigma}\right)^4\right]$$

$$+ \frac{1}{1296}\left[\left\{\sum_{k=1}^n p_k q_k (q_k - p_k)^3\right\}\left(\frac{it}{\sigma}\right)^9 + 9\left\{\sum_{k=1}^n p_k q_k (q_k - p_k)\right\} \cdot \left\{\sum_{k=1}^n p_k q_k (1 - 6p_k q_k)\left(\frac{it}{\sigma}\right)^7\right]$$

$$(4.\ 5) \quad + \frac{1}{24 \times 6^4}\left\{\sum_{k=1}^n p_k q_k \cdot (q_k - p_k)\right\}^4 \left(\frac{t}{\sigma}\right)^{12} \cdot \vartheta$$

$$+ \frac{1}{72 \times 24}\left[\left\{\sum_{k=1}^n p_k q_k (q_k - p_k)\right\}^2 \cdot \left\{\sum_{k=1}^n p_k q_k (1 - 6p_k q_k)\right\}\right]\left(\frac{t}{\sigma}\right)^{10} \cdot \vartheta$$

$$+ \frac{1}{1152}\left\{\sum_{k=1}^n p_k q_k (1 - 6p_k q_k)\right\}^2 \left(\frac{t}{\sigma}\right)^8 \cdot \exp\left\{\frac{1}{24}\sum_{k=1}^n p_k q_k |1 - 6p_k q_k| \cdot \left(\frac{t}{\sigma}\right)^4\right\} \cdot \vartheta$$

and

$$Y = R \cdot \vartheta \cdot e^{|R|} + R \cdot \frac{1}{24}\sum_{k=1}^n p_k q_k (1 - 6p_k q_k) \cdot \left(\frac{t}{\sigma}\right)^4 \cdot \vartheta e^{|R|}$$

$$(4.\ 6) \qquad \cdot \exp\left\{\frac{1}{24}\sum_{k=1}^n p_k q_k \cdot (1 - 6p_k q_k)\left(\frac{it}{\sigma}\right)^4\right\}$$

$$= R \cdot \vartheta \cdot e^{|R|} \cdot \exp\left\{\frac{1}{24}\sum_{k=1}^n p_k q_k (1 - 6p_k q_k) \cdot \left(\frac{it}{\sigma}\right)^4\right\}.$$

2. *Expansion of* $f_{S'}(t)\{\sin(t/2\sigma)\}^{-1}$.

From (4. 5), (4. 6) and Taylor expansion of $\{\sin(t/2\sigma)\}^{-1}$ (see the Lemma 3 of [12] p. 40), we have

$$f_{S'}(t)\left(\sin\frac{t}{2\sigma}\right)^{-1}=e^{-t^2/2}(L+M+N) \tag{4. 7}$$

where

$$L=\frac{2\sigma}{t}+\frac{1}{3}\sum_1^n p_kq_k(q_k-p_k)\frac{1}{\sigma}\cdot i(it)^2+\frac{1}{36\sigma}\left[\left\{\sum_1^n p_kq_k(q_k-p_k)\frac{1}{\sigma^2}\right\}\cdot(it)^5 + 3\sum_1^n p_kq_k(1-6p_kq_k)\frac{1}{\sigma^2}(it)^3-3(it)\right] \tag{4. 8}$$

M and N have the analoguous forms as in [7], (2. 2) except the first term of M from which $t/12\sigma$ adds to L.

3. *Theorems.*

We are now in the position to prove the following

THEOREM 4. *For* $\sigma>5$ *or* $\sum_{k=1}^n p_kq_k>25$ *and* $p_k<1/2$, *we have*

$$P(l'+1\leqq S\leqq l)=\frac{1}{\sqrt{2\pi}}\int_{\xi_1}^{\xi_2}e^{-u^2/2}\,du+\frac{A}{6\sqrt{2\pi}\,\sigma}\left[(1-\xi_2^2)e^{-\xi_2^2/2}-(1-\xi_1^2)e^{-\xi_1^2/2}\right]+\omega \tag{4. 9}$$

where

$$\xi_1=\frac{l'+1/2-\mu}{\sigma},\quad \xi_2=\frac{l+1/2-\mu}{\sigma},\quad A=\frac{\sigma^2-2\sum_1^n p_k^2q_k}{\sigma^2}=\frac{\sum_1^n p_kq_k(q_k-p_k)}{\sigma^2} \tag{4. 10}$$

and

$$|\omega|\leqq 0.08\frac{1}{\sigma^2}+0.23\frac{1}{\sigma^3}\left\{\sum p_kq_k(q_k-p_k)\frac{1}{\sigma^2}\right\}+0.75\frac{1}{\sigma^4}+0.62\frac{1}{\sigma^5}+e^{-3\sigma/2}. \tag{4. 11}$$

