ON CURVATURES OF SPACES WITH NORMAL GENERAL CONNECTIONS, I

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In a previous paper [7], the curvature tensor of a space with a general connection was defined by formulas analogous to the classical ones for the spaces with affine connections. As is well known, the Ricci's formula

$$V^j{}_{,hk} - V^j{}_{,kh} = -R_i{}^j{}_{hk} V^i$$

is fundamental for the theory of differential geometry in the large, for instance, the holonomy group.

In this paper, the author will investigate the formula for spaces with normal general connections, making use of the results obtained in [12] regarding basic curves in such spaces. He will use the notations in [8], [10], [11] and [12].

§1. The curvature tensor of a space with a general connection.

Let $\mathfrak{X}$ be an n-dimensional differentiable manifold with a general connection Γ written in terms of local coordinates u^i as

$$\Gamma = \partial u_j \otimes (P_i^j d^2u^i + \Gamma^j_{ih} du^i \otimes du^h).$$

By (6.28) in [7], the components of the curvature tensor of the space are given by

$$\begin{aligned} R_i{}^j{}_{hk} = &\left\{ P_l^j \left(\frac{\partial \Gamma^l_{mk}}{\partial u^h} - \frac{\partial \Gamma^l_{mh}}{\partial u^k} \right) + \Gamma^j_{lh}\Gamma^l_{mk} - \Gamma^j_{lk}\Gamma^l_{mh} \right\} P_i^m \\ &-\delta^j_{m,h} \Lambda^m_{ik} + \delta^j_{m,k} \Lambda^m_{ih}, \end{aligned} \tag{1.1}$$

where

$$\Lambda^j_{ih} = \Gamma^j_{ih} - \frac{\partial P_i^j}{\partial u^h}$$

and δ_i^j are the Kronecker's δ. The formulas can be written as follows:

$$\begin{aligned} R_i{}^j{}_{hk} = &\left\{ P_l^j \left(\frac{\partial \Lambda^l_{mk}}{\partial u^h} - \frac{\partial \Lambda^l_{mh}}{\partial u^k} \right) + \Gamma^j_{lh}\Gamma^l_{mk} - \Gamma^j_{lk}\Gamma^l_{mh} \right\} P_i^m \\ &-(\Gamma^j_{lh} P^l_m - P^j_l \Lambda^l_{mh}) \Lambda^m_{ik} + (\Gamma^j_{lk} P^l_m - P^j_l \Lambda^l_{mk}) \Lambda^m_{ih} \end{aligned}$$

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$$=P_i^j\left\{\left(\frac{\partial\Lambda^l_{mk}}{\partial u^h}-\frac{\partial\Lambda^l_{mh}}{\partial u^k}\right)P^m_i+\Lambda^l_{mh}\Lambda^m_{ik}-\Lambda^l_{mk}\Lambda^m_{ih}\right\}$$

$$+\Gamma^j_{ih}(\Gamma^l_{mk}P^m_i-P^l_m\Lambda^m_{ik})-\Gamma^j_{ik}(\Gamma^l_{mh}P^m_i-P^l_m\Lambda^m_{ih}),$$

hence

$$R_{i}{}^{j}{}_{hk}=P_l^j\left\{\left(\frac{\partial\Lambda^l_{mk}}{\partial u^h}-\frac{\partial\Lambda^l_{mh}}{\partial u^k}\right)P^m_i+\Lambda^l_{mh}\Lambda^m_{ik}-\Lambda^l_{mk}\Lambda^m_{ih}\right\}$$

$$+\Gamma^j_{ih}\delta^l_{i,k}-\Gamma^j_{ik}\delta^l_{i,h}. \tag{1.2}$$

(1.1) is of contravariant form and (1.2) is of covariant form of the component $R_{i}{}^{j}{}_{hk}$.

§2. The curvature tensors of the contravariant part and the covariant part of a normal general connection.

Let Γ be a normal general connection, i. e. let the tensor

$$P=\partial u_j\otimes P^j_i du^i=\lambda(\Gamma)$$

be normal.[1] Let Q be the tensor such that $Q=P^{-1}$ on the image of P and $Q=P$ on the kernel of P regarding P as a homomorphism of the tangent bundle $T(\mathfrak{X})$ of $\mathfrak{X}$. Let $A=QP=PQ$ with local components A^j_i be the projection of $T(\mathfrak{X})$ onto the image of P and $N=1-A$ with local components N^j_i be the projection of $T(\mathfrak{X})$ onto the kernel of P. We say A the *canonical projection* of Γ.

