ON CURVATURES OF SPACES WITH NORMAL GENERAL CONNECTIONS, II

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In this paper, the author makes a formula (§2) related to the curvature tensors of a normal general connection γ and $B\gamma B$, where B is a tensor field of type (1, 1) satisfying some conditions, making use of the results in a previous paper [14], and then he shows that the formula applied to the case in which γ is a classical affine connection is a generalization of the Gauss' equations in the theory of subspaces of Riemannian geometry (§4). He also shows that regarding the set of general connections as a vector space over the algebra of all tensor fields of type (1, 1), the calculations in connection with the above purpose can be simplified.

§1. Preliminaries.

Let $\mathfrak{X}$ be an n-dimensional differentiable manifold. Let γ be a general connection given on $\mathfrak{X}$ which is written in terms of local coordinates u^i of $\mathfrak{X}$ as

$$\gamma=\partial u_j\otimes(P_i^j d^2u^i+\Gamma_{ih}^j du^i\otimes du^h), \tag{1.1}$$

where $\partial u_j=\partial/\partial u^j$. We denote the tensor of type (1, 1) with local components P_i^j by $\lambda(\gamma)$ and denote the components P_i^j, Γ_{ih}^j of γ with respect to u^i by $P_i^j(\gamma)$, $\Gamma_{ih}^j(\gamma)$ respectively, in case of treating several general connections. Let $Q=\partial u_j\otimes Q_i^j du^i$ be a tensor of type (1, 1), then the products $Q\gamma$ and γQ of γ and Q are general connections given by

$$Q\gamma=\partial u_k Q_j^k\otimes(P_i^j d^2u^i+\Gamma_{ih}^j du^i\otimes du^h) \tag{1.2}$$

and

$$\gamma Q=\partial u_j\otimes(P_k^j d(Q_i^k du^i)+\Gamma_{kh}^j(Q_i^k du^i)\otimes du^h),^{1)} \tag{1.3}$$

that is

$$P_i^j(Q\gamma)=Q_k^j P_i^k(\gamma),\ \Gamma_{ih}^j(Q\gamma)=Q_k^j\Gamma_{ih}^k(\gamma) \tag{1.2'}$$

and

$$P_i^j(\gamma Q)=P_k^j(\gamma)Q_i^k,\ \Gamma_{ih}^j(\gamma Q)=\Gamma_{kh}^j(\gamma)Q_i^k+P_k^j(\gamma)\frac{\partial Q_i^k}{\partial u^h}. \tag{1.3'}$$

Lemma 1.1. *Let γ be a general connection and $Q=\partial u_j\otimes Q_i^j du^i$ be a tensor field. The covariant derivatives $Q_{i,h}^j$ of Q_i^j with respect to γ can be written as*

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1) See [11], §1.

$$Q^j_{i,h}=\Gamma^j_{ih}(\gamma Q\lambda(\gamma)-\lambda(\gamma)Q\gamma). \tag{1.4}$$

Proof. By virtue of (2.15) in [7], we have by definition

$$Q^j_{i,h}=P^j_l\frac{\partial Q^l_k}{\partial u^h}P^k_i+\Gamma^j_{lh}Q^l_kP^k_i-P^j_lQ^l_k\Lambda^k_{ih},$$

where we put

$$\Lambda^j_{ih}(\gamma)=\Gamma^j_{ih}(\gamma)-\frac{\partial P^j_i(\gamma)}{\partial u^h}.$$

The right of the above equation can be written as

$$\Gamma^j_{lh}Q^l_kP^k_i+P^j_l\frac{\partial(Q^l_kP^k_i)}{\partial u^h}-P^j_lQ^l_k\Gamma^k_{ih}$$

$$=\Gamma^j_{ih}(\gamma QP)-\Gamma^j_{ih}(PQ\gamma)=\Gamma^j_{ih}(\gamma Q\lambda(\gamma)-\lambda(\gamma)Q\gamma).$$ q.e.d.

LEMMA 1.2. *A necessary and sufficient condition in order that the tensor field I with the components δ^j_i is covariantly constant with respect to a general connection γ is that γ is commutative with $\lambda(\gamma)$.*

Proof. By means of LEMMA 1.1, we have

$$\delta^j_{i,h}=\Gamma^j_{ih}(\gamma\lambda(\gamma)-\lambda(\gamma)\gamma)=\Gamma^j_{ih}(\gamma P-P\gamma) \tag{1.5}$$

and

$$\lambda(\gamma P-P\gamma)=0.$$

These relations lead to the assertion of this lemma. q.e.d.

By (6.28) in [7], the components of the curvature tensor of γ are given by

$$\begin{aligned}R_i{}^j{}_{hk}=&\left\{P^j_l\left(\frac{\partial\Gamma^l_{mk}}{\partial u^h}-\frac{\partial\Gamma^l_{mh}}{\partial u^k}\right)+\Gamma^j_{lh}\Gamma^l_{mk}-\Gamma^j_{lk}\Gamma^l_{mh}\right\}P^m_i\\&-\delta^j_{m,h}\Lambda^m_{ik}+\delta^j_{m,k}\Lambda^m_{ih}.\end{aligned} \tag{1.6}$$

Now, let γ be normal and let Q be the tensor such that Q is the inverse of P on its image and identical with P on its kernel regarding P as a homomorphism of the tangent bundle $T(\mathfrak{X})$ of $\mathfrak{X}$. Then the components $'R_i{}^j{}_{hk}$ and $''R_i{}^j{}_{hk}$ of the curvature tensors of the contravariant part $'\gamma=Q\gamma$ and the covariant part $''\gamma=\gamma Q$ of γ can be written respectively as[2]

$$'R_i{}^j{}_{hk}=A^j_l\left(\frac{\partial'\Lambda^l_{mk}}{\partial u^h}-\frac{\partial'\Lambda^l_{mh}}{\partial u^k}+{}'\Lambda^l_{th}{}'\Lambda^t_{mk}-{}'\Lambda^l_{tk}{}'\Lambda^t_{mh}\right)A^m_i \tag{1.7}$$

and

$$''R_i{}^j{}_{hk}=A^j_l\left(\frac{\partial''\Gamma^l_{mk}}{\partial u^h}-\frac{\partial''\Gamma^l_{mk}}{\partial u^k}+{}''\Gamma^l_{th}{}''\Gamma^t_{mk}-{}''\Gamma^l_{tk}{}''\Gamma^t_{mh}\right)A^m_i, \tag{1.8}$$