Proof. From the Levy's inversion formula, we have

$$\begin{aligned}
P(l'+1\leqq S\leqq l)&=\sum_{k=l'+1}^{l}P(S=k)\\
&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=l'+1}^{l}e^{-ikt}\right)\cdot\prod_{k=1}^{n}(p_ke^{it}+q_k)\,dt\\
&=\frac{1}{2\pi\sigma}\int_{-\pi\sigma}^{\pi\sigma}f_{S'}(t)\frac{e^{-i\xi_1t}-e^{-i\xi_2t}}{2i\sin(t/2\sigma)}dt\\
&=\frac{1}{2\pi\sigma}\left\{\int_{-\beta\sqrt{\sigma}}^{\beta\sqrt{\sigma}}+\int_{\pi\sigma>|t|>\beta\sqrt{\sigma}}\right\}f_{S'}(t)\frac{e^{-i\xi_1t}-e^{-i\xi_2t}}{2i\sin(t/2\sigma)}dt\\
&=J_1+J_2,\quad\text{say,}
\end{aligned} \tag{4. 12}$$

hence we have the results of 2,

$$J_1=\frac{1}{2\pi\sigma}\int_{-\beta\sqrt{\sigma}}^{\beta\sqrt{\sigma}}\left\{\frac{\sigma}{it}+\frac{1}{6}\sum_{k=1}^{n}p_kq_k(q_k-p_k)\left(\frac{it}{\sigma}\right)^2\right\}e^{-t^2/2}\cdot(e^{-i\xi_1t}-e^{-i\xi_2t})\,dt$$

$$+\frac{1}{2\pi\sigma}\int_{-\beta\sqrt{\sigma}}^{\beta\sqrt{\sigma}}\frac{1}{36}\left\{\left(\sum_{k=1}^{n}p_kq_k(q_k-p_k)\right)^2\left(\frac{it}{\sigma}\right)^5\right.$$

$$\left.+3\sum_{1}^{n}p_kq_k(1-6p_kq_k)\left(\frac{it}{\sigma}\right)^3-3\left(\frac{it}{\sigma}\right)\right\}\cdot e^{-t^2/2}(e^{-i\xi_1t}-e^{-i\xi_2t})\,dt$$

(4. 13)

$$+\frac{1}{2\pi\sigma}\int_{-\beta\sqrt{\sigma}}^{\beta\sqrt{\sigma}}(M+N)e^{-t^2/2}\frac{(e^{-i\xi_1t}-e^{-i\xi_2t})}{2i}\,dt$$

$$=\frac{1}{2\pi\sigma}\int_{-\infty}^{\infty}\left\{\frac{\sigma}{it}+\frac{1}{6}\sum_{1}^{n}p_kq_k(q_k-p_k)\left(\frac{it}{\sigma}\right)^2\right\}e^{-t^2/2}\cdot(e^{-i\xi_1t}-e^{-i\xi_2t})\,dt$$

$$+R'+S+T \qquad \text{say,}$$

where

$$R'=\frac{1}{2\pi\sigma}\int_{\infty>|t|>\beta\sqrt{\sigma}}\left\{\frac{\sigma}{it}+\frac{1}{6}\sum_{1}^{n}p_kq_k(q_k-p_k)\left(\frac{it}{\sigma}\right)^2\right\}e^{-t^2/2}\cdot(e^{-i\xi_1t}-e^{-i\xi_2t})\,dt \tag{4. 14}$$

and S and T are the second term and the third term of (4. 13) respectively and we shall take $\sqrt{3}$ for β when we calculate the evaluation of the error terms R, S and T.

It is also easy to see from the results of Uspensky [15] and [7] that

$$|J_2|+|R'|\leqq e^{-3\sigma/2}. \tag{4. 15}$$

Absolute value of S and T can be majorated by the similar method as we have adopted in [7]. From the evaluation of S, thus we have

$$|S|\leqq\frac{1}{36\pi\sigma^2}\int_0^{\infty}\left|\left\{\frac{\sum_1^n p_kq_k(q_k-p_k)}{\sigma^2}\right\}^2t^5-\frac{3\sum_1^n p_kq_k(q_k-p_k)}{\sigma}t^3\right|\cdot e^{-t^2/2}\,dt$$