Let $'\Gamma=Q\Gamma$ be the contravariant part of Γ and $''\Gamma=\Gamma Q$ be the covariant part of Γ which are written as [2]

$$'\Gamma=\partial u_j\otimes(A^j_i d^2u^i+{}'\Gamma^j_{ih}du^i\otimes du^h)$$

and

$$''\Gamma=\partial u_j\otimes(A^j_i d^2u^i+{}''\Gamma^j_{ih}du^i\otimes du^h),$$

and we have

$$'\Gamma^j_{ih}=Q^j_l\Gamma^l_{ih}\quad\text{and}\quad ''\Lambda^j_{ih}=\Lambda^j_{lh}Q^l_i,$$

where

$$''\Lambda^j_{ih}={}''\Gamma^j_{ih}-\frac{\partial A^j_i}{\partial u^h}.$$

Now, we denote the components of the curvature tensors of the two normal general connections $'\Gamma$ and $''\Gamma$ by $'R_{i}{}^{j}{}_{hk}$ and $''R_{i}{}^{j}{}_{hk}$, which are defined by the formulas (1.1) for them respectively.

LEMMA 2.1. *For the curvature tensor of the contravariant part $'\Gamma=Q\Gamma$ of any normal general connection Γ we have*

$$A^j_l\left(\frac{\partial'\Lambda^l_{mk}}{\partial u^h}-\frac{\partial'\Lambda^l_{mh}}{\partial u^k}+{}'\Lambda^l_{th}{}'\Lambda^t_{mk}-{}'\Lambda^l_{tk}{}'\Lambda^t_{mh}\right)A^m_i={}'R_{i}{}^{j}{}_{hk}.\text{[3]} \tag{2.1}$$

1) See [10], §1.
2) See [10], §3 and [11], §1.
3) $'\Lambda^j_{ih}={}'\Gamma^j_{ih}-\dfrac{\partial A^j_i}{\partial u^h}$.

Proof. We have

$$A^j_l\left(\frac{\partial' \Lambda^l_{mk}}{\partial u^h}-\frac{\partial' \Lambda^l_{mh}}{\partial u^k}+'\Lambda^l_{th}{}'\Lambda^t_{mk}-'\Lambda^l_{tk}{}'\Lambda^t_{mh}\right)A^m_i$$

$$=\left\{A^j_l\left(\frac{\partial' \Gamma^l_{mk}}{\partial u^h}-\frac{\partial' \Gamma^l_{mh}}{\partial u^k}\right)+'\Gamma^j_{lh}{}'\Gamma^l_{mk}-'\Gamma^j_{lk}{}'\Gamma^l_{mh}\right\}A^m_i$$

$$-\left('\Gamma^j_{lh}\frac{\partial A^l_m}{\partial u^k}-'\Gamma^j_{lk}\frac{\partial A^l_m}{\partial u^h}\right)A^m_i-A^j_l\left(\frac{\partial A^l_t}{\partial u^h}\,'\Lambda^t_{mk}-\frac{\partial A^l_t}{\partial u^k}\,'\Lambda^t_{mh}\right)A^m_i.$$

Making use of $'R_i{}^j{}_{hk}$, these can be written as

$$='R_i{}^j{}_{hk}+('\Gamma^j_{lh}A^l_m-A^j_l{}'\Lambda^l_{mh})'\Lambda^m_{ik}-('\Gamma^j_{lk}A^l_m-A^j_l{}'\Lambda^l_{mk})'\Lambda^m_{ih}$$

$$-\left('\Gamma^j_{lh}\frac{\partial A^l_m}{\partial u^k}-'\Gamma^j_{lk}\frac{\partial A^l_m}{\partial u^h}\right)A^m_i-A^j_l\left(\frac{\partial A^l_t}{\partial u^h}\,'\Lambda^t_{mk}-\frac{\partial A^l_t}{\partial u^k}\,'\Lambda^t_{mh}\right)A^m_i.$$

On the other hand, we have easily

$$'\Gamma^j_{lh}\left(A^l_m{}'\Lambda^m_{ik}-\frac{\partial A^l_m}{\partial u^k}A^m_i\right)='\Gamma^j_{lh}{}'\Lambda^l_{ik}$$

and

$$A^j_l{}'\Lambda^l_{mh}{}'\Lambda^m_{ik}+A^j_l\frac{\partial A^l_t}{\partial u^h}\,'\Lambda^t_{mk}A^m_i$$

$$=A^j_l{}'\Gamma^l_{mh}{}'\Lambda^m_{ik}-A^j_l\frac{\partial A^l_m}{\partial u^h}('\Lambda^m_{ik}-'\Lambda^m_{tk}A^t_i)$$

$$='\Gamma^j_{lh}{}'\Lambda^l_{ik}-\frac{\partial A^j_l}{\partial u^h}N^l_m\left('\Gamma^m_{tk}N^t_i-A^m_t\frac{\partial A^t_i}{\partial u^k}\right)='\Gamma^j_{lh}{}'\Lambda^l_{ik}.$$

Hence, the terms but $'R_i{}^j{}_{hk}$ of the last side of the above equation cancel with each others and so we get (2.1). q. e. d.