2) See [14], LEMMAS 2.1 and 2.2.

where we put $'\Gamma^j_{ih}=\Gamma^j_{ih}('\gamma)$, $''\Gamma^j_{ih}=\Gamma^j_{ih}(''\gamma)$ and $A=PQ=QP$ is the canonical projection of the normal general connection γ.

Putting $N=1-A$, we have

$$(1.9)\qquad 'N^j_{ih}=N^j_l\Gamma^l_{ih}=\Gamma^j_{ih}(N\gamma) \quad\text{and}\quad ''N^j_{ih}=A^j_{lh}N^l_i=\Gamma^j_{ih}(\gamma N).$$

Let $'D$ and $''D$ be the covariant differential operators of $'\gamma$ and $''\gamma$ respectively. By means of LEMMA 1.1 and $\lambda('\gamma)=\lambda(''\gamma)=A$, we have

$$\frac{'DP^j_i}{\partial u^h}=\Gamma^j_{ih}('\gamma\cdot P\lambda('\gamma)-\lambda('\gamma)P\cdot{}'\gamma)=\Gamma^j_{ih}('\gamma\cdot PA-AP\cdot{}'\gamma)$$

and

$$\frac{''DP^j_i}{\partial u^h}=\Gamma^j_{ih}(''\gamma\cdot P\lambda(''\gamma)-\lambda(''\gamma)P\cdot{}''\gamma)=\Gamma^j_{ih}(''\gamma\cdot PA-AP\cdot{}''\gamma),$$

that is

$$(1.10)\qquad \frac{'DP^j_i}{\partial u^h}=\Gamma^j_{ih}(Q\gamma P-A\gamma) \quad\text{and}\quad \frac{''DP^j_i}{\partial u^h}=\Gamma^j_{ih}(\gamma A-P\gamma Q).$$

Accordingly, we get from (1.9) and (1.10)

$$(1.11)\qquad \frac{'DP^j_i}{\partial u^h}-{}'N^j_{ih}=\Gamma^j_{ih}(Q\gamma P-\gamma) \quad\text{and}\quad \frac{''DP^j_i}{\partial u^h}+{}''N^j_{ih}=\Gamma^j_{ih}(\gamma-P\gamma Q).$$

Now, making use of these relations for (3.3) in [14], we have

$$\begin{aligned}
&P^j_l\frac{'DP^l_t}{\partial u^h}\left(\frac{'DP^t_i}{\partial u^k}-{}'N^t_{ik}\right)+{}'N^j_{lh}P^l_t\frac{'DP^t_i}{\partial u^k}-\left(\frac{'DP^j_l}{\partial u^h}-{}'N^j_{lh}\right){}'N^l_{mk}P^m_i\\
&=P^j_l\Gamma^l_{th}(Q\gamma P-A\gamma)\Gamma^t_{ik}(Q\gamma P-\gamma)+\Gamma^j_{lh}(N\gamma)P^l_t\Gamma^t_{ik}(Q\gamma P-A\gamma)-\Gamma^j_{lh}(Q\gamma P-\gamma)\Gamma^l_{mk}(N\gamma)P^m_i\\
&=\Gamma^j_{th}(A\gamma P-P\gamma)\Gamma^t_{ik}(Q\gamma P-\gamma)+\Gamma^j_{th}(N\gamma P)\Gamma^t_{ik}(Q\gamma P-\gamma)-\Gamma^j_{lh}(Q\gamma P-\gamma)\Gamma^l_{ik}(N\gamma P)\\
&=\Gamma^j_{th}(A\gamma P-P\gamma+N\gamma P)\Gamma^t_{ik}(Q\gamma P-\gamma)-\Gamma^j_{lh}((Q\gamma P-\gamma)N)\Gamma^l_{ik}(\gamma P)\\
&=\Gamma^j_{th}(\gamma P-P\gamma)\Gamma^t_{ik}(Q\gamma P-\gamma)+\Gamma^j_{lh}(\gamma N)\Gamma^l_{ik}(\gamma P),^{3)}
\end{aligned}$$

hence

$$(1.12)\qquad \begin{aligned}R_i{}^j{}_{hk}=P^j_lP^t_i{}'R_m{}^l{}_{hk}P^m_t&+\Gamma^j_{lh}(\gamma P-P\gamma)\Gamma^l_{ik}(Q\gamma P-\gamma)-\Gamma^j_{lk}(\gamma P-P\gamma)\Gamma^l_{ih}(Q\gamma P-\gamma)\\
&+\Gamma^j_{lh}(\gamma N)\Gamma^l_{ik}(N\gamma P)-\Gamma^j_{lk}(\gamma N)\Gamma^l_{ih}(N\gamma P).\end{aligned}$$

Analogously, from (3.6) in [14], we get

$$(1.13)\qquad \begin{aligned}R_i{}^j{}_{hk}=P^j_l{}''R_m{}^l{}_{hk}P^m_tP^t_i&+\Gamma^j_{lh}(\gamma-P\gamma Q)\Gamma^l_{ik}(\gamma P-P\gamma)-\Gamma^j_{lk}(\gamma-P\gamma Q)\Gamma^l_{ih}(\gamma P-P\gamma)\\
&+\Gamma^j_{lh}(P\gamma N)\Gamma^l_{ik}(N\gamma)-\Gamma^j_{lk}(P\gamma N)\Gamma^l_{ih}(N\gamma).\end{aligned}$$

3) Since $\lambda(\gamma N)=PN=0$, γN is a tensor of type (1, 2). The second term can be written as $\Gamma^j_{lh}(\gamma N)\Gamma^l_{mk}(N\gamma P)$.