(4. 16)

$$+\frac{3}{36\pi\sigma^2}\left|\int_0^{\infty}\left(2\cdot\frac{\sum_1^n(p_kq_k)^2}{\sigma^2}t^3-t\right)e^{-t^2/2}\,dt\right|$$

$$=S_1+S_2 \qquad \text{say,}$$

and

$$S_1\leqq\frac{1}{36\pi\sigma^6}\sum_{1}^{n}p_kq_k\cdot\sum_{1}^{n}p_lq_l\cdot(q_k-p_k)^2\cdot\frac{a_{kl}}{3}\int_0^{\infty}|t^5-a_{kl}t^3|e^{-t^2/2}\,dt$$

or

$$S_1\leqq\frac{1}{36\pi\sigma^6}\sum_{1}^{n}p_kq_k\cdot\sum_{1}^{n}p_lq_l\cdot(q_l-p_l)^2\cdot\frac{3}{a_{kl}}\int_0^{\infty}|t^5-a_{kl}t^3|e^{-t^2/2}\,dt$$

where $a_{kl}=3(q_l-p_l)/(q_k-p_k)$.

The delicated and longsome calculus shows that S_1 is bounded by $0.053/\sigma^2$ and S_2 can be majorated by the second term of the last expression of the inequality in our previous paper (cf. Remark 1). Hence we can say

$$|S|\leqq\frac{0.053+0.027}{\sigma^2}=\frac{0.08}{\sigma^2} \tag{4. 17}$$

and for T we can also apply the discussions of the paper cited above [7] and have the result

$$|T|\leqq\frac{1}{\sigma^2}\sum_1^n p_kq_k(q_k-p_k)\cdot\frac{0.23}{\sigma^3}+\frac{0.88}{\sigma^4}. \tag{4. 18}$$

Summarizing (4. 12~4.18) the proof of our Theorem is completed.

REMARKS. 1) For the result of our theorem, $|\omega|$ can be majorated in the following form also;

$$|\omega|\leqq\frac{0.16}{\sigma^2}+e^{-3\sigma/2}. \tag{4. 19}$$

The fact may be clear from our proofs of Theorem 4.

2) The evaluation of the error term ω seems to be comparable to the one obtained in [7] which is the special case in which all the p_k's are equal, and seems to be nearly best possible in this form in the sense that the more examined calculus may serve to improve only the minor details of our evaluations.

§5. Applications to the theory of the sampling inspection.

In this paragragh, we shall discuss the theory of sampling inspection applying the previous result on the Poisson approximation to binomial and negative binomial distribution.

1. *Average outgoing quality* (AOQ) *and Dodge and Romig's table of sampling inspection.*

The probabilities of finding Y defectives in n samples drawn from the lot of size N and of percent defective p are

$$P(r, n\mid N, p)=\frac{\binom{N-Np}{n-r}\cdot\binom{Np}{r}}{\binom{N}{n}}, \qquad (0\leqq r\leqq \min(n, Np)) \tag{5. 1}$$

If the condition $N\gg n$ is satisfied, (5. 1) can be replaced by the binomial distribution

$$P(r\mid n, p)=b(r\colon n, p)=\binom{n}{r}p^r(1-p)^{n-r}$$

and if the conditions $p\ll 1$ but $np=\lambda=$const. is added to it, then we have

(5. 2)
$$P(r \mid n, p) \doteqdot p(r;\ \lambda) = \frac{e^{-\lambda}\lambda^r}{r!}.$$

The detailed approximation theory concerning the relation (5. 2) is discussed in §2 and [11].

In 1928, Dodge and Romig have constructed the valuable tables of sampling inspection using the above equality (5. 2). They have introduced some principal notions "Lot Tolerance Percent Defective" (LTPD) and "Average Outgoing Quality" (AOQ) and "AOQ limit" (AOQL). Under the some restriction on LTPD or on AOQL, they decided the sample size n and its acceptance number c of the sampling plan so as to minimize the average amount of inspection I at average percent defective $\bar{p}$.

2. AOQ *and* AOQL *with due regard to prior distribution of* p.

Dodge and Romig [1] have calculated AOQ and AOQL as

(5. 3)
$$\text{AOQ} = p \cdot L(p)$$

and

(5. 3′)
$$\text{AOQL} = \max_{p} p \cdot L(p)$$

where

(5. 4)
$$L(p) = \sum_{r=0}^{c} P(r \mid L) \qquad (h = np)$$

is the probability of accepting the lot submitted to the inspection, assuming that the percent defectives of the lots do not vary from the lot to the lot, which is the same to assume that prior distribution is the one point distribution with a mass 1 at p, and it is easy to see that this assumption gives the maximized AOQ. Hence if we wish to determine the optimal sampling inspection plan under the restriction that this AOQL is equal to the preassigned quantity, then the values of AOQL in many practical cases are smaller than the AOQL. This fact was pointed out and discussed by Hamaker [5] who proposed the utilization of prior distribution of p.