LEMMA 2.2. *For the curvature tensor of the covariant part $''\Gamma=\Gamma Q$ of any normal general connection Γ, we have*

$$A^j_l\left(\frac{\partial'' \Gamma^l_{mk}}{\partial u^h}-\frac{\partial'' \Gamma^l_{mh}}{\partial u^k}+''\Gamma^l_{th}{}''\Gamma^t_{mk}-''\Gamma^l_{tk}{}''\Gamma^t_{mh}\right)A^m_i=''R_i{}^j{}_{hk}. \tag{2.2}$$

Proof. We have

$$A^j_l\left(\frac{\partial'' \Gamma^l_{mk}}{\partial u^h}-\frac{\partial'' \Gamma^l_{mh}}{\partial u^k}+''\Gamma^l_{th}{}''\Gamma^t_{mk}-''\Gamma^l_{tk}{}''\Gamma^t_{mh}\right)A^m_i$$

$$=\left\{A^j_l\left(\frac{\partial'' \Gamma^l_{mk}}{\partial u^h}-\frac{\partial'' \Gamma^l_{mh}}{\partial u^k}\right)+''\Gamma^j_{th}{}''\Gamma^t_{mk}-''\Gamma^j_{tk}{}''\Gamma^t_{mh}\right\}A^m_i$$

$$-N^j_l(''\Gamma^l_{th}{}''\Gamma^t_{mk}-''\Gamma^l_{tk}{}''\Gamma^t_{mh})A^m_i$$

$$=''R_i{}^j{}_{hk}+(''\Gamma^j_{th}A^t_m-A^j_t{}''\Lambda^t_{mh})''\Lambda^m_{ik}-(''\Gamma^j_{tk}A^t_m-A^j_t{}''\Lambda^t_{mk})''\Lambda^m_{ih}$$

$$-N^j_l(''\Gamma^l_{th}{}''\Gamma^t_{mk}-''\Gamma^l_{tk}{}''\Gamma^t_{mh})A^m_i.$$

On the other hand, we have easily

$$''\Gamma^j_{th}A^t_m - A^{j\prime\prime}_t A^t_{mh} = ''A^j_{mh} + \frac{\partial A^j_t}{\partial u^h} A^t_m - A^{j\prime\prime}_t A^t_{mh} = N^{j\prime\prime}_t \Gamma^t_{mh}$$

and

$$N^{j\prime\prime}_t \Gamma^t_{mh}(''A^m_{ik} - ''\Gamma^m_{lk}A^l_i) = -N^{j\prime\prime}_t \Gamma^t_{mh}\frac{\partial A^m_l}{\partial u^k} A^l_i$$

$$= -N^{j\prime\prime}_t \Gamma^t_{mh} N^m_l \frac{\partial A^l_i}{\partial u^h} = -N^j_t \frac{\partial A^t_m}{\partial u^h} N^m_l \frac{\partial A^l_i}{\partial u^h} = 0.$$

Hence, the terms but $''R_i{}^j{}_{hk}$ of the last side of the above equation cancel with each others and so we get (2.2). q. e. d.

We denote the curvature forms of the normal general connection Γ, its contravariant part $'\Gamma$ and its covariant part $''\Gamma$ by

$$\Omega^j_i = \frac{1}{2} R_i{}^j{}_{hk} du^h \wedge du^k, \qquad '\Omega^j_i = \frac{1}{2}\, 'R_i{}^j{}_{hk} du^h \wedge du^k$$

and

$$''\Omega^j_i = \frac{1}{2}\, ''R_i{}^j{}_{hk} du^h \wedge du^k$$

respectively. We say a tensor is *A-invariant* if it is invariant under the homomorphism on the tensor bundles over $\mathfrak{X}$ induced from A. From these lemmas, we obtain easily the following

THEOREM 1. *The curvature forms of the contravariant part and the covariant part of a normal general connection Γ are A-invariant on any A-invariant 2-dimensional tangent plane, where A is the canonical projection of Γ.*

§ 3. Some relations between the curvature tensors of a normal general connection and its contravariant and covariant parts.