§ 2. The curvature tensor of a general connection $B\gamma B$.

Let γ be a normal general connection on $\mathfrak{X}$ as in § 1. Let B be a projection of $T(\mathfrak{X})$ such that

$$AB = BA. \tag{2.1}$$

Putting

$$\bar{A} = AB, \tag{2.2}$$

$\bar{A}$ is a projection of $T(\mathfrak{X})$ such that

$$A\bar{A}=\bar{A}A=\bar{A},\quad B\bar{A}=\bar{A}B=\bar{A},\quad N\bar{A}=\bar{A}N=0,\quad NB=BN=B-\bar{A}. \tag{2.3}$$

Let be assumed furthermore that

$$PB = BP, \tag{2.4}$$

then we have easily

$$P\bar{A}=\bar{A}P,\quad QB=BQ,\quad Q\bar{A}=\bar{A}Q,\quad Q\bar{N}=\bar{N}Q, \tag{2.5}$$

where $\bar{N}=1-AB$.

Now, let us consider a general connection $\bar{\gamma}=B\gamma B$, then $\bar{\gamma}$ is normal, because putting

$$\bar{P}=\lambda(\gamma)=BPB \quad \text{and} \quad \bar{Q}=BQB, \tag{2.6}$$

we have easily

$$\bar{P}\bar{Q}=\bar{Q}\bar{P}=\bar{A},\qquad \bar{P}\bar{N}=\bar{N}\bar{P}=0.$$

Now, let $'\bar{\gamma}=\bar{Q}\bar{\gamma}$ and $''\bar{\gamma}=\bar{\gamma}\bar{Q}$ be the contravariant part and the covariant part of the normal general connection $\bar{\gamma}$. By virtue of (2.5), we get easily

$$'\bar{\gamma}=\bar{Q}\bar{\gamma}=B(Q\gamma)B=B('\gamma)B \quad \text{and} \quad ''\bar{\gamma}=\bar{\gamma}\bar{Q}=B(\gamma Q)B=B(''\gamma)B. \tag{2.7}$$

Let $'\bar{R}_i{}^j{}_{hk}$ and $''\bar{R}_i{}^j{}_{hk}$ be the components of the curvature tensors of the contravariant part and the covariant part of the general connection $\bar{\gamma}$ respectively. By means of (1.2′), (1.3′) and

$$\lambda('\bar{\gamma})=B\lambda('\gamma)B=BAB=\bar{A},$$

we have

$$\begin{aligned}
'\bar{\Lambda}^j_{ih} &= \Gamma^j_{ih}('\bar{\gamma})-\frac{\partial P^j_i('\bar{\gamma})}{\partial u^h}\\
&= B^j_k\left\{\Gamma^k_{lh}('\gamma)B^l_i+P^k_l('\gamma)\frac{\partial B^l_i}{\partial u^h}\right\}-\frac{\partial \bar{A}^j_i}{\partial u^h}\\
&= \bar{A}^j_k{}'\Gamma^k_{lh}B^l_i+\bar{A}^j_l\frac{\partial B^l_i}{\partial u^h}-\frac{\partial \bar{A}^j_i}{\partial u^h}\\
&= \bar{A}^j_k{}'\Lambda^k_{lh}B^l_i+\bar{A}^j_k\frac{\partial A^k_l}{\partial u^h}B^l_i+\bar{A}^j_l\frac{\partial B^l_i}{\partial u^h}-\frac{\partial \bar{A}^j_i}{\partial u^h}.
\end{aligned}$$

Making use of (2.3) we get easily

$$'\bar{\Lambda}^j_{ih}=\bar{A}^j_k{}'\Lambda^k_{lh}B^l_i-\bar{N}^j_l\frac{\partial\bar{A}^l_i}{\partial u^h}. \tag{2.8}$$

Applying the formula (1.7) for the contravariant part $'\bar{\gamma}$ of $\bar{\gamma}$, we have

$$'\bar{R}_i{}^j{}_{hk}=\bar{A}^j_l\left(\frac{\partial'\bar{\Lambda}^l_{mk}}{\partial u^h}-\frac{\partial'\bar{\Lambda}^l_{mh}}{\partial u^k}+'\bar{\Lambda}^l_{th}{}'\bar{\Lambda}^t_{mk}-'\bar{\Lambda}^l_{tk}{}'\bar{\Lambda}^t_{mh}\right)\bar{A}^m_i.$$

Substituting (2.8) into the right, it can be written as

$$'\bar{R}_i{}^j{}_{hk}=B^j_p{}'R_q{}^p{}_{hk}B^q_i-\bar{A}^j_t\left('\Lambda^t_{sh}\bar{N}^s_p+\frac{\partial\bar{N}^t_p}{\partial u^h}\right)\left('\Lambda^p_{qk}\bar{A}^q_i+\frac{\partial\bar{A}^p_i}{\partial u^k}\right)$$

$$+\bar{A}^j_t\left('\Lambda^t_{sk}\bar{N}^s_p+\frac{\partial\bar{N}^t_p}{\partial u^k}\right)\left('\Lambda^p_{qh}\bar{A}^q_i+\frac{\partial\bar{A}^p_i}{\partial u^h}\right).$$