3. *Distribution of the lot percent defective* p *and* AOQ.

From the author's experiences, the lot percent defectives p seem to distribute as gamma-distribution in many cases, and this distribution can be decided by its two parameters which are the lot average percent defective $\bar{p}$ and coefficient of variation v_p. Hald [4] has calculated the loss function to find optimal inspection plans in many important cases.

When the probability density function of prior distribution $\varphi(p)$ is given by the following one of beta distribution

(5. 5)
$$\varphi(p) = \frac{1}{B(\lambda_1, \lambda_2)} p^{\lambda_1 - 1}(1-p)^{\lambda_2 - 1}$$

we have AOQ as

$$\int_0^1 pL(p)\varphi(p)\,dp$$

where $L(p)=\sum_{r=0}^{c} p(r\mid n,p)$. Hence, we have

$$\text{AOQ}=\frac{\lambda_1}{\lambda_1+\lambda_2}\sum_{r=0}^{c}\binom{n}{r}\frac{(\lambda_1+1)\cdots(\lambda_1+r)\cdot\lambda_2(\lambda_2+1)\cdots(\lambda_2+n-r-1)}{(\lambda_1+\lambda_2+1)(\lambda_1+\lambda_2+2)\cdots(\lambda_1+\lambda_2+n)} \tag{5. 6}$$

and the probability of accepting the lot as

$$L=\int_0^1 L(p)\varphi(p)\,dp=\sum_{r=0}^{c}\binom{n}{r}\frac{\lambda_1(\lambda_1+1)\cdots(\lambda_1+\overline{r-1})\lambda_2(\lambda_2+1)\cdots\lambda_2+\overline{n-r-1})}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+1)\cdots(\lambda_1+\lambda_2+n-1)}. \tag{5. 7}$$

Now, if we assume that

$$\bar{p}=\int_0^1 p\varphi(p)\,dp=\frac{\lambda_1}{\lambda_1+\lambda_2}$$

is small but $n\bar{p}=h$ and $n\cdot(\sigma_p^2/\bar{p})=d$ are moderate where

$$\sigma_p^2=\int_0^1 (p-\bar{p})^2\varphi(p)\,dp=\frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_2+1)}$$

we can approximate the beta distribution by gamma distribution and deform the above quantities (5. 6) and (5. 7) as follows:

$$\text{AOQ}=\bar{p}\cdot\sum_{r=0}^{c}\frac{h'(h'+d)\cdots(h'+\overline{r-1}\,d)}{r\,!}(1+d)^{-h'/d-r}, \tag{5. 6'}$$

$$L=\sum_{r=0}^{c}\frac{h(h+d)\cdots(h+\overline{r-1}\,d)}{r\,!}(1+d)^{-h/d-r} \tag{5. 7'}$$

where $h'=h+d$ and $\sqrt{d/h}=\sqrt{p}$ is the coefficient of variation of the gamma distribution of the lot percent defectives. We can also obtain the following equality for average amount of inspection I as

$$I=n+(N-n)\cdot L\doteqdot n+N\cdot L \qquad (N\gg n) \tag{5. 8}$$

where L can be calculated for $v_p\geqq 1/2$ by our approximation formula in §1.

4. *Construction of the table of sampling inspection.*

As noted in 2, that $\text{AOQL}=\max_p \text{AOQ}=\text{AOQ}]_{np_l=x}$ is a monotonely increasing function of v_p is easily seen, while Dodge and Romig's table corresponds to the case when $v_p=0$. Hence if we have any information concernig with the distribution of the lot percent defectives, we can decide the sampling inspection plan to minimize I under the restriction $\text{AOQL}\leqq y$. As Campbell (see [1]) calculated and constructed the table showing the relation of x and y based on the Poisson distribution in (5. 2), (5. 3) and (5. 4), we can tabulate it for $v_p=1$, $\sqrt{2}/2$, $1/2$ starting from the (5. 6′) and (5. 7′), replacing Poisson distribution by negative binomial distribution and have the table (Table I). For the numerical calculation of AOQL or AOQ, we can use the binomial expression for negative binomial distribution