For a normal general connection Γ, let be

$$'N_i{}^j{}_h = N^j_l \Gamma^l_{ih} \quad \text{and} \quad ''N_i{}^j{}_h = A^j_{kh} N^k_i,$$

which are the local components of the general connections $N\Gamma$ and ΓN respectively. Since $\lambda(N\Gamma) = \lambda(\Gamma N) = 0$, $N\Gamma$ and ΓN are both tensor fields of type (1, 2) on $\mathfrak{X}$.

From the formula (1.1), we have

$$R_i{}^j{}_{hk} = \left\{ P^j_l \left(\frac{\partial P^l_t\, '\Gamma^t_{mk}}{\partial u^h} - \frac{\partial P^l_t\, '\Gamma^t_{mh}}{\partial u^k} \right) + \Gamma^j_{lh} P^l_t\, '\Gamma^t_{mk} - \Gamma^j_{lk} P^l_t\, '\Gamma^t_{mh} \right\} P^m_i$$

$$+ \left\{ P^j_l \left(\frac{\partial\, 'N_m{}^l{}_k}{\partial u^h} - \frac{\partial\, 'N_m{}^l{}_h}{\partial u^k} \right) + \Gamma^j_{lh}\, 'N_m{}^l{}_k - \Gamma^j_{lk}\, 'N_m{}^l{}_h \right\} P^m_i$$

$$- \delta^j_{m,h} A^m_{ik} + \delta^j_{m,k} A^m_{ih}.$$

Now, we denote the covariant differential operator of $'\Gamma = Q\Gamma$ by $'D$. Since P is A-invariant, we have

(3.1) $$\frac{'DP_i^j}{\partial u^h}=\frac{\partial P_i^j}{\partial u^h}+'\Lambda_{lh}^j P_i^l-P_l^j\,'\Gamma_{ih}^l=A_l^j\,\frac{\partial P_m^l}{\partial u^h}\,A_i^m+'\Gamma_{lh}^j P_i^l-P_l^j\,'\Lambda_{ih}^l.$$

Since

$$\begin{aligned}\delta_{i,h}^j&=\Gamma_{lh}^j P_i^l-P_l^j\Lambda_{ih}^l\\&=P_l^j\left('\Lambda_{th}^l P_i^t+\frac{\partial A_t^l}{\partial u^h}\,P_i^t-P_t^l\,'\Gamma_{ih}^t+\frac{\partial P_i^l}{\partial u^h}\right)+'N_{lh}^j P_i^l,\end{aligned}$$

we have

(3.2) $$\delta_{i,h}^j=P_l^j\,\frac{'DP_i^l}{\partial u^h}+'N_l{}^j{}_h P_i^l.$$

Making use of (2.1), (3.1) and (3.2), the above equations can be written as

$$\begin{aligned}R_i{}^j{}_{hk}=&M_l^j\,'R_m{}^l{}_{hk}P_i^m+P_l^j\left\{\frac{'DP_t^l}{\partial u^h}('\Lambda_{mk}^t P_i^m-\Lambda_{ik}^t)-\frac{'DP_t^l}{\partial u^k}('\Lambda_{mh}^t P_i^m-\Lambda_{ih}^t)\right\}\\&+'N_l{}^j{}_h P_t^l('\Gamma_{mk}^t P_i^m-\Lambda_i{}^t{}_k)-'N_l{}^j{}_k P_t^l('\Gamma_{mh}^t P_i^m-\Lambda_{ih}^t)\\&-\left\{\left(\frac{'DP_l^j}{\partial u^h}-'N_l{}^j{}_h\right)'N_m{}^l{}_k-\left(\frac{'DP_l^j}{\partial u^k}-'N_l{}^j{}_k\right)'N_m{}^l{}_h\right\}P_i^m,\end{aligned}$$

where $M_i^j=P_l^jP_i^l$. Furthermore

$$\begin{aligned}'\Lambda_{mk}^t P_i^m-\Lambda_{ik}^t&=\frac{\partial P_i^t}{\partial u^k}-P_m^t\,'\Gamma_{ik}^m+'\Lambda_{mk}^t P_i^m-'N_i{}^t{}_k\\&=\frac{'DP_i^t}{\partial u^k}-'N_i{}^t{}_k\end{aligned}$$