On the other hand, we have

$$\Gamma^j_{ih}('\gamma\bar{A})=\Gamma^j_{lh}('\gamma)\bar{A}^l_i+A^j_l\frac{\partial\bar{A}^l_i}{\partial u^h}$$

$$='\Lambda^j_{lh}\bar{A}^l_i+\frac{\partial A^j_l}{\partial u^h}\bar{A}^l_i+A^j_l\frac{\partial\bar{A}^l_i}{\partial u^h}='\Lambda^j_{lh}\bar{A}^l_i+\frac{\partial A^j_i}{\partial u^h}$$

and

$$\Gamma^j_{ih}(\bar{A}\cdot'\gamma\bar{N})=\bar{A}^j_k\Gamma^k_{ih}('\gamma\bar{N})=\bar{A}^j_k\left('\Gamma^k_{lh}\bar{N}^l_i+A^k_l\frac{\partial\bar{N}^l_i}{\partial u^h}\right)$$

$$=\bar{A}^j_k\left('\Lambda^k_{lh}\bar{N}^l_i+\frac{\partial\bar{N}^k_i}{\partial u^h}+\frac{\partial A^k_l}{\partial u^h}\bar{N}^l_i\right).$$

Making use of these, the above equation can be written as

$$'\bar{R}_i{}^j{}_{hk}=B^j_p{}'R_q{}^p{}_{hk}B^q_i-\left(\Gamma^j_{ph}(\bar{A}\cdot'\gamma\bar{N})-\bar{A}^j_t\frac{\partial A^t_l}{\partial u^h}\bar{N}^l_p\right)\Gamma^p_{ik}('\gamma A)$$

$$+\left(\Gamma^j_{pk}(\bar{A}\cdot'\gamma\bar{N})-\bar{A}^j_t\frac{\partial A^t_l}{\partial u^k}\bar{N}^l_p\right)\Gamma^p_{ih}('\gamma\bar{A}).$$

We have

$$\bar{A}^j_t\frac{\partial A^t_l}{\partial u^h}\bar{N}^l_pA^p_i=\frac{\partial\bar{A}^j_t}{\partial u^h}N^t_l\bar{N}^l_pA^p_i=0$$

and

$$\Gamma^p_{ik}('\gamma\bar{A})=A^p_l\Gamma^l_{ik}('\gamma\bar{A}),$$

since $A('\gamma\bar{A})=A(Q\gamma\bar{A})=(AQ)\gamma\bar{A}=Q\gamma\bar{A}='\gamma\bar{A}$. Therefore, the right of the above equation can be written as

$$'\bar{R}_i{}^j{}_{hk}=B^j_p{}'R_q{}^p{}_{hk}B^q_i-\Gamma^j_{ph}(\bar{A}Q\gamma\bar{N})\Gamma^p_{ik}(Q\gamma\bar{A})+\Gamma^j_{pk}(\bar{A}Q\gamma\bar{N})\Gamma^p_{ih}(Q\gamma\bar{A}).$$

Now, regarding the second term and the third term of the right of the equation,

$\Gamma^j_{ph}(\bar{A}Q\gamma\bar{N})$ are the components of a tensor of type $(1,2)$ but $\Gamma^p_{ih}(Q\gamma\bar{A})$ are not so, because $\lambda(\bar{A}Q\gamma\bar{N})=\bar{A}QP\bar{N}=\bar{A}A\bar{N}=\bar{A}\bar{N}=0$ but $\lambda(Q\gamma\bar{A})=QP\bar{A}=\bar{A}\neq 0$. Since $\bar{A}Q\gamma\bar{N}$ is a tensor and $\bar{N}^2=\bar{N}$, we have

$$\Gamma^j_{ph}(\bar{A}Q\gamma\bar{N})=\Gamma^j_{lh}(\bar{A}Q\gamma\bar{N})\bar{N}^l_p$$

and since $\lambda(\bar{N}Q\gamma\bar{A})=\bar{N}\bar{A}=0$, $\bar{N}Q\gamma\bar{A}$ is a tensor. Accordingly, we have the formula of $'\bar{R}_i{}^j{}_{hk}$ in tensorial form as follows:

$$'\bar{R}_i{}^j{}_{hk}=B^j_p{}'R_q{}^p{}_{hk}B^q_i-\Gamma^j_{ph}(\bar{A}Q\gamma\bar{N})\Gamma^p_{ik}(\bar{N}Q\gamma\bar{A})+\Gamma^j_{pk}(\bar{A}Q\gamma\bar{N})\Gamma^p_{ih}(\bar{N}Q\gamma\bar{A}). \tag{2.9}$$

Analogously, we obtain

$$''\bar{R}_i{}^j{}_{hk}=B^j_p{}''R_q{}^p{}_{hk}B^q_i-\Gamma^j_{ph}(\bar{A}\gamma Q\bar{N})\Gamma^p_{ik}(\bar{N}\gamma Q\bar{A})+\Gamma^j_{pk}(\bar{A}\gamma Q\bar{N})\Gamma^p_{ih}(\bar{N}\gamma Q\bar{A}). \tag{2.10}$$

Lastly, making use of (2.9) and (1.12), we compute the components of $\bar{R}_i{}^j{}_{hk}$ of the curvature tensor of the general connection $\bar{\gamma}=B\gamma B$ in terms of the components of γ and B. We have easily

$$\bar{\gamma}\bar{P}-\bar{P}\bar{\gamma}=B(\gamma P-P\gamma)B,\ \bar{Q}\bar{\gamma}\bar{P}-\bar{\gamma}=B(Q\gamma P-\gamma)B,$$
$$\bar{\gamma}\bar{N}=B\gamma NB,\ \bar{N}\bar{\gamma}\bar{P}=BN\gamma PB.$$