TABLE I

c	x	y	x	y	x	y	x	y
0	1.0	0.37	1.0	0.25	1.0	0.30	1.0	0.33
1	1.62	0.84	1.55	0.53	1.57	0.64	1.59	0.73
2	2.27	1.37	2.10	0.82	2.15	1.00	2.19	1.15
3	2.95	1.95	2.66	1.11	2.73	1.39	2.80	1.59
4	3.64	2.54	3.21	1.40	3.31	1.74	3.41	2.02
5	4.35	3.17	3.77	1.70	3.89	2.13	4.03	2.49
6	5.07	3.81	4.32	1.99	4.49	2.51	4.66	2.96
7	5.80	4.47	4.88	2.29	5.07	2.89	5.28	3.42
8	6.55	5.15	5.44	2.59	5.66	3.28	5.91	3.89
9	7.30	5.84	5.99	2.88	6.25	3.66	6.53	4.35
10	8.06	6.54	6.55	3.18	6.84	4.05	7.16	4.82
11	9.22	7.23	7.11	3.48	7.43	4.43	7.79	5.29
12	9.59	7.95	7.66	3.78	8.02	4.82	8.42	5.78
13	10.37	8.68	8.22	4.08	8.61	5.25	9.04	6.22
	No. 1		No. 2		No. 3		No. 4	

$v_p=$	0,	1,	$\sqrt{2}/2$,	1/2
No.	1,	2,	3,	4

TABLE II

1. $v_p=1$, $v'_p=1$

N \ $\bar{p}$	AOQL=1%				AOQL=5%			
	0.1%		1%		1%		5%	
	n	c	n	c	n	c	n	c
500	25	0	53	1	11	1	16	2
1,000	25	0	82	2	11	1	28	4
2,000	53	1	111	3	16	2	40	6
3,000	53	1	111	3	16	2	52	8
4,000	53	1	140	4	16	2	58	9
5,000	53	1	140	4	22	3	64	10
10,000	53	1	170	5	22	3	88	14

2. $v_p=\sqrt{2}/2$, $v'_p=1/2$

N \ $\bar{p}$	AOQL=2%				AOQL=10%			
	0.1%		1%		1%		5%	
	n	c	n	c	n	c	n	c
500	17	0	17	0	4	0	16	3
1,000	17	0	58	2	4	0	21	4
2,000	37	1	58	2	12	2	21	4
3,000	37	1	80	3	16	3	21	4
4,000	37	1	101	4	16	3	25	5
5,000	37	1	101	4	16	3	25	5
10,000	37	1	101	4				

[13] and we can apply the Poisson approximation stated in §2 $v_p/n\bar{p}\geqq 1/2$ for simple calculation.

Under this restriction on AOQL, we decide the sampling inspection plan (n, c) which minimize I (Table II), where v_p' stands for the coefficient of variation of the distributioh with the mean at $\bar{p}$, Table II shows a part of the table thus constracted, and complete table will be published (for example see [10]).

For example, if we have some information that prior distribution of p has lot average defective $\bar{p}=0.01$ and coefficient of variation $v_p=\sqrt{2}/2$ at usual production process, but seems to have the coefficient of variation to be $v_p'=1/2$ at worst, then we have for $N=3{,}000$ the required sample inspection plan (80,3) when AOQL $=2\%$.

5. *Variation of the outgoing quality.*

So far we have treated the average outgoing quality (OQ, say) which means the outgoing quality for the long run or for the sufficiently large number of lots. But for finite number of the lots, we must consider the variation of OQ, or the distribution of OQ. This study were taken up by Steck and Owen firstly. We shall discuss this problem for the cases when the prior distribution is given by the gamma distribution.

Put L be the number of lots submitted to the inspection and $\bar{p}$ and v_p are the average outgoing quality and the coefficient of variation of prior distribution respectively. Then we can show by calculating the characteristic function of OQ that for sufficiently large L the distribution of OQ is approximated by normal distribution with mean $A=\text{AOQ}$ and variance σ_{OQ}^2 where

$$A=\text{AOQ}=\bar{p}\cdot\sum_{r=0}^{c}\frac{(h+d)(L+2d)\cdots(L+d+\overline{r-1}\,d)}{r!}(1+d)^{-(h+d)/d-r} \tag{5. 9}$$

$$\sigma_{\text{OQ}}^2=\frac{B-A^2}{L} \qquad \text{and}$$

$$B=\bar{p}^2\cdot\left(\frac{h+d}{h}\right)\cdot\sum_{r=0}^{c}\frac{(h+2d)(h+3d)\cdots(h+2d+\overline{r-1}\,d)}{r!}(1+d)^{-(h+2d)/d-r}, \tag{5. 10}$$

For the purposes to obtain the quantities A and σ_{OQ}^2, we can use the Patil's expressions for $v_p=1$, $\sqrt{2}/2$ and $1/2$ and when $v_p\geqq 1/2$, our approximation formula (2. 3), (2. 4) are useful for quick checks of the variation of OQ.

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