and

$$'\Gamma_{mk}^t P_i^m-\Lambda_{ik}^t=\frac{'DP_i^t}{\partial u^k}-'N_i{}^t{}_k+\frac{\partial A_m^t}{\partial u^k}\,P_i^m,$$

the above equations can be written as

(3.3) $$\begin{aligned}R_i{}^j{}_{hk}=&M_l^j\,'R_m{}^l{}_{hk}P_i^m\\&+P_l^j\left\{\frac{'DP_t^l}{\partial u^h}\left(\frac{'DP_i^t}{\partial u^k}-'N_i{}^t{}_k\right)-\frac{'DP_t^l}{\partial u^k}\left(\frac{'DP_i^t}{\partial u^h}-'N_i{}^t{}_h\right)\right\}\\&+'N_l{}^j{}_h P_t^l\,\frac{'DP_i^t}{\partial u^k}-'N_l{}^j{}_k P_t^l\,\frac{'DP_i^t}{\partial u^h}\\&-\left\{\left(\frac{'DP_l^j}{\partial u^h}-'N_l{}^j{}_h\right)'N_m{}^l{}_k-\left(\frac{'DP_l^j}{\partial u^k}-'N_l{}^j{}_k\right)'N_m{}^l{}_h\right\}P_i^m.\end{aligned}$$

We call attention to the fact that the right side of the above formula is written in terms of the quantities of $Q\Gamma$ and $N\Gamma$.

In the next place, we will try to describe $R_i{}^j{}_{hk}$ in terms of the quantities of ΓQ and ΓN. Analogously to (3.1) and (3.2), we have

(3.4) $$\frac{''DP_i^j}{\partial u^h}=\frac{\partial P_i^j}{\partial u^h}+''\Lambda_{lh}^j P_i^l-P_l^j\,''\Gamma_{ih}^l=A_l^j\,\frac{\partial P_m^l}{\partial u^h}\,A_i^m+''\Gamma_{lh}^j P_i^l-P_l^j\,''\Lambda_{ih}^l$$

and

$$\begin{aligned}\delta^j_{i,h} &= \Gamma^j_{l h} P^l_i - P^j_l \Lambda^l_{ih} \\ &= \left(\frac{\partial P^j_l}{\partial u^h} + {}''\Lambda^j_{th} P^t_l - P^j_t {}''\Gamma^t_{lh} + P^j_t \frac{\partial A^t_l}{\partial u^h} \right) P^l_i - P^j_l {}''N_i{}^l{}_h,\end{aligned}$$

that is

$$\delta^j_{i,h} = \frac{''DP^j_l}{\partial u^h} P^l_i - P^j_l {}''N_i{}^l{}_h. \tag{3.5}$$

Making use of (2.2), (3.4) and (3.5), from the formula (1.2) we have

$$\begin{aligned} R_i{}^j{}_{hk} &= P^j_l \left\{ \left(\frac{\partial {}''\Lambda^l_{tk} P^t_m}{\partial u^h} - \frac{\partial {}''\Lambda^l_{th} P^t_m}{\partial u^k} \right) P^m_i + {}''\Lambda^l_{th} P^t_m \Lambda^m_{ik} - {}''\Lambda^l_{tk} P^t_m \Lambda^m_{ih} \right\} \\ &\quad + P^j_l \left\{ \left(\frac{\partial {}''N_m{}^l{}_k}{\partial u^h} - \frac{\partial {}''N_m{}^l{}_h}{\partial u^k} \right) P^m_i + {}''N_m{}^l{}_h \Lambda^m_{ik} - {}''N_m{}^l{}_k \Lambda^m_{ih} \right\} \\ &\quad + \Gamma^j_{lh} \delta^l_{i,k} - \Gamma^j_{lk} \delta^l_{i,h} \\ &= P^j_l {}''R_m{}^l{}_{hk} M^m_i \\ &\quad + \left\{ (-P^j_l {}''\Gamma^l_{th} + \Gamma^j_{th}) \frac{''DP^t_m}{\partial u^k} - (-P^j_l {}''\Gamma^l_{tk} + \Gamma^j_{tk}) \frac{''DP^t_m}{\partial u^h} \right\} P^m_i \\ &\quad - (-P^j_l {}''\Lambda^l_{th} + \Gamma^j_{th}) P^t_m {}''N_i{}^m{}_k + (-P^j_l {}''\Lambda^l_{tk} + \Gamma^j_{tk}) P^t_m {}''N_i{}^m{}_h \\ &\quad + P^j_l \left\{ {}''N_m{}^l{}_h \left(\frac{''DP^m_i}{\partial u^k} + {}''N_i{}^m{}_k \right) - {}''N_m{}^l{}_k \left(\frac{''DP^m_i}{\partial u^h} + {}''N_i{}^m{}_h \right) \right\}. \end{aligned}$$