Accordingly, we have

$$\begin{aligned}
\bar{R}_i{}^j{}_{hk}=&\bar{P}^j_t\bar{P}^t_l{}'\bar{R}^l_{m\,hk}\bar{P}^m_i\\
&+\Gamma^j_{lh}(\bar{\gamma}\bar{P}-\bar{P}\bar{\gamma})\Gamma^l_{ik}(\bar{Q}\bar{\gamma}\bar{P}-\bar{\gamma})-\Gamma^j_{lk}(\bar{\gamma}\bar{P}-\bar{P}\bar{\gamma})\Gamma^l_{ih}(\bar{Q}\bar{\gamma}\bar{P}-\bar{\gamma})\\
&+\Gamma^j_{lh}(\bar{\gamma}\bar{N})\Gamma^l_{ik}(\bar{N}\bar{\gamma}\bar{P})-\Gamma^j_{lk}(\bar{\gamma}\bar{N})\Gamma^l_{ih}(\bar{N}\bar{\gamma}\bar{P})\\
=&\{B^j_pP^p_tP^t_l{}'R_m{}^l{}_{hk}P^m_qB^q_i-\Gamma^j_{ph}(\bar{P}^2\bar{A}Q\gamma\bar{N})\Gamma^p_{ik}(\bar{N}Q\gamma\bar{A}\bar{P})\\
&+\Gamma^i_{pk}(\bar{P}^2\bar{A}Q\gamma\bar{N})\Gamma^p_{ih}(\bar{N}Q\gamma\bar{A}\bar{P})\}\\
&+\Gamma^j_{lh}(\bar{\gamma}\bar{P}-\bar{P}\bar{\gamma})\Gamma^l_{ik}(\bar{Q}\bar{\gamma}\bar{P}-\bar{\gamma})-\Gamma^j_{lk}(\bar{\gamma}\bar{P}-\bar{P}\bar{\gamma})\Gamma^l_{ih}(\bar{Q}\bar{\gamma}\bar{P}-\bar{\gamma})\\
&+\Gamma^j_{lh}(\bar{\gamma}\bar{N})\Gamma^l_{ik}(\bar{N}\bar{\gamma}\bar{P})-\Gamma^j_{lk}(\bar{\gamma}\bar{N})\Gamma^l_{ih}(\bar{N}\bar{\gamma}\bar{P})\\
=&B^j_p\{P^p_tP^t_l{}'R_m{}^l{}_{hk}P^m_q\\
&+\Gamma^p_{lh}(\gamma P-P\gamma)B^l_m\Gamma^m_{qk}(Q\gamma P-\gamma)-\Gamma^p_{lk}(\gamma P-P\gamma)B^l_m\Gamma^m_{qh}(Q\gamma P-\gamma)\\
&+\Gamma^p_{lh}(\gamma N)B^l_m\Gamma^m_{qk}(N\gamma P)-\Gamma^p_{lk}(\gamma N)B^l_m\Gamma^m_{qh}(N\gamma P)\}B^q_i\\
&-\Gamma^j_{ph}(\bar{P}^2\bar{A}Q\gamma\bar{N})\Gamma^p_{ik}(\bar{N}Q\gamma\bar{A}\bar{P})+\Gamma^j_{pk}(\bar{P}^2\bar{A}Q\gamma\bar{N})\Gamma^p_{ih}(\bar{N}Q\gamma\bar{A}\bar{P}).
\end{aligned}$$

By virtue of (1.12), the right of the equation can be written as

$$\begin{aligned}
=&B^j_pR_q{}^p{}_{hk}B^q_i-\Gamma^j_{ph}(B(\gamma P-P\gamma))\Gamma^p_{ik}((1-B)(Q\gamma P-\gamma)B)\\
&+\Gamma^j_{pk}(B(\gamma P-P\gamma))\Gamma^p_{ih}((1-B)(Q\gamma P-\gamma)B)\\
&-\Gamma^j_{ph}(B\gamma N)\Gamma^p_{ik}((1-B)N\gamma PB)+\Gamma^j_{pk}(B\gamma N)\Gamma^p_{ih}((1-B)N\gamma PB)\\
&-\Gamma^j_{ph}(\bar{P}^2\bar{A}Q\gamma\bar{N})\Gamma^p_{ik}(\bar{N}Q\gamma\bar{A}\bar{P})+\Gamma^j_{pk}(\bar{P}^2\bar{A}Q\gamma\bar{N})\Gamma^p_{ih}(\bar{N}Q\gamma\bar{A}\bar{P}).
\end{aligned}$$

By means of (2.2)~(2.5), we have $\bar{P}^2\bar{A}Q\gamma\bar{N}=PB\gamma(1-AB)$ and $\bar{N}Q\gamma\bar{A}\bar{P}=Q(1-AB)\gamma BP$. Making use of the property that the general connections in the parentheses belonging to each Γ^j_{ih} of the right of the above equation are tensors and simply writing $\Gamma^j_{ih}(\gamma_1)\Gamma^l_{ik}(\gamma_2)$ by $\{\gamma_1\}\{\gamma_2\}$, we can take the following changes:

$$\begin{aligned}
&-\{B(\gamma P-P\gamma)\}\{(1-B)(Q\gamma P-\gamma)B\}\\
&-\{B\gamma N\}\{(1-B)N\gamma PB\}\\
&-\{PB\gamma(1-AB)\}\{Q(1-AB)\gamma BP\}\\
=&-\{B\gamma(1-B)P-PB\gamma(1-B)\}\{Q\gamma BP-\gamma B\}\\
&-\{B\gamma(1-B)\}\{(1-A)\gamma BP\}-\{PB\gamma(1-B)\}\{Q(1-B)\gamma BP\}\\
=&-\{B\gamma(1-B)\}\{PQ\gamma BP-P\gamma B+(1-A)\gamma BP\}\\
&+\{PB\gamma(1-B)\}\{Q\gamma BP-\gamma B-Q(1-B)\gamma BP\}\\
=&-\{B\gamma(1-B)\}\{\gamma BP\}-\{PB\gamma(1-B)\}\{\gamma B\}+\{B\gamma(1-B)\}\{P\gamma B\}\\
=&-\{B\gamma(1-B)\}\{(1-B)\gamma BP\}-\{PB\gamma(1-B)\}\{(1-B)\gamma B\}\\
&+\{B\gamma(1-B)\}\{P(1-B)\gamma B\}.
\end{aligned}$$

Thus, we obtain a formula showing a relation between the curvatures of the normal general connections γ and $B\gamma B$:

$$\begin{aligned}
\bar{R}_i{}^j{}_{hk}=&B^j_pR_q{}^p{}_{hk}B^q_i\\
&-P^l_i\{\Gamma^j_{ph}(B\gamma(1-B))\Gamma^p_{lk}((1-B)\gamma B)-\Gamma^l_{pk}(B\gamma(1-B))\Gamma^p_{ih}((1-B)\gamma B)\}\\
&-\{\Gamma^j_{ph}(B\gamma(1-B))\Gamma^p_{lk}((1-B)\gamma B)-\Gamma^j_{pk}(B\gamma(1-B))\Gamma^p_{lh}((1-B)\gamma B)\}P^l_i\\
&+\Gamma^j_{lh}(B\gamma(1-B))P^l_m\Gamma^m_{ik}((1-B)\gamma B)-\Gamma^j_{lk}(B\gamma(1-B))P^l_m\Gamma^m_{ih}((1-B)\gamma B).
\end{aligned}\tag{2.11}$$

§3. Induced general connections.

Let γ be a general connection of $\mathfrak{X}$ given by (1.1) in terms of local coordinates u^i of $\mathfrak{X}$. Let $\mathfrak{Y}$ be an m-dimensional submanifold of $\mathfrak{X}$ with the imbedding map ι: $\mathfrak{Y}\rightarrow\mathfrak{X}$.

Let us take a field Z of $(n-m)$-dimensional tangent subspaces of $\mathfrak{X}$ given on $\iota(\mathfrak{Y})$ such that $\iota_*(T_y(\mathfrak{Y})$ and $Z(\iota(y))$ is complement with each other in $T_{\iota(y)}(\mathfrak{X})$ for any point y of $\mathfrak{Y}$. In local coodinates v^α, $\alpha=1, \cdots, m$, of $\mathfrak{Y}$, let ι be written as

$$u^j = u^j(v^\alpha). \tag{3.1}$$

Let $\{X_\alpha, X_\lambda\}$, $\alpha=1, \cdots, m$, $\lambda=m+1, \cdots, n$, be a local field of n-frames of $\mathfrak{X}$ on $\iota(\mathfrak{Y})$ such that

$$X_\alpha=X^j_\alpha\partial/\partial u^j,\quad X^j_\alpha=\partial u^j/\partial v^\alpha\quad\text{and}\quad X_\lambda=X^j_\lambda\partial/\partial u^j\in Z \tag{3.2}$$

and $\{Y^\alpha, Y^\lambda\}$ with local components Y^α_i, Y^λ_i, be its dual. Then, we say the general connection of $\mathfrak{Y}$:

$$\gamma^*=\partial v_\beta\otimes Y^\beta_j\iota^*(P^j_i d^2u^i+\Gamma^j_{ih}du^i\otimes du^h)\,^{4)} \tag{3.3}$$

4) For the differential forms d^2u^i of order 2, $\iota^*d^2u^i$ are naturally defined by

$$\iota^*d^2u^i=\frac{\partial u^i}{\partial v^\alpha}d^2v^\alpha+\frac{\partial^2u^i}{\partial v^\beta\partial v^\alpha}dv^\alpha\otimes dv^\beta.$$

the induced general connection on $\mathfrak{Y}$ from γ by means of the complementary field Z. We can easily prove that the general connection γ^* does not depend on the local coordinates u^i, v^α and it is determined only by the submanifold $(\iota, \mathfrak{Y})$ of $\mathfrak{X}$, Z and γ.

THEOREM 1. *Let $(\iota, \mathfrak{Y})$ be an m-dimensional submanifold of $\mathfrak{X}$ and Z be a field of tangent subspaces of $\mathfrak{X}$ defined on $\iota(\mathfrak{Y})$ complementary to $\iota_*(T(\mathfrak{Y}))$. Let γ be a general connection of $\mathfrak{X}$ and B be a projection of $T(\mathfrak{X})$ such that the image and the kernel of B at each point of $\iota(\mathfrak{Y})$ are identical with the tangent space of $\iota(\mathfrak{Y})$ and Z, respectively. Let γ^* and $(B\gamma B)^*$ be the induced general connections on $\mathfrak{Y}$ from γ and $B\gamma B$ by means of Z, respectively. Then, we have $\gamma^*=(B\gamma B)^*$.*

Proof. By the assumptions in the theorem, we have

$$B_i^j X_\alpha^i = X_\alpha^j \qquad \text{and} \qquad B_i^j X_\lambda^i = 0 \tag{3.4}$$

on $\iota(\mathfrak{Y})$, hence we have

$$B_i^j = X_\alpha^j Y_i^\alpha. \tag{3.5}$$

On the other hand, representing γ by (1.1), $B\gamma B$ can be written in terms of local coordinates as

$$B\gamma B = \partial u_j \otimes B_l^j \{P_k^l d(B_i^k du^i) + \Gamma_{kh}^l B_i^k du^i \otimes du^h\},$$

hence we have

$$\begin{aligned}(B\gamma B)^* &= \partial v_\beta \otimes Y_j^\beta \iota^*(B_l^j \{P_k^l d(B_i^k du^i) + \Gamma_{kh}^l B_i^k du^i \otimes du^h\}) \\ &= \partial v_\beta \otimes Y_l^\beta \iota^* \{P_k^l d(B_i^k du^i) + \Gamma_{kh}^l B_i^k du^i \otimes du^h\}.\end{aligned}$$

Since $\iota^*(B_i^k du^i) = B_i^k X_\alpha^i dv^\alpha = X_\alpha^k dv^\alpha = \iota^* du^k$, we get

$$(B\gamma B)^* = \partial v_\alpha \otimes Y_l^\alpha \iota^* \{P_k^l d^2 u^k + \Gamma_{kh}^l du^k \otimes du^h\} = \gamma^*. \qquad \text{q.e.d.}$$