Since

$$\begin{aligned} -P^j_l {}''\Gamma^l_{th} + \Gamma^j_{th} &= \frac{\partial P^j_t}{\partial u^h} + {}''\Lambda^j_{lh} P^l_t - P^j_l {}''\Gamma^l_{th} + {}''N_t{}^j{}_h \\ &= \frac{''DP^j_t}{\partial u^h} + {}''N_t{}^j{}_h \end{aligned}$$

and

$$-P^j_l {}''\Lambda^l_{th} + \Gamma^j_{th} = \frac{''DP^j_t}{\partial u^h} + {}''N_t{}^j{}_h + P^j_l \frac{\partial A^l_t}{\partial u^h},$$

the above equations can be written as

$$\begin{aligned} R_i{}^j{}_{hk} &= P^j_l {}''R_m{}^l{}_{hk} M^m_i \\ &\quad + \left\{ \left(\frac{''DP^j_t}{\partial u^h} + {}''N_t{}^j{}_h \right) \frac{''DP^t_m}{\partial u^k} - \left(\frac{''DP^j_t}{\partial u^k} + {}''N_t{}^j{}_k \right) \frac{''DP^t_m}{\partial u^h} \right\} P^m_i \\ &\quad - \frac{''DP^j_t}{\partial u^h} P^t_m {}''N_i{}^m{}_k + \frac{''DP^j_t}{\partial u^k} P^t_m {}''N_i{}^m{}_h \\ &\quad + P^j_l \left\{ {}''N_m{}^l{}_h \left(\frac{''DP^m_i}{\partial u^k} + {}''N_i{}^m{}_k \right) - {}''N_m{}^l{}_k \left(\frac{''DP^m_i}{\partial u^h} + {}''N_i{}^m{}_h \right) \right\}. \end{aligned} \tag{3.6}$$

THEOREM 2. *Let Γ be any normal general connection and $'\Omega^j_i$ and $''\Omega^j_i$ be the curvature forms of its contravariant part and its covariant part respectively, then*

$$P^j_l\,'\Omega^l_i = ''\Omega^j_l P^l_i, \tag{3.7}$$

where $P=\partial u_j \otimes P^j_i du^i = \lambda(\Gamma)$.

Proof. Making use of (2.1) and (2.2), from (3.3) and (3.6) we get the equation:

$$P^j_l\,'\Omega^l_i + A^j_l\,'DP^l_t \wedge ('DP^t_m - 'N_m{}^t{}_k du^k)Q^m_i - Q^j_l\,'DP^l_t \wedge 'N_m{}^t{}_k du^k \cdot A^m_i$$
$$= ''\Omega^j_l P^l_i + Q^j_l(''DP^l_t + ''N_t{}^l{}_h du^h) \wedge ''DP^t_m \cdot A^m_i + A^j_l\,''N_t{}^l{}_h du^h \wedge ''DP^t_m \cdot Q^m_i.$$

By means of (3.1) and (3.4), we get easily

$$A^j_l \cdot 'DP^l_i = 'DP^j_i \quad \text{and} \quad ''DP^j_l \cdot A^l_i = ''DP^j_i. \tag{3.8}$$

By means of (3.2) and (3.5), we get similarly

$$'DP^j_i = Q^j_l \cdot D\delta^l_i \quad \text{and} \quad ''DP^j_i = D\delta^j_l \cdot Q^l_i. \tag{3.9}$$

Substituting these into the above equation, we get

$$P^j_l\,'\Omega^l_i + Q^j_l D\delta^l_t \wedge (Q^t_s D\delta^s_m - 'N_m{}^t{}_k du^k)Q^m_i - Q^j_s Q^s_l D\delta^l_t \wedge 'N_m{}^t{}_k du^k \cdot A^m_i$$
$$= ''\Omega^j_l P^l_i + Q^j_l(D\delta^l_s \cdot Q^s_t + ''N_t{}^l{}_h du^h) \wedge D\delta^t_m \cdot Q^m_i + A^j_l\,''N_t{}^l{}_h du^h \wedge D\delta^t_s \cdot Q^s_m Q^m_i.$$

On the other hand, we have easily

$$\delta^j_{l,h}\,'N_i{}^l{}_k = -P^j_t A^t_{lh} N^l_m \Gamma^m_{ik}$$

and

$$''N_l{}^j{}_h \delta^l_{i,k} = A^j_{lh} N^t_l \Gamma^l_{mk} P^m_i.$$

Making use of the equations, the terms but $P^j_l\,'\Omega^l_i$ and $''\Omega^j_l P^l_i$ in the last equation cancel with each others. Hence we have

$$P^j_l\,'\Omega^l_i = ''\Omega^j_l P^l_i. \qquad \text{q. e. d.}$$

§ 4. The Ricci's formula for spaces with normal general connections.

In [12], the author proved that for a basic curve C: $u^j = u^j(t)$ in a space with a normal general connection Γ, any A-invariant vector V^j_0 of $\mathfrak{X}$ at $u^j_0 = u^j(t_0)$ can be uniquely translated parallel along C such that the parallelly translated vector $V^j(t)$ is A-invariant at each point of the curve, if Γ is contravariantly proper, that is

$$N^j_k \Gamma^k_{lm} A^l_i A^m_h = 0, \tag{4.1}$$

and the same fact holds good for A-invariant covariant vectors, if Γ is covariantly proper, that is

$$A^j_k A^k_{lm} N^l_i A^m_h = 0.^{4)} \tag{4.2}$$

Let $\bar{D}$ be the basic covariant differential operator of Γ. In [12], the author proved that along a basic curve the two conditions for a contravariant vector V^j:

4) See [12], Theorem 3.1 and Theorem 3.2.

$$A_i^j V^i = V^j, \qquad \frac{DV^j}{dt} = 0$$

and

$$A_i^j V^i = V^j, \qquad \frac{\bar{D}V^j}{dt} = 0$$

are equivalent to each others, if Γ is contravariantly proper. Analogously, the two conditions for a covariant vector W_i:

$$A_i^j W_j = 0, \qquad \frac{DW_i}{dt} = 0$$

and

$$A_i^j W_j = 0, \qquad \frac{\bar{D}W_i}{dt} = 0$$

are equivalent to each others, if Γ is covariantly proper.[5]

Now, we say that a general connection Γ is *integrable* when the distribution of the tangent subspace $P_x = P(T_x(\mathfrak{X}))$, $P = \lambda(\Gamma)$, $x \in \mathfrak{X}$, is completely integrable. If Γ is normal, the condition that Γ is integrable is equivalent to that for any two A-invariant vector fields $X^j = P_i^j x^i$ and $Y^j = P_i^j y^i$, we have

$$N_i^j [X, Y]^i = 0,$$

that is

$$N_i^j \left(P_l^h \frac{\partial P_k^i}{\partial u^h} - P_k^h \frac{\partial P_l^i}{\partial u^h} \right) x^l y^k = 0.$$

x^j and y^j are components of arbitrary contravariant vectors, hence the above condition can be replaced by

$$N_i^j \left(P_l^h \frac{\partial P_k^i}{\partial u^h} - P_k^h \frac{\partial P_l^i}{\partial u^h} \right) = 0$$

or

$$\left(\frac{\partial N_i^j}{\partial u^h} - \frac{\partial N_h^j}{\partial u^i} \right) A_k^i A_l^h = 0. \tag{4.3}$$

Now, let Γ be a contravariantly proper and integrable normal general connection. On an integral submanifold $\mathfrak{Y}$ of the distribution P_x, take a curve and consider an A-invariant contravariant vector field V^j parallelly translated along this curve, then we have

$$\bar{D}V^j = dV^j + {}'\Lambda_{ih}^j V^i du^h = 0 \tag{4.4}$$

by means of (4.8) in [10]. Take a two-dimensional cell $u^i = u^i(t, w)$ on $\mathfrak{Y}$, $(t, w) \in [-1, 1] \times [-1, 1]$. We denote the variations corresponding to t and w by d and δ respectively. Then, the infinitesimal difference between the vectors obtained by parallel translating an A-invariant contravariant vector V^j at the initial point $u_0^j = u^j(0, 0)$ firstly along a t-curve and secondly a w-curve and firstly along a w-

5) See [12], Theorem 4.1 and Theorem 4.2.

curve and secondly a t-curve is given by

$$(d\delta-\delta d)V^j=-\left(\frac{\partial' A^j_{ik}}{\partial u^h}-\frac{'A^j_{ih}}{\partial u^k}+'A^j_{lh}{}'A^l_{ik}-'A^j_{lk}{}'A^l_{ih}\right)V^i du^h \delta u^k.$$

The left hand side are the components of an A-invariant infinitesimal contravariant vector from the above mentioned fact, and so we have

$$(d\delta-\delta d)V^j=-A^j_l\left(\frac{\partial' A^l_{mk}}{\partial u^h}-\frac{\partial' A^l_{mh}}{\partial u^k}+'A^l_{ph}{}'A^p_{mk}-'A^l_{pk}{}'A^p_{mh}\right)A^m_i V^i du^h \delta u^k,$$

which can be written as

$$(d\delta-\delta d)V^j=-'R_i{}^j{}_{hk}V^i du^h \delta u^k \tag{4.5}$$

by means of (2.1).