THEOREM 2. *Under the assumptions in* THEOREM 1, *let $\bar{R}_i{}^j{}_{hk}$ be the components of the curvature tensor of the general connection $B\gamma B$, then $Y_j^\beta \bar{R}_i{}^j{}_{hk} X_\alpha^i X_\sigma^h X_\tau^k$ are the components of the curvature tensor of the induced general connection γ^*.*

Proof. Let us take a family of m-dimensional surfaces such that it is written as

$$u^j = u^j(v^1, \cdots, v^m; v^{m+1}, \cdots, v^n) \tag{3.6}$$

which are identical with (3.1), when $v^{m+1} = \cdots = v^n = 0$, and the family simply covers a neighborhood of $\mathfrak{X}$. Then, $v^1, \cdots, v^n$ can be regarded as local coordinates of $\mathfrak{X}$. Making use of the coordinates, we have on the surface $\iota(\mathfrak{Y})$

$$X_\alpha^j = \delta_\alpha^j, \quad B_\alpha^j = \delta_\alpha^j, \quad B_i^\lambda = 0, \quad Y_\alpha^\beta = \delta_\alpha^\beta, \quad \alpha, \beta = 1, \cdots, m; \ \lambda = m+1, \cdots, n.$$

Now, we put

$$B\gamma B = \partial v_j \otimes \{\bar{P}_i^j d^2 v^i + \bar{\Gamma}_{ih}^j dv^i \otimes dv^h\},$$

then we get on the surface $\iota(\mathfrak{Y})$

$$\bar{P}_i^\lambda = B_j^\lambda P_l^j B_i^l = 0,$$

$$\bar{\Gamma}^{\lambda}_{ij}=B^{\lambda}_{l}\left\{\Gamma^{l}_{kh}B^{k}_{i}+P^{l}_{k}\frac{\partial B^{k}_{i}}{\partial v^{h}}\right\}=0,\qquad \bar{\Lambda}^{\lambda}_{ih}=0$$

and so

$$\begin{aligned}\bar{R}_{\alpha}{}^{\beta}{}_{\sigma\tau}&=\left\{\bar{P}^{\beta}_{l}\left(\frac{\partial\bar{\Gamma}^{l}_{m\tau}}{\partial v^{\sigma}}-\frac{\partial\bar{\Gamma}^{l}_{m\sigma}}{\partial v^{\tau}}\right)+\bar{\Gamma}^{\beta}_{l\sigma}\bar{\Gamma}^{l}_{m\tau}-\bar{\Gamma}^{\beta}_{l\tau}\bar{\Gamma}^{l}_{m\sigma}\right\}\bar{P}^{m}_{\alpha}\\&\quad-\delta^{\beta}_{m,\sigma}\bar{\Lambda}^{m}_{\alpha\tau}+\delta^{\beta}_{m,\tau}\bar{\Lambda}^{m}_{\alpha\sigma}\\&=\left\{\bar{P}^{\beta}_{\rho}\left(\frac{\partial\bar{\Gamma}^{\rho}_{\delta\tau}}{\partial v^{\sigma}}-\frac{\partial\bar{\Gamma}^{\rho}_{\delta\sigma}}{\partial v^{\tau}}\right)+\bar{\Gamma}^{\beta}_{\rho\sigma}\bar{\Gamma}^{\rho}_{\delta\tau}-\bar{\Gamma}^{\beta}_{\rho\tau}\bar{\Gamma}^{\rho}_{\delta\sigma}\right\}\bar{P}^{\delta}_{\alpha}\\&\quad-\delta^{\beta}_{\delta,\sigma}\bar{\Lambda}^{\delta}_{\alpha\tau}+\delta^{\beta}_{\delta,\tau}\bar{\Lambda}^{\delta}_{\alpha\sigma},\qquad \delta^{\beta}_{\delta,\sigma}=-\bar{P}^{\beta}_{\rho}\bar{\Lambda}^{\rho}_{\delta\sigma}+\bar{\Gamma}^{\beta}_{\rho\sigma}\bar{P}^{\rho}_{\delta},\end{aligned}$$

where indices l, m run on $1, 2, \cdots, n$ and indices α, β, δ, σ, τ, ρ run on $1, 2, \cdots, m$.

On the other hand, in the local coordinates $v^{1}\cdots v^{m}$ of $\mathfrak{Y}$, $(B\gamma B)^{*}$ can be written as

$$\begin{aligned}(B\gamma B)^{*}&=\partial v_{\beta}\otimes Y^{\beta}_{j}\iota^{*}\{\bar{P}^{j}_{i}d^{2}v^{i}+\bar{\Gamma}^{j}_{ih}dv^{i}\otimes dv^{h}\}\\&=\partial v_{\beta}\otimes Y^{\beta}_{\rho}\{\bar{P}^{\rho}_{\alpha}d^{2}v^{\alpha}+\bar{\Gamma}^{\rho}_{\alpha\sigma}dv^{\alpha}\otimes dv^{\sigma}\}\\&=\partial v_{\beta}\otimes\{\bar{P}^{\beta}_{\alpha}d^{2}v^{\alpha}+\bar{\Gamma}^{\beta}_{\alpha\sigma}dv^{\alpha}\otimes dv^{\sigma}\}.\end{aligned}$$

Hence, the components of the curvature tensor of $(B\gamma B)^{*}=\gamma^{*}$ with respect to the coordinates v^{α} are $\bar{R}_{\alpha}{}^{\beta}{}_{\sigma\tau}$. Accordingly, if $\bar{R}_{i}{}^{j}{}_{hk}$ are the components of the curvature tensor of $B\gamma B$ with respect to the coordinates $u^{1}, \cdots, u^{n}$, they are given by $Y^{\beta}_{j}\bar{R}_{i}{}^{j}{}_{hk}X^{i}_{\alpha}X^{h}_{\sigma}X^{k}_{\tau}$. q.e.d.

§4. The Gauss' equation and the general connection $B\gamma B$.

In this section, we apply the formula (2.11) to the case, in which γ is an affine connection, that is $\lambda(\gamma)=1$. Then $P=Q=A=1$, (2.11) turns in

$$\begin{aligned}\bar{R}_{i}{}^{j}{}_{hk}&=B^{j}_{p}R_{q}{}^{p}{}_{hk}B^{q}_{i}\\&\quad-\Gamma^{j}_{ph}(B\gamma(1-B))\Gamma^{p}_{ik}((1-B)\gamma B)+\Gamma^{j}_{pk}(B\gamma(1-B))\Gamma^{p}_{ih}((1-B)\gamma B).\end{aligned}$$

Now, $B\gamma(1-B)$ and $(1-B)\gamma B$ are tensors of type (1, 2). We write the components of these tensors in terms of γ. By means of LEMMA 1.1, we have

$$\begin{aligned}\Gamma^{j}_{ih}(B\gamma(1-B))&=\Gamma^{j}_{ih}(B(\gamma(1-B)-(1-B)\gamma))=B^{j}_{l}\Gamma^{l}_{ih}(\gamma(1-B)-(1-B)\gamma)\\&=B^{j}_{l}(\delta^{l}_{i,h}-B^{l}_{i,h})=-B^{j}_{l}B^{l}_{i,h},\\ \Gamma^{j}_{ih}((1-B)\gamma B)&=\Gamma^{j}_{ih}(((1-B)\gamma-\gamma(1-B))B)=\Gamma^{j}_{lh}((1-B)\gamma-\gamma(1-B))B^{l}_{i}\\&=-(\delta^{j}_{l,h}-B^{j}_{l,h})B^{l}_{i}=B^{j}_{l,h}B^{l}_{i}.\end{aligned}$$

Hence, the above equation can be written as

$$\bar{R}_{i}{}^{j}{}_{hk}=B^{j}_{p}(R_{q}{}^{p}{}_{hk}+B^{p}_{l,h}B^{l}_{q,k}-B^{p}_{l,k}B^{l}_{q,h})B^{q}_{i}. \tag{4.1}$$

Now, let $\mathfrak{Y}$ be an m-dimensional submanifold of $\mathfrak{X}$ with the imbedding map ι: $\mathfrak{Y}\rightarrow\mathfrak{X}$ such that $\iota_*(T_y(\mathfrak{Y}))$, $y\in\mathfrak{Y}$, is the image of $T_{\iota(y)}(\mathfrak{X})$ under B. Let $\iota(\mathfrak{Y})$ be locally written by (3.1) and let $\{X_\alpha, X_\lambda\}$, $\alpha=1, \cdots, m$, $\lambda=m+1, \cdots, n$, be a local field of n-frames of $\mathfrak{X}$ such that $B(X_\alpha)=X_\alpha$, $B(X_\lambda)=0$ and putting $X_\alpha=X_\alpha^j\partial/\partial u^j$, $X_\lambda=X_\lambda^j\partial/\partial u^j$, $X_\alpha^j=\partial u^j/\partial v^\alpha$ on $\iota(\mathfrak{Y})$. Taking the dual frame $\{Y^\alpha, Y^\lambda\}$, $Y^\alpha=Y_i^\alpha du^i$, $Y^\lambda=Y_i^\lambda du^i$, we have

$$B_i^j=X_\alpha^j Y_i^\alpha \quad \text{and} \quad \delta_i^j=B_i^j+X_\lambda^i Y_i^\lambda.$$

Since we have

$$B_p^j B_{l,h}^p B_{q,k}^l B_i^q=B_p^j(X_\lambda^p Y_l^\lambda)_{,h}(X_\mu^l Y_q^\mu)_{,k}B_i^q=B_p^j X_{\lambda,h}^p Y_{q,k}^\lambda B_i^q,$$

we get from (4.1) the equation

$$\begin{aligned} Y_j^\beta \bar{R}_i{}^j{}_{hk}X_\alpha^i X_\sigma^h X_\tau^k &= Y_j^\beta R_i{}^j{}_{hk}X_\alpha^i X_\sigma^h X_\tau^k \\ &\quad + Y_j^\beta(X_{\lambda,h}^j Y_{i,k}^\lambda - X_{\lambda,k}^j Y_{i,h}^\lambda)X_\alpha^i X_\sigma^h X_\tau^k. \end{aligned} \tag{4.2}$$

Putting

$$Y_j^\beta X_{\lambda,h}^j X_\sigma^h = H_{(\lambda)\sigma}^\beta, \qquad Y_{i,k}^\lambda X_\alpha^i X_\tau^k = H_{\alpha\tau}^{(\lambda)}, \tag{4.3}$$

we get

$$Y_j^\beta \bar{R}_i{}^j{}_{hk}X_\alpha^i X_\sigma^h X_\tau^k = Y_j^\beta R_i{}^j{}_{hk}X_\alpha^i X_\sigma^h X_\tau^k + H_{(\lambda)\sigma}^\beta H_{\alpha\tau}^{(\lambda)} - H_{(\lambda)\tau}^\beta H_{\alpha\sigma}^{(\lambda)}, \tag{4.4}$$

$$\alpha, \beta, \sigma, \tau=1, 2, \cdots, m; \ \lambda=m+1, \cdots, n.$$

As is well known, on $\iota(\mathfrak{Y})$ $H_{\alpha\beta}^{(\lambda)}$, $H_{(\lambda)\alpha}^\beta$ are the components of the second fundamental tensor of the surface $\iota(\mathfrak{Y})$, in case of Riemannian geometry. By virtue of THEOREM 2, the left of (4.4) are the components of the curvature tensor of the induced connection γ^* from γ on $\mathfrak{Y}$ by means of the field $1-B$. Accordingly, the formula (4.4) is the Gauss' equation in classical differential geometry. Thus, we can regard the formula (2.11) as a generalization of the Gauss' equation.

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