Analogously consider the case in which Γ is a covariantly proper and integrable normal general connection. On an integral submanifold $\mathfrak{Y}$ of the distribution P_x, take a curve and an A-invariant covariant vector field W_i parallelly translated along this curve, then we have

$$\bar{D}W_i=dW_i-''\Gamma^j_{ih}W_j du^h=0 \tag{4.6}$$

by means of (4.8) in [10]. Take a two-dimensional cell on $\mathfrak{Y}$ and consider the same thing for an A-invariant vector W_i at the initial point as mentioned obove, then we have easily

$$(d\delta-\delta d)W_i=\left(\frac{\partial''\Gamma^j_{ik}}{\partial u^h}-\frac{\partial''\Gamma^j_{ih}}{\partial u^k}+''\Gamma^j_{lh}{}''\Gamma^l_{ik}-''\Gamma^j_{lk}{}''\Gamma^l_{ih}\right)W_j du^h \delta u^k.$$

The left hand side are the components of an A-invariant infinitesimal covariant vector and so we have

$$(d\delta-\delta d)W_i=A^j_l\left(\frac{\partial''\Gamma^l_{mk}}{\partial u^h}-\frac{\partial''\Gamma^l_{mh}}{\partial u^k}+''\Gamma^l_{th}{}''\Gamma^t_{mh}-''\Gamma^l_{tk}{}''\Gamma^t_{mh}\right)A^m_i W_j du^h \delta u^k,$$

which can be written as

$$(d\delta-\delta d)W_i=''R_i{}^j{}_{hk}W_j du^h \delta u^k \tag{4.7}$$

by means of (2.2).

From the form of the left hand side of (4.5) and (4.7), these correspond to the formula of Ricci in the classical theory of affine connections.

We define for a normal and integrable general connection Γ its holonomy groups as follows: Take a point x_0 of $\mathfrak{X}$ and the integral submanifold $\mathfrak{Y}$ through x_0 of the distribution of P_x. Take a curve of piece-wise class C^1 starting from x_0 and ending at the same point and parallelly translate A-invariant contravariant vectors or covariant vectors at x_0 along the curve when Γ is contravariantly proper or covariantly proper respectively. Then we get an isomorphism on P_{x_0} or its dual space $P_{x_0}{}^*$ corresponding to the curve. The set of such isomorphisms makes a group which we call *the basic homogeneous holonomy group at* x_0 *of the first kind or the second kind* and we denotes it by $BH'(x_0)$ or $BH''(x_0)$.

By means of the above mentioned method of definition of the basic homogeneous holonomy group, the groups of the same kind at points of $\mathfrak{Y}$ are clearly homologous to each others.

THEOREM 3. *Let Γ be a normal and integrable general connection of a differentiable manifold $\mathfrak{X}$. If Γ is contravariantly proper (covariantly proper) and the fundamental group $\pi_1(\mathfrak{Y})$ has at most a countable number of elements, then the connected component $BH_0'(x)$ $(BH_0''(x))$ of the basic homogeneous holonomy group of the first (second) kind $BH'(x)$ $(BH''(x))$ of Γ at a point x of $\mathfrak{X}$ is generated by*

$$'R_i{}^j{}_{hk} X^h Y^k \quad (''R_i{}^j{}_{hk} X^h Y^k),$$

where $'R_i{}^j{}_{hk}$ $(''R_i{}^j{}_{hk})$ are the components of the curvature tensor of the contravariant (covariant) part $'\Gamma$ $(''\Gamma)$ of Γ with respect to frames parallelly translated from a standard frame at x along any basic curves starting from x and X^j, Y^j are any A-invariant contravariant vectors.

Proof. By means of the definition $BH'(x)$ or $BH''(x)$, they are determined by the contravariant or covariant part of Γ. Hence, by virtue of the formulas (4.5) or (4.7), the connected component of the basic homogeneous holonomy group are generated by the elements mentioned in this theorem respectively as in the case of the classical affine connections.